Noncollapsed degeneration and desingularization of Einstein 4-manifolds

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Numerical and Geometric Methods for Ricci-flat Metrics and Flows
Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics

Einstein metrics in dimension 4

An **Einstein** metric satisfies

$$\Lambda \quad \mathbb{R},\ \mathrm{Ric}(g) = \Lambda g.$$

- **In dimension** 4, when they exist, Einstein metrics minimize

$$g \to \int_{M^4} |\mathrm{Rm}_g|^2 dv_g = \underbrace{8\pi^2\chi(M^4)}_{topological} + \underbrace{\int_{M^4} |\mathrm{Ric}^0_g|^2 dv_g}_{>0}.$$

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Moduli space of Einstein metrics

The Einstein equation is invariant by **reparametrization** and **rescaling**. The **moduli space of Einstein metrics** on M is

$$\mathbf{E}(M) := \Big\{(M,g) \mid \ \Lambda \quad \mathbb{R},\ \mathrm{Ric}(g) = \Lambda g,\ \mathrm{Vol}(M,g) = 1\Big\}/D(M),$$

where $D(M)$ is the group of diffeomorphisms from M to M acting on the metric by pull-back.

The **Gromov-Hausdorff distance** d_{GH} is the natural distance on $\mathbf{E}(M)$.

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What are the global properties of $\mathbf{E}(M)$? Can it be compactified ? With some structure ?

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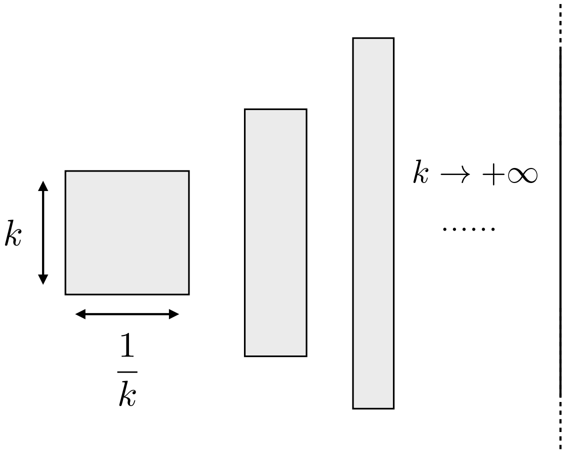
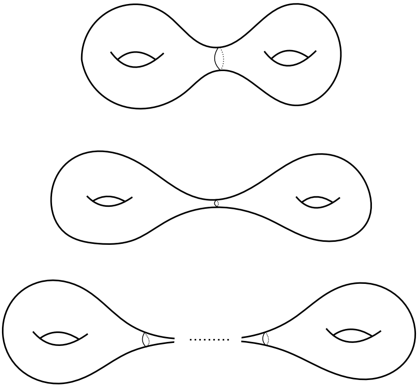
Degeneration in $\mathbf{E}(M^2)$

Collapsing



Degeneration in $\mathbf{E}(M^2)$

Cusp formation



Compactification of $\mathbf{E}(M^4)$

Theorem (Anderson ('89,'92), Bando-Kasue-Nakajima ('89), Cheeger-Tian ('06))

Let M^4 be a compact 4-dimensional differentiable manifold. There exists compactification of $(\mathbf{E}(M^4), d_{GH})$, denoted $\overline{\mathbf{E}(M^4)}_{GH}$.

- *We have a decomposition*

$$\overline{\mathbf{E}(M^4)}_{GH} = \mathbf{E}(M^4) \qquad \partial_\infty \mathbf{E}(M^4) \qquad \partial_o \mathbf{E}(M^4).$$

- *$\partial_\infty \mathbf{E}(M^4)$: limits of **collapsing** or formation of **cusps**,*
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Here, we will focus on the d_{GH}-**completion** of $\mathbf{E}(M^4)$ which is $\mathbf{E}(M^4) \quad \partial_o \mathbf{E}(M^4)$.

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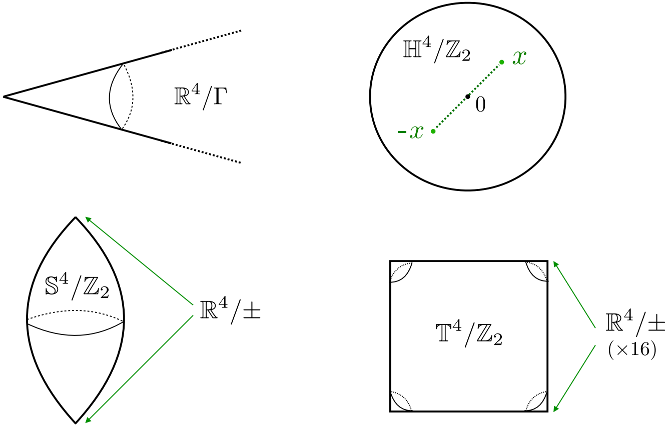
Examples of Einstein orbifolds

Einstein orbifolds (with isolated singularities) may have a **finite** number of singularities modelled on $\mathbb{R}^4/\Gamma$, for Γ $SO(4)$ acting freely on $\mathbb{S}^3$.

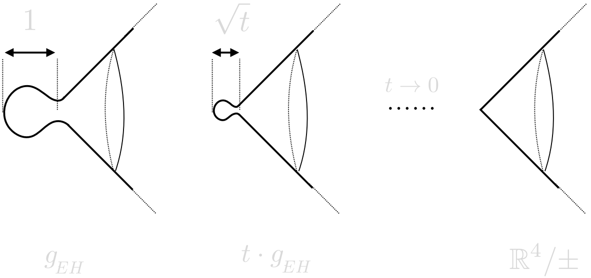


Rescaling of Ricci-flat ALE manifolds

Orbifold singularity formation

A Ricci-flat **Asymptotically Locally Euclidean (ALE)** satisfies $\mathrm{Ric} \equiv 0$ and is asymptotic to a cone $\mathbb{R}^4/\Gamma$ for Γ $SO(4)$ acting freely on $\mathbb{S}^3$.

Example : The Eguchi-Hanson metric ('79), g_{EH}, is Ricci-flat, asymptotic to $\mathbb{R}^4/\{\pm \mathrm{Id}\}$ and defined on T $\mathbb{S}^2$.



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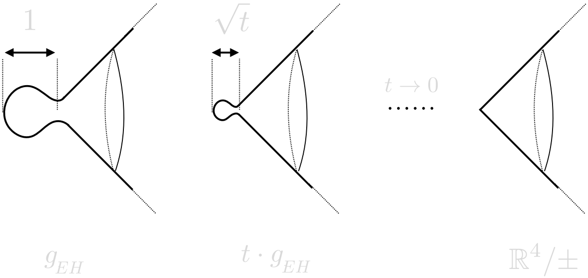

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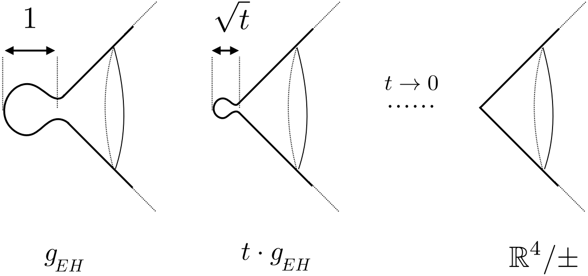
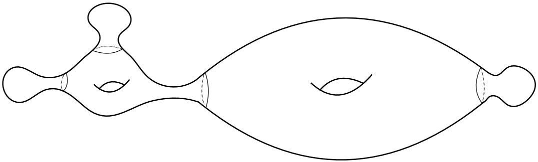


Tree of singularities

In general : there can be formation of **trees** of Ricci-flat ALE **orbifolds** (Bando '90).



Known Ricci-flat ALE orbifolds

The only known examples of Ricci-flat ALE orbifolds in dimension 4 are quotient of **hyperkähler** metrics called **gravitational instantons** (Kronheimer '89).

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And limits of Einstein metrics

Any d_{GH}-limit of a d_{GH}-**bounded** sequence of metrics in $\mathbf{E}(M^4)$ is an Einstein orbifold.

Question : Are **all** Einstein orbifolds d_{GH}-limit of Einstein metrics ?
Question : Can we d_{GH}-desingularize **any** Einstein orbifold by smooth Einstein metrics ?

O. ('19, '21) : No. The orbifold $\mathbb{S}^4/\mathbb{Z}_2$ is not d_{GH}-limit of smooth Einstein metrics.

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Question : What are the possible **compact** limits of sequences of Einstein metrics bubbling out Eguchi-Hanson metrics ?

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Structure of the moduli space

near its boundary $\partial_o\mathbf{E}(M^4)$

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- The **convergence** holds in a (weighted) C^∞ sense including in the bubble and neck regions.
- Any smooth Einstein metric sufficiently close to $\partial_o\mathbf{E}(M)$ is the result of a **gluing-perturbation** procedure.
- The moduli space is the **zero set** of a C^1-function on a C^1-manifold of metrics.
- The **dimension** of the moduli space is bounded by a function of the diameter and the Euler characteristic (or lower bound on scalar curvature) in dimension 4.

Question (Anderson) : Is the set of singular metrics $\partial_o\mathbf{E}(M)$ of codimension 2 in the moduli space $\mathbf{E}(M) \quad \partial_o\mathbf{E}(M)$? It is true up to second order (O. '20).

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Gluing-perturbation

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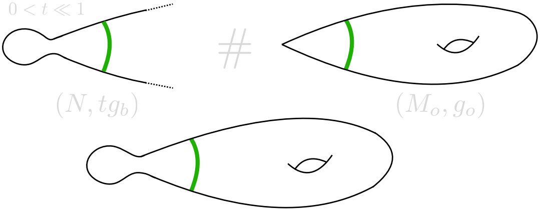
- $(N, g_b)\ {}_{\infty}\ \mathbb{R}^4/\Gamma$ with $\mathrm{Ric}(g_b) \equiv 0$,
- $(M_o, g_o)\ {}_{p}\ \mathbb{R}^4/\Gamma$ with $\mathrm{Ric}(g_o) = \Lambda g_o$.

We define the **naïve desingularization** (M, g^D_t).



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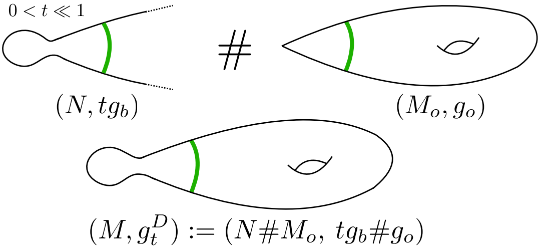

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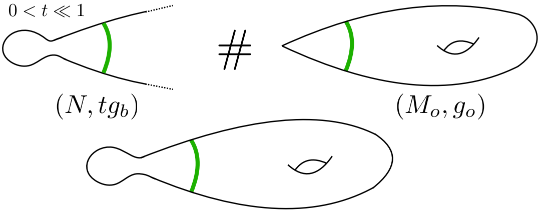

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From d_{GH} to weighted C^∞

Theorem (O. '19)

Let (M_o, g_o) $\partial_o \mathbf{E}(M)$. $\varepsilon > 0$, $\delta > 0$ *such that if* (M, g) $\mathbf{E}(M)$ *satisfies*

$$d_{GH}((M,g),(M_o,g_o)) \; 6 \; \delta,$$

then, there ***exists*** *a naïve desingularization of* (M_o, g_o) *by Ricci-flat ALE at scales* $t = (t_j)_j$ *denoted* (M, g^D_t) *such that we have*

$$g - g^D_t \quad {}_{C^{2,\alpha}_\beta(g^D_t)} \; 6 \; \varepsilon.$$

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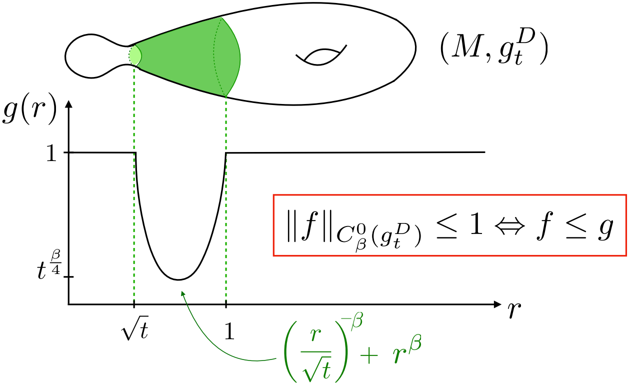
Weighted norm $C^{2,\alpha}_\beta$

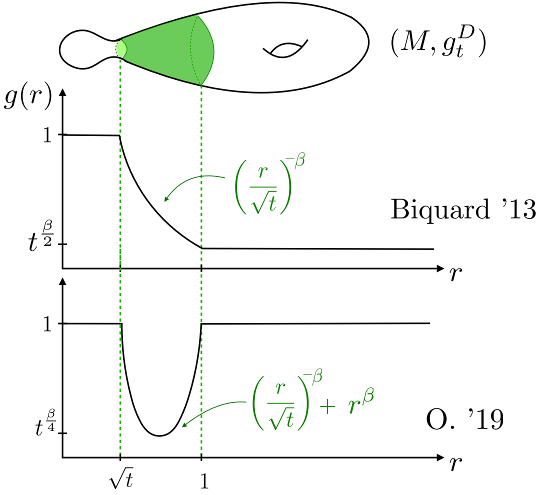
For any tensor s on M, one defines

$$s_{\ C^k_\beta(g^D_t)} = \sup_M \ g(r)^{-1}\Big(\sum_{i=0}^{k} r^i \big|\ {}^{i}_{g^D_t} s\big|_{g^D_t}\Big).$$

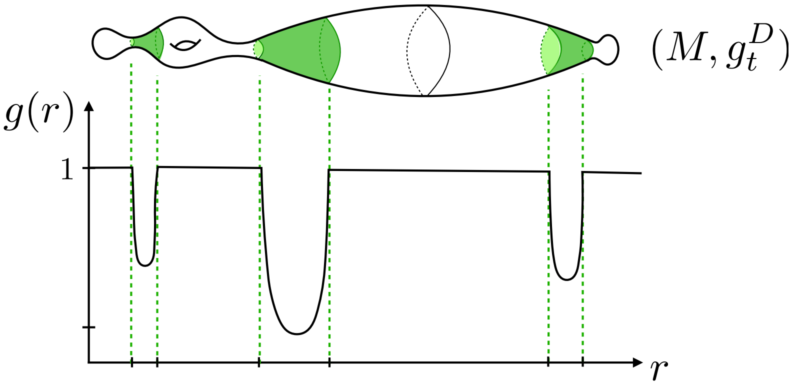


Weighted norm $C^{2,\alpha}_\beta$

Comparison with $\tilde{C}^{2,\alpha}_\beta$

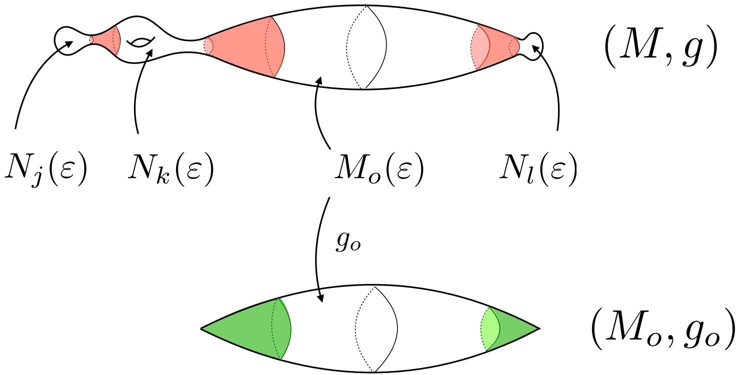


Weighted norm $C^{2,\alpha}_\beta$ on a tree of singularities



ε-regularity

(M,g) $\mathbf{E}(M)$ and $d_{GH}((M,g),(M_o,g_o))$ 1.



Neck regions, $A_k(t,\varepsilon)$

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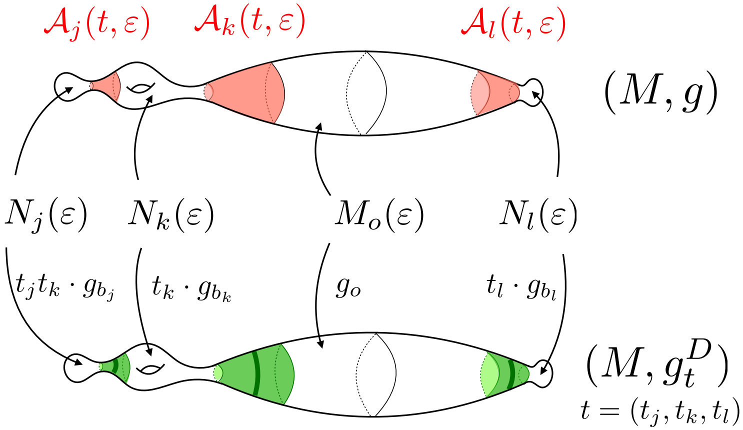

Coordinates in the neck regions

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Theorem (Partial converse, O.'19)

For any naïve desingularization (M, g^D_t) with $t \quad 1$, there exists an (unique if a gauge is fixed) Einstein ***modulo obstructions*** *metric $\hat{g}_t$ with :*

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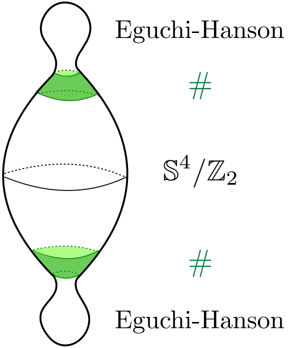

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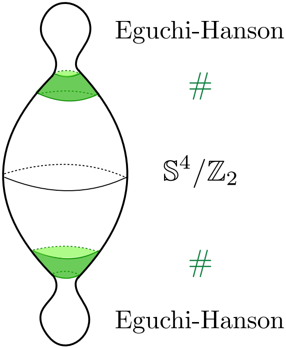

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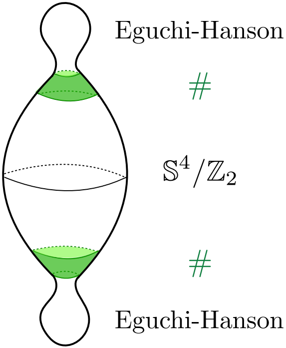

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Obstruction for spin manifolds

In dimension 4, a manifold is **spin** if its intersection pairing is even.

Corollary (O. ('19))

Assume that M^4 is **spin**. If n, g_n $\mathbf{E}(M^4)$ and $(M^4, g_n) \xrightarrow[n\to\infty]{GH} (M_o, g_o)$ then, for any p M_o with singularity $\mathbb{R}^4/\Gamma$ for Γ $SU(2)$, we have

$$\det \mathbf{R}^+_{g_o}(p) = 0 \text{ i.e. } \dim\ker \mathbf{R}^+_{g_o}(p) > 1.$$

General obstruction

without any assumption on the bubbles

Theorem (O. ('21))

*Let (M_o, g_o) be a compact **spherical** or **hyperbolic** orbifold with at least one singularity $\mathbb{R}^4/\mathbb{Z}_2$ (for instance $\mathbb{S}^4/\mathbb{Z}_2$). Then it **cannot** be d_{GH}-desingularized by smooth Einstein 4-manifold.*

- Analogy with obstructions to the **integrability** of infinitesimal Einstein deformations (inspired by Taub's preserved quantities in general relativity, Arms-Fischer-Marsden-Moncrief '80-'82).
- Study of the **obstruction** $\mathrm{Hess}_0\, u$ "coker"(Ric) of Ricci-flat ALE spaces (with $\Delta u = 8$, u r^2) as in Biquard-Hein ('19) and some variations of Schoen's Pohozaev identity ('88).
- Crucially uses the notion of **renormalized volume** and the coordinates of Biquard-Hein ('19).

1. From Gromov-Hausdorff to weighted C^∞
2. Desingularization of Einstein metrics
3. **Higher order obstructions to the desingularization**
4. Obstructions to the desingularization of $\mathbb{T}^4/\mathbb{Z}_2$
5. Conclusion and perspectives

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Assumptions of Biquard ('13), only assuming a d_{GH}-convergence.

Theorem (O', ('20))

Consider an Einstein orbifold (M_o^4, g_o) which

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and assume that $(M_o, g_o) \quad \mathbf{E}(M)$ *for* $M = M_o \# T \; \mathbb{S}^2$ *(like a desingularization by* $(T \; \mathbb{S}^2, g_{EH})$*). Then, we have*

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Nondegenerate desingularization

In the spirit of Spotti ('14), Biquard-Rollin ('15)

The obstruction $\dim\ker \mathbf{R}^+_{g_o}(p) > 2$ also holds under the assumptions :

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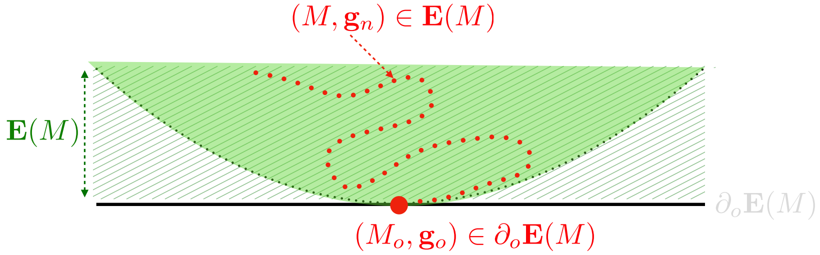

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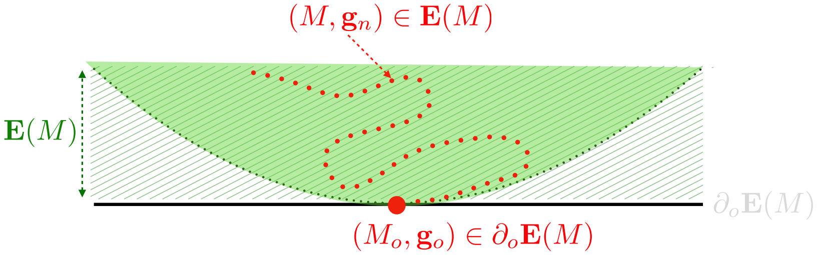

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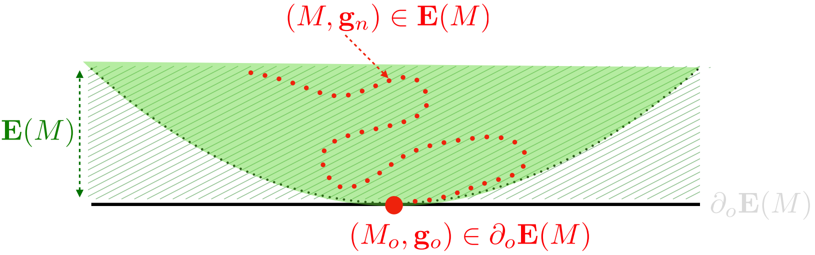

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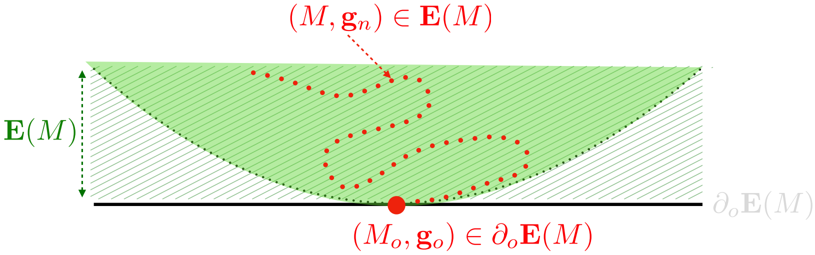

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In the spirit of Spotti ('14), Biquard-Rollin ('15)

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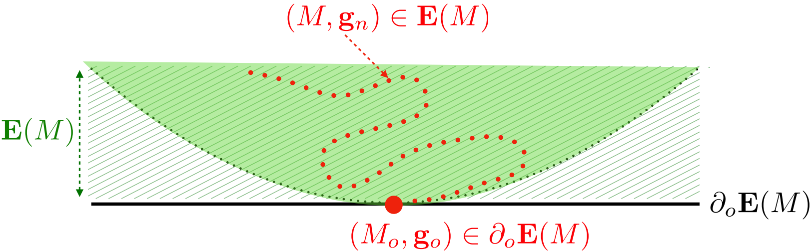

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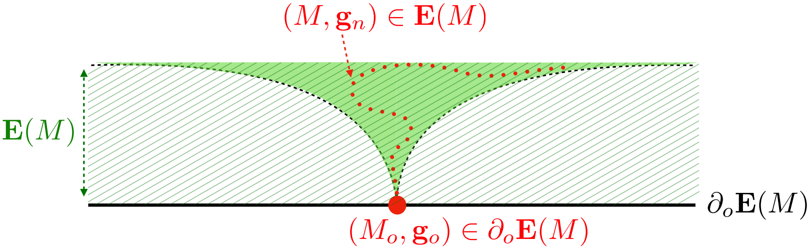

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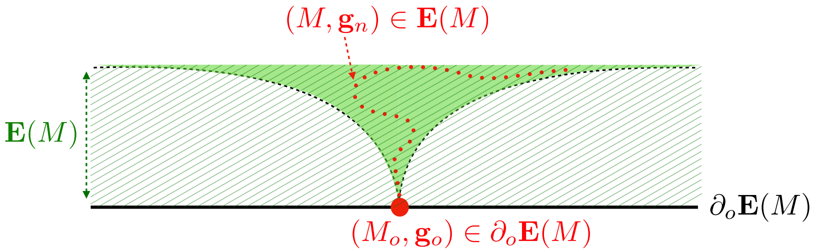

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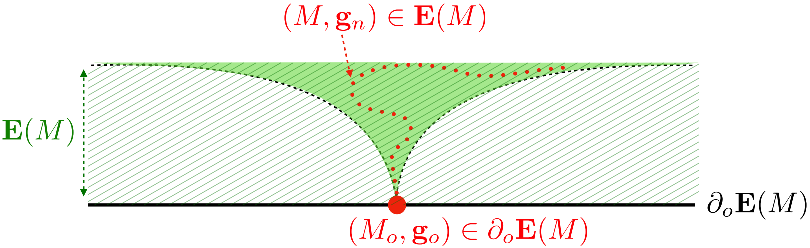

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Page-Pope ('87) : The AdS Taub-Bolt metrics on $T\ \mathbb{S}^2$ are Asymptotically Hyperbolic (AH) Einstein metrics degenerating to an AH Einstein orbifold bubbling out **one** Eguchi-Hanson metric.
This orbifold has only **one** singularity $\mathbb{R}^4/\pm$ at which, we have

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A question/conjecture

about the orbifolds in $\partial_o\mathbf{E}(M^4)$

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Conjecture (Anderson – "It is a filling")

The subspace $\partial_o\mathbf{E}(M^4)$ is of codimension 2 in $\overline{\mathbf{E}(M^4)}_{GH}$.

Remark : false in the AH setting.

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For any metric $(M_o, g_o) \quad \partial_o\mathbf{E}(M^4)$ there is a 2-dimensional set of Einstein desingularizations **transverse** to $\partial_o\mathbf{E}(M^4)$.

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2. Desingularization of Einstein metrics
3. Higher order obstructions to the desingularization
4. Obstructions to the desingularization of $\mathbb{T}^4/\mathbb{Z}_2$
5. Conclusion and perspectives

Naïve desingularization

by Eguchi-Hanson metrics

For $t > 0$ and $\varphi \quad O(4) = \mathrm{Isom}(\mathbb{R}^4/\pm)$, we glue :

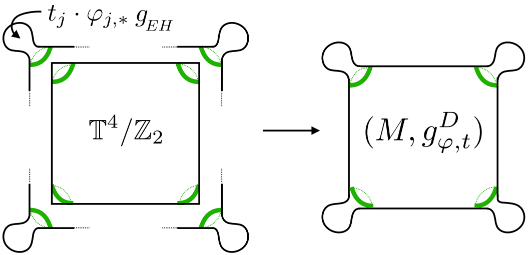
$$t \cdot g_b = t \cdot \varphi \;\; g_{EH}.$$

- **Positive** orientation : $\varphi \quad SO(4)$ **−→ obstruction :** $\det \mathbf{R}^+$ and $\dim\ker \mathbf{R}^+ > 2$,
- **Negative** orientation : $\varphi \quad O(4)\backslash SO(4)$ **−→ obstruction :** $\det \mathbf{R}^-$ and $\dim\ker \mathbf{R}^- > 2$,

Hyperkähler metrics on the K3 surface

Idea from Page ('78), Gibbons-Pope ('79)

Consider the desingularization $g^D_{\varphi,t}$ of a flat orbifold $\mathbb{T}^4/\mathbb{Z}_2$ by Eguchi-Hanson metrics at scales $t = (t_j)_j$ and $\varphi = (\varphi_j)_j$ **in the same orientation** (that is for $\varphi_j \quad SO(4)$).



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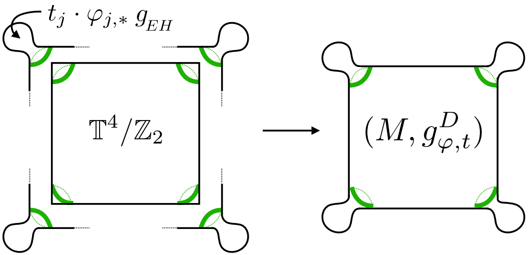

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General desingularization of $\mathbb{T}^4/\mathbb{Z}_2$

by Eguchi-Hanson metrics

Once a configuration of points with **positive** and **negative** orientations is chosen (among thousands of possibilities), there are 57 degrees of freedom :

- 9-dimensional space of flat deformations of $\mathbb{T}^4/\mathbb{Z}_2$ with fixed volume,
- 16 Eguchi-Hanson metrics with 3-dimensional spaces of Einstein deformations : one from a scaling factor $t > 0$ and two from $\varphi \in SO(4)/U(2)$ (since g_{EH} is $U(2)$-invariant).

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Obstructions on $\mathbb{T}^4/\mathbb{Z}_2$

Theorem (O. '20)

- *There are* 57 *obstructions to a* ***Ricci-flat*** *desingularization of* $\mathbb{T}^4/\mathbb{Z}_2$ *by Eguchi-Hanson metrics in different orientations.* 48 *are analogous to* $\det \mathbf{R}^{\pm} = 0$*.*
- *There are* 84 *obstructions to a* ***nondegenerate*** *or* ***stable Ricci-flat*** *desingularization.* 80 *are analogous to* $\mathbf{R}^{\pm} = 0$*.*

This indicates that for almost all flat metric on $\mathbb{T}^4/\mathbb{Z}_2$, the desingularization should be obstructed.

1. Consider the hyperkähler **partial** desingularizations.
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Obstructed situation

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*It is **impossible** to d_{GH}-desingularize the **regular** $\mathbb{T}^4/\mathbb{Z}_2$ coming from the lattice $\mathbb{Z}^4$ by Ricci-flat metrics thanks to*

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- 15 *negatively oriented Eguchi-Hanson metrics.*

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Brendle-Kapouleas ('17) configuration

Consider the following 1-dimensional set of desingularization configurations depending on $t > 0$:

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- a "**chessboard**" configuration of points with positive and negative orientations,
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- For all j, $t_{(j)} = t > 0$, $\varphi_{(j)} = \mathrm{Id} \quad SO(4)$ or $\varphi_{(j)} = (x_1, x_2, x_3, x_4) \to (-x_1, x_2, x_3, x_4) \ / \ SO(4)$.

Brendle-Kapouleas ('17) : There exists one nonvanishing **obstruction** to the desingularization. It is strikingly used to construct an intriguing **ancient solution to the Ricci flow**.

In this configuration, **none** of our 57 obstructions is satisfied.

Brendle-Kapouleas ('17) configuration

Consider the following 1-dimensional set of desingularization configurations depending on $t > 0$:

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Unobstructed situation ?

Consider the 48-dimensional situation with :

- the **regular** torus $\mathbb{T}^4/\mathbb{Z}_2$ coming from the lattice $\mathbb{Z}^4$,
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Then, there exists a 14-dimensional subspace of desingularization configurations satisfying **all** of the 84 obstructions.

Conjecture

Higher order obstructions should prevent this gluing.

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1. From Gromov-Hausdorff to weighted C^∞
2. Desingularization of Einstein metrics
3. Higher order obstructions to the desingularization
4. Obstructions to the desingularization of $\mathbb{T}^4/\mathbb{Z}_2$
5. Conclusion and perspectives

Conclusion (O. '19)

I have

- described the d_{GH}-neighborhood of $\partial_o\mathbf{E}(M)$ in $\overline{\mathbf{E}(M)}_{GH}$ in a **smooth** sense,
- extended the obstruction $\det \mathbf{R}(p) = 0$ to the conjecturally general situation
  - assuming only a d_{GH}-convergence,
  - allowing the orbifold to have **infinitesimal deformations** and **several** singularities of **general type**,
  - considering **any** gravitational instanton and quotient,
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Conclusion (O. '20)

I have partially answered and motivated 4 questions.

- Does a d_{GH}-limit of Einstein metric bubbling out gravitational instantons satisfy $\dim\ker \mathbf{R}^\pm > 2$ at its singular points ?
- Are the d_{GH}-limits of sequences of Einstein metrics bubbling out gravitational instantons **Kähler-Einstein** orbifolds ?
- Should $\partial_o\mathbf{E}(M^4)$ be thought of as a filling of missing pieces in $\mathbf{E}(M^4)$ rather than a boundary ?
- Can we desingularize $\mathbb{T}^4/\mathbb{Z}_2$ by a Ricci-flat but not hyperkähler metric by perturbing one of the configurations of the 14-dimensional set satisfying **all** of the first 84 identified obstructions ?

Conclusion (O. '21)

- Analogy between the problem of **desingularization** of Einstein metrics and the question of **integrability** of infinitesimal Einstein deformations.
- Some obstructions can be recovered from the **conformal Killing** vector fields of the cones in the neck regions.
- It is impossible to d_{GH}-desingularize spherical and hyperbolic orbifolds with $\mathbb{R}^4/\mathbb{Z}_2$ singularities.

Thank you for your attention !