Ricci solitons, conical singularities, and nonuniqueness

Numerical and Geometric Methods for Ricci-flat Metrics and Flows:

Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics

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- $\exists$ few known conditions that preclude finite-time singularities.
- "Most" singularities of compact solutions are modeled locally by ($\varkappa$-noncollapsed shrinking gradient) *solitons*: stationary solutions of the RF dynamical system on $\mathfrak{Met}/(\mathfrak{Diff} \times \mathbb{R}_+)$.

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- Perelman (2002): "It is likely that by passing to the limit in this construction one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that."

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- One might regard these as *weak solutions* of Ricci flow.
- *Under what circumstances can one expect weak solutions to be unique?*

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They prove that under suitable conditions, a sequence $\mathcal{M}_j$ of Ricci flows with surgery such that $\delta_j \searrow 0$ subconverges to a Ricci flow spacetime that should be regarded as a *weak solution* of the PDE.

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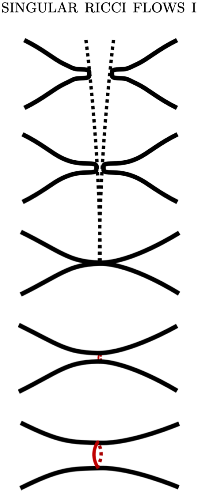

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- *What about higher dimensions?*

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- $\mathfrak{t} > 0$: each $\mathcal{M}^k_{>0}$ and $\tilde{\mathcal{M}}^k_{>0}$ is isometric to the flow of a distinct expanding $(n+1)$-dimensional soliton — that is, the spacetimes $\mathcal{M}^j_{>0}$ and $\tilde{\mathcal{M}}^k_{>0}$ are isometric if and only if $j = k$.

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Conclusion: weak solutions will not be unique in dimensions ≥ 5, not even topologically.

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- Seek $x_1(s)$, $x_2(s)$ and $X = f(s)\frac{\partial}{\partial s}$ that yield Ricci solitons:

$$-2\,\mathrm{Rc}[g] + \mathcal{L}_X(g) = \lambda g.$$

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$$-2\,\mathrm{Rc}[g] + \mathcal{L}_X(g) = \lambda g.$$

- Writing $x_\alpha = s^2 e^{-2u_\alpha}$ and $v = f - \sum_\alpha p_\alpha \frac{\mathrm{d}u_\alpha}{\mathrm{d}s}$, we obtain

$$\frac{\mathrm{d}^2 u_\alpha}{\mathrm{d}s^2} = v\frac{\mathrm{d}u_\alpha}{\mathrm{d}s} + e^{-2u_\alpha} + \lambda \quad (\alpha = 1, 2),$$
$$\frac{\mathrm{d}v}{\mathrm{d}s} = \sum_\alpha p_\alpha \left(\frac{\mathrm{d}u_\alpha}{\mathrm{d}s}\right)^2 - \lambda,$$

which we study as a first-order ODE system on $\mathbb{R}^5 \times \mathbb{R}$.

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$$x_\alpha = s^2 e^{-2u_\alpha}, \quad y_\alpha = s\frac{\mathrm{d}u_\alpha}{\mathrm{d}s} - 1, \quad \Gamma = sv + \lambda s^2, \quad \sigma = s^2.$$

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- Then setting $\tau = \log s$ and $' = \frac{\mathrm{d}}{\mathrm{d}\tau} = s\frac{\mathrm{d}}{\mathrm{d}s}$, we get our *soliton system*:

$$\begin{aligned} x_\alpha' &= -2x_\alpha y_\alpha, \\ y_\alpha' &= x_\alpha + \left(\Gamma + 1 - \lambda\sigma\right) y_\alpha + \Gamma + 1, \\ \Gamma' &= \Gamma + \textstyle\sum_\alpha p_\alpha (1 + y_\alpha)^2, \\ \sigma' &= 2\sigma. \end{aligned}$$

“Gluing” two special solutions

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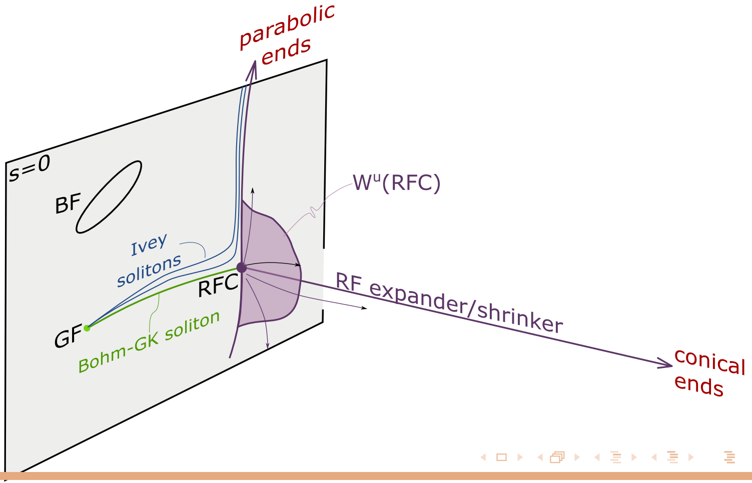

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- The linearization at the RFC stationary solution has a 2-dimensional invariant subspace with eigenvalues $-A \pm i\Omega$, where

$$A = \frac{n-1}{2}, \qquad \Omega = \frac{\sqrt{(n-1)(9-n)}}{2}.$$

Averaged & Difference variables

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- Better variables: define $\xi_\alpha = x_\alpha - (n-1)$ and coordinates

$$\Phi = \Big(\xi = \frac{1}{n}\sum_\alpha p_\alpha \xi_\alpha,\ y = \tfrac{1}{n}\sum p_\alpha y_\alpha,\ \gamma = \Gamma + n,$$
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- Then $\Phi' = \mathcal{A}(s)\Phi + \mathcal{N}(\Phi,\Phi)$, where

$$\mathcal{A}(s) = \begin{bmatrix} 0 & -2(n-1) & 0 & & \\ 1 & -(n-1+\lambda s^2) & 1 & & \\ 0 & 2n & 1 & & \\ & & & 0 & -2(n-1) \\ & & & 1 & -(n-1+\lambda s^2) \end{bmatrix}.$$

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- Oscillatory subsystem $\Rightarrow$ winding behavior near $s = 0$.

Visualization revisited

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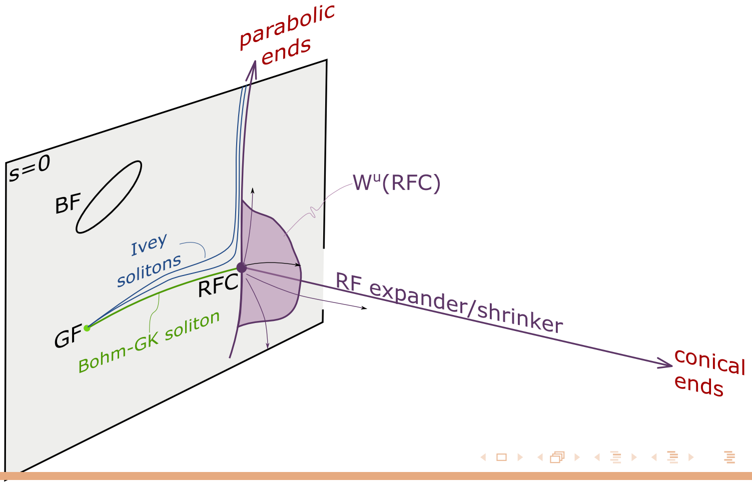

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THANK YOU!