Discussion: Ricci flat metrics of generic holonomy.

Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics Workshop 24-27 May 2021

B. Acharya (ICTP/KCL)

Fundamental Question: do Ricci flat metrics of GENERIC holonomy exist on compact manifolds?

Since the splitting theorem asserts that the cpt mfld M^n is isometric to a local product:

$$M^n = \frac{(M^{d_1} \times M^{d_2} \times \dots M^{d_k} \times T^p)}{\Gamma}$$

- $\sum d_i + p = n$ Γ: finite group
- metric is flat on T^p, $\pi_1(M^{d_i}) = \mathbb{1}$.
- We can reduce the question to:

Do compact, simply-connected manifolds admit Ricci flat metrics of generic holonomy?

No known counterexamples.

All known examples have special holonomy and parallel spinor(s).

$\Updownarrow$

SUPERSYMMETRY

Topological obstructions Exist

eg in dimensions $n = 4k$

can use Lichnerowicz - Hitchin...

by squaring the Dirac operator

$$\not{D}^2 = \nabla^2 + \frac{1}{4}R$$

If $R = 0$ Dirac zero modes are parallel $\Longleftrightarrow$ special holonomy

In dimension 4, topological 4-mflds are $M^4_s(m,n) = \#_m K3 \#_n S^2 \times S^2$ (spin)

$M^4_{ns}(p,q) = \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2}$ (non spin)

Since $\operatorname{Index} \not{D}(M_s(m,n)) = 2m$ we get that eg $M_s(m \geq 1, n \geq 1)$ has no Ricci flat metric.

But for $M(0,n)$ the argument fails.

So does $S^2 \times S^2$ have Ricci flat metrics?

For the non-spin $M^4_{ns}(p,q)$ can use Hitchin-Thorpe inequality to rule out Einstein metrics when $q \gg p$.

So far, topological obstructions are essentially limited to $4k$, $4k+1$, $4k+2$ dimensions.

Even here if the invariants are zero, not useful.

Other methods? $\Rightarrow$ Explicit attempts

Numerics?

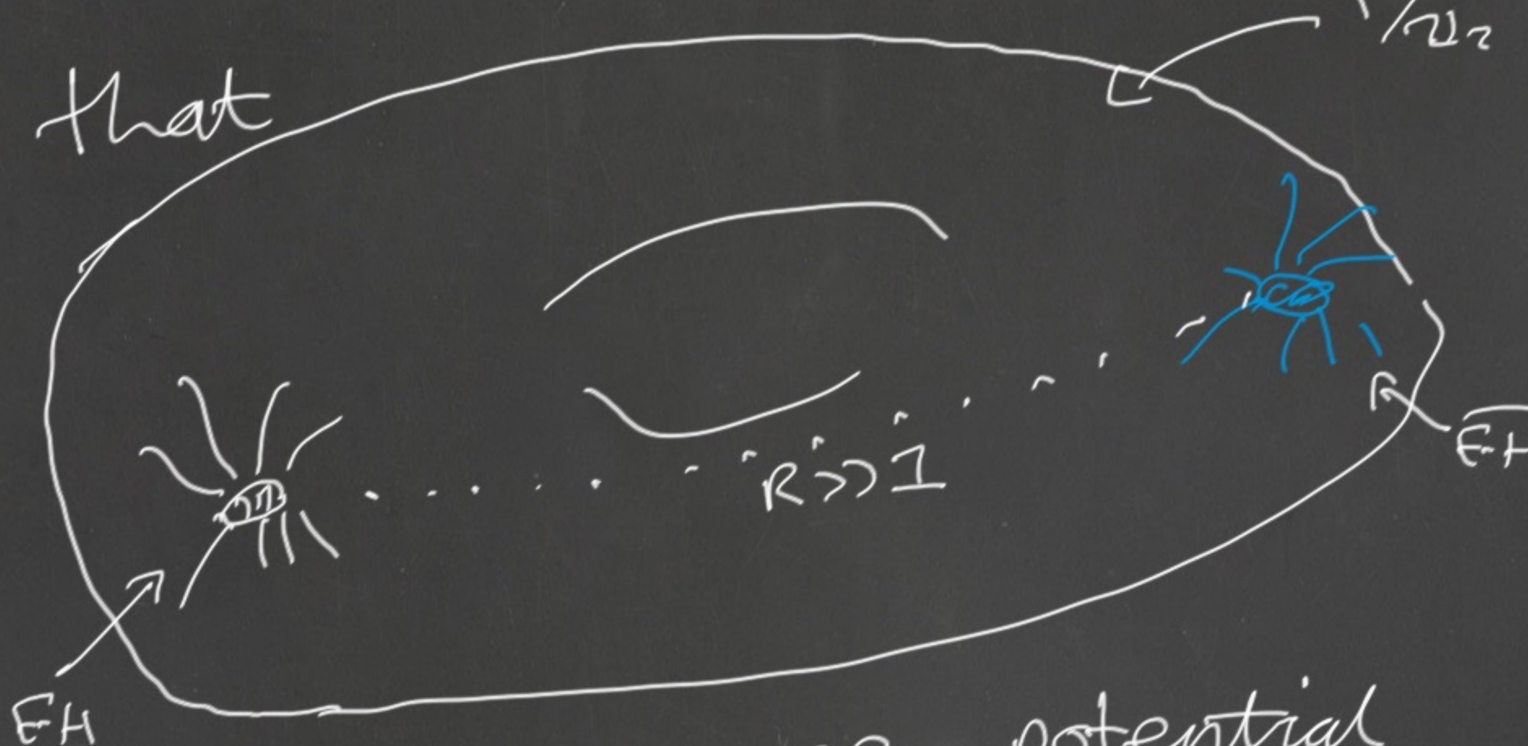
In the talks of Nicos and Tristan we saw that attempts to find Ricci flat metrics using gluing for

$$\frac{T^4}{\mathbb{Z}_2} = \#_p \mathbb{CP}^2 \#_q \overline{\mathbb{CP}^2} \qquad n_{EH} > 0$$

$$p = 3 + n_{EH} \qquad q = 3 + n_{\overline{EH}}$$

obeys Hitchin-Thorpe

From those results we see that



Very long range potential develops between EH and $\overline{EH}$.

It could be interesting to try and understand if this potential leads to some general intuition/physics which would give "deeper" reasons for why these metrics could never be Ricci flat.

Questions about gluing / direct construction:

- More examples needed
  - local models eg $\frac{\mathbb{R}^3 \times S^1}{\mathbb{Z}_2}$.
    or ALE versions
- Higher dimensional versions of the Kummer case...

Physically, one is interested in STABLE Ricci flat metrics.

The conjecture that all stable Ricci flat metrics on compact manifolds have special holonomy and parallel spinors implies that superstring/M-theory in the geometric regime "predicts" low energy supersymmetry.

Non-Simply connected cases are also interesting....

- Witten '80's: $S^1 \times$ Minkowski Spacetime is unstable with antiperiodic spin structure.

- BSA 2020: <u>All</u> compact, flat 3-mflds without parallel spinors have a generalised Witten Instability (except one case, which was proven by Garcia-Extebarria, Montero, Sousa, Valenzuela 2020)

This proves the conjecture for 3-manifolds!

- What about $M^n = \dfrac{\widetilde{M}^n}{\Gamma}$ with $\pi_1(\widetilde{M}^n) = \mathbb{1}$?
- If $(\widetilde{M}^n, g)$ is Ricci flat <u>special</u> holonomy, but Γ does <u>not</u> preserve parallel spinors, what happens?

eg if $M^4 = \text{Enriques Surface} = \dfrac{K3}{\mathbb{Z}_2}$?

- Here one would hope to find a generalised Witten / Bubble of Nothing instability.