Analyticity and Resurgence

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1. Stability structures on graded Lie algebras

1.1. Setup: grading lattice: $\Gamma = \mathbb{Z}^n$ Lie algebra over $\mathbb{Q}$: $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_{\gamma}$

$\text{Stab}(\mathfrak{g}) := \{(Z, (a_\gamma)_{\gamma \in \Gamma}) \mid Z : \Gamma \to \mathbb{C}, a_\gamma \in \mathfrak{g}_{\gamma}, \text{obeys support axiom}\}$

Support axiom: $\exists$ non-degenerate quadratic form $Q$ on $\Gamma \otimes \mathbb{R}$ such that

\begin{align*}
1. Q|_{\text{Ker}(Z \otimes \mathbb{R}) - \{0\}} &< 0 \\
2. Q(\gamma) &> 0 \quad \forall \gamma, a_\gamma \neq 0
\end{align*}

$\implies \{Z(\gamma) \mid a_\gamma \neq 0\} \subset \mathbb{C}$ is discrete
1.2. Alternative description of $\text{Stab}(g)$ for given $Z$

Define a group $G_V$ for any open sector $V \subset \mathbb{C}$, $0^\circ < \text{angle}(V) < 180^\circ$:

$$G_V := \varprojlim G_{C_V} \text{ where } C_V \subset \Gamma \otimes \mathbb{R} \text{ is a strict closed convex cone such that }$$

$$Z(C_V - \{0\}) \subset V \text{ (this implies that } Z|_{C_V} \text{ is a proper map),}$$

and $G_{C_V} := \mathbb{Q}$-pronilpotent group with $\text{Lie}(G_V) := \prod_{\gamma \in (C_V \cap \Gamma)} g_\gamma$
Choose a covering $C - \{0\} = \bigcup_{i \in \mathbb{Z}/N\mathbb{Z}} V_i$ s.t. $V_i \cap V_{i+1} \neq \emptyset, V_i \cap V_{i+2} = \emptyset$:

We get a diagram of inclusions of groups:

\[
\begin{array}{ccc}
\cdots & G_{V_{i} \cap V_{i+1}} & \cdots \\
\downarrow & & \downarrow \\
G_{V_i} & G_{V_{i+1}} & \\
\end{array}
\]

**Theorem:** Fix $Z$ and a covering $(V_i)$. Then

\[
\text{Stab}(g) = \cdots \times G_{V_{i-1} \cap V_i} \times G_{V_i \cap V_{i+1}} \times G_{V_{i+1} \cap V_{i+2}} \times \cdots 
\]
1.3. Topology on $\text{Stab}(g)$

Choose a covering $(V_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$, stability structure, and its representative given by collections $(C_{V_i})$ and $(A_i \in G_{C_{V_i}})$. Fundamental system of neighborhoods: vary a bit $Z : \Gamma \to \mathbb{C}$ keeping both $(C_{V_i})$ and $(A_i \in G_{C_{V_i}})$ constant.

In this way $\text{Stab}(g)$ becomes a Hausdorff complex manifold such that the forgetting map $\text{Stab}(g) \to \Gamma^* \otimes \mathbb{C} \simeq \mathbb{C}^n$, $(Z, (a_\gamma)) \mapsto Z$ is a local analytic isomorphism.

**Generalization**: replace $\Gamma$ by a local system of lattices on $\mathbb{C} - \{0\}$, and $g$ also by a local system of Lie algebras.

Appears in real life: WKB limits for irregular singularities.
2. Analytic stability structures

From now on we assume $g = \text{polynomial vector fields on } (\mathbb{C}^\times)^n$:

$$g = \left\{ \sum_{\text{finite}} a_{\gamma,j} x^\gamma \cdot \frac{x_j \partial}{\partial x_j} \right\}, \quad \dim_{\mathbb{C}} g_\gamma = n \quad \forall \gamma \in \mathbb{Z}^n$$

Sufficient for 2d-4d wall-crossing.

2.1. Definition: An **analytic** stability structure is such that $\exists$ a representative $(A_i \in G_{C_{V_i}})$ such that each transformation $A_i$ is analytic: $\exists \ const > 0$

$$A_i : x_j \mapsto x_j \cdot \left( 1 + \sum_{\gamma \in C_{V_i} \cap \Gamma \neq 0} c_{\gamma,j}^{(i)} x^\gamma \right) \text{ obeys } |c_{\gamma,j}^{(i)}| < const |\gamma|$$
2.2. (Potential) examples and non-examples

If we talk about DT-invariants for 3CY categories, there are several groups of examples:

- Quivers with potentials:
  - derived equivalent to acyclic quivers: yes
  - corresponding to 2CY categories associates with non-oriented graphs (related to Kac polynomials and hyperbolic Kac-Moody algebras): no
  - associated with "Strebel points" in Hitchin bases: yes

- Fukaya categories of certain noncompact 3CY: yes

- Fukaya categories for compact 3CY: no.
  - Ooguri-Strominger-Vafa conjecture: \( |DT(\gamma)| \sim e^{const}|\gamma|^2 \)
2.3. Main result

**Theorem:** *the property of stability structure to be analytic is open and closed.*

From the proof we will see that analytic stability structures give rise to a complex analytic object.

Later we will propose another construction, producing a second analytic space which gives (hypothetically) a special class of Écalle resurgence.

What analytic object can be associated with $\exp\left(O(|\gamma|^2)\right)$ growth?

For a compact 3CY physics predicts that *refined* DT invariants give rise to a quaternionic-Kähler manifold, which gives a holomorphically contact manifold.
2.4. Proof

Openess is obvious. The closedness (and also the independence of the choice of covering $(V_i)$) is not.

Cones $C_{V_i}$ in $\Gamma \otimes \mathbb{R} = \mathbb{R}^n$ can overlap. We can always modify them such that each $C_{V_i}$ is rational polyhedral cone, and $C_{V_i} \cap C_{V_{i+1}}$ is a $(n - 1)$-dimensional common face of $C_{V_i}$ and $C_{V_{i+1}}$ for each $i$.

Get a fan, hence a non-compact toric variety, which contains a wheel of $\mathbb{C}P^1$-s, stratified by toric strata. Formal stability structure $\iff$ deformation of the formal neighborhood of the wheel preserving stratification.

We will see that analytic stability structures are analytic germs.
Canonical representatives:

\[ \ldots, A_{C_{V_i}}, A_{C_{V_i} \cap C_{V_{i+1}}}, A_{C_{V_{i+1}}}, \ldots \]

Claim: Analyticity \iff all canonical representatives are analytic.

Basic lemma: For any rational polyhedral cone \( C \) and its cut by a rational hyperplane \( C = C_- \cup C_0 \cup C_+ \) we have

\[
G_{C}^{an} = G_{C_-}^{an} \cdot G_{C_0}^{an} \cdot G_{C_+}^{an}
\]
Proof of lemma is based on a geometric interpretation of decomposition as above. After additional blow-ups we get two divisors, a chain of \( \mathbb{C}P^1 \)-s which deform by the versal deformation to a \((n - 1)\)-parameter family of smooth \( \mathbb{C}P^1 \)-s covering locally the space. This is true formally, to go to analytic theory we use Douady spaces.
For Hitchin systems, algebraically it is quite interesting:

a wheel of $\mathbb{C}P^1$-s at infinity of a partial compactification of Betti moduli spaces (character varieties). Families of smooth $\mathbb{C}P^1$-s are representations

$$\pi_1(\text{surface}) \rightarrow GL(r, \mathbb{C}[t, t^{-1}])$$
3. Second analytic space and relation to resurgence

3.1. Construction

For an analytic stability structure, draw several cuts (3 is enough) on $\mathbb{h}$-plane. Consider the trivial holomorphic bundle

$$(\mathbb{C}^\times)^n_{x_1,\ldots,x_n} \times \mathbb{C}_\mathbb{h} \rightarrow \mathbb{C}_\mathbb{h}$$

and modify it along cuts by

$$x_j \mapsto x_j \cdot \left( 1 + \sum_{\gamma \in C_{V_i} \cap \Gamma-0} e^{-\frac{Z(\gamma)}{\mathbb{h}}} c^{(i)}_{\gamma,j} x^\gamma \right)$$

which is $A_i$ conjugated by the multiplicative shift $x_j \mapsto e^{-\frac{z_j}{\mathbb{h}}} x_j$. 
Rigorous construction: use the oriented real blow-up at $\hbar = 0$ and Newlander-Nirenberg theorem.

In this way we get a holomorphic submersion $\mathcal{X} \to \mathbb{C}_\hbar$ together with a formal trivialization over $\mathbb{C}[[\hbar]]$,

$$\mathcal{X} \times_{\mathbb{C}_\hbar} \text{Spec} \mathbb{C}[[\hbar]] \simeq (\mathbb{C}^\times)^n \times \text{Spec} \mathbb{C}[[\hbar]]$$

More precisely, $\mathcal{X}$ is only a germ at the central fiber $(\mathbb{C}^\times)^n \times \{\hbar = 0\}$.

Real life: consider blow-up of the twistor family at a subvariety of the Hitchin moduli space which is a fiber of Hitchin system (an abelian variety). Then the germ of the total space at the exceptional divisor is our "second analytic space".
3.2 Relation to resurgence.

**Master conjecture**: *in the above situation, for any germ of a section of $\mathcal{X} \to \mathbb{C}_\hbar$, in the formal trivialization written as*

\[ x_j(\hbar) = x_j(0) \cdot \exp \left( \sum_{k \geq 1} c_{j,k} \hbar^k \right) \]

*each series $\sum_k c_{j,k} \hbar^k$ is resurgent* (Borel transform has all nice properties).

\[ n = 1 \iff \text{Écalle-Voronin theory.} \]

General $n$: work in progress with David Sauzin: sum over trees, estimates.

This will be a geometric formulation of a very common class of resurgent series, including all relevant ones for WKB.