Analyticity and Resurgence

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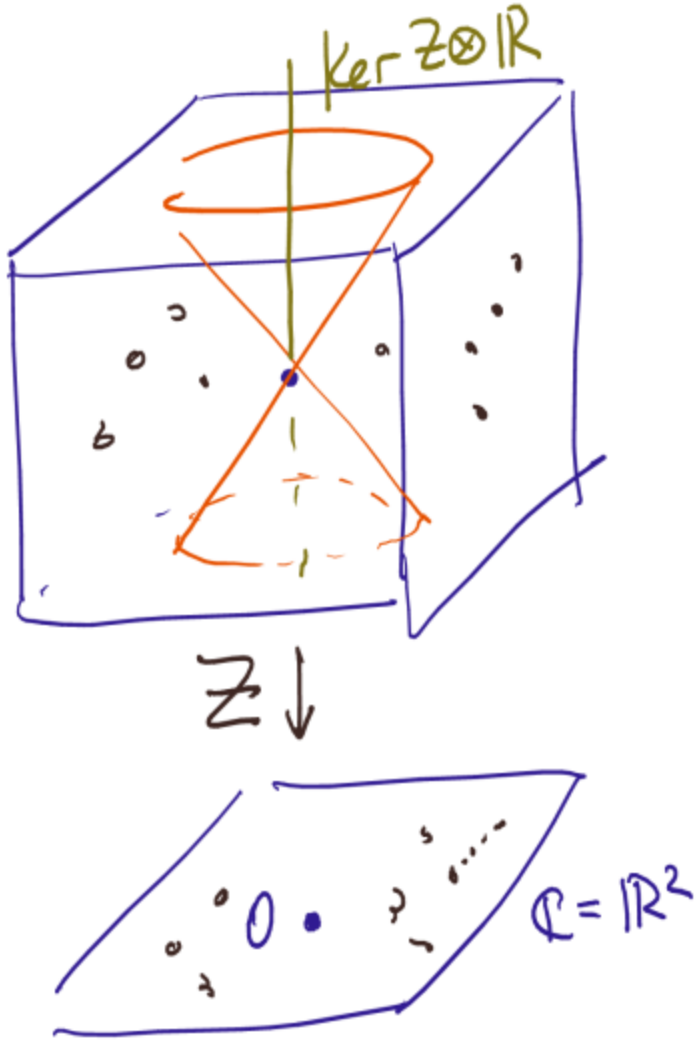
1. Stability structures on graded Lie algebras

1.1. Setup: grading lattice : $\Gamma = \mathbb{Z}^n$ Lie algebra over $\mathbb{Q}$: $\mathfrak{g} = \oplus_{\gamma\in\Gamma}\mathfrak{g}_\gamma$

$$\mathsf{Stab}(\mathfrak{g}) := \{(Z, (a_\gamma)_{\gamma\in\Gamma}) \,|\, Z : \Gamma \to \mathbb{C},\ a_\gamma \in \mathfrak{g}_\gamma,\ \text{obeys support axiom}\}$$

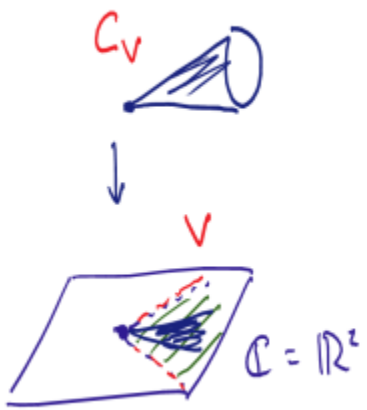
Support axiom: $\exists$ *non-degenerate quadratic form* Q *on* $\Gamma \otimes \mathbb{R}$ *such that*

$$\left.\begin{array}{l} 1.\, Q_{|\mathrm{Ker}(Z\otimes\mathbb{R})-\{0\}} < 0 \\ 2.\, Q(\gamma) > 0 \quad \forall\gamma,\ a_\gamma \neq 0 \end{array}\right\} \implies \{Z(\gamma)|a_\gamma \neq 0\} \subset \mathbb{C} \text{ is discrete}$$


Ker Z⊗ℝ
Z
0
ℂ = ℝ²

1.2. Alternative description of $\mathsf{Stab}(\mathfrak{g})$ for given Z

Define a group G_V for any *open* sector $V \subset \mathbb{C}, \quad 0^o < \mathrm{angle}(V) < 180^o$:
$G_V := \varinjlim_{C_v} G_{C_V}$ where $C_V \subset \Gamma \otimes \mathbb{R}$ is a strict *closed* convex cone such that
$Z(C_V - \{0\}) \subset V$ (this implies that $Z_{|C_V}$ is a proper map),



and $G_{C_V} :=$ $\mathbb{Q}$-pronilpontent group with $Lie(G_V) := \prod_{\gamma \in (C_V \cap \Gamma)} \mathfrak{g}_\gamma$

Choose a covering $\mathbb{C} - \{0\} = \cup_{i \in \mathbb{Z}/N\mathbb{Z}} V_i$ s.t. $V_i \cap V_{i+1} \neq \emptyset, V_i \cap V_{i+2} = \emptyset$:

We get a diagram of inclusions of groups

$$\begin{tikzcd}
\cdots \arrow[dr] & & G_{V_i \cap V_{i+1}} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \\
& G_{V_i} & & G_{V_{i+1}} &
\end{tikzcd}$$

Theorem: *Fix Z and a covering (V_i). Then*

$$\mathsf{Stab}(\mathfrak{g}) = \ldots \underset{G_{V_{i-1} \cap V_i}}{\times} G_{V_i} \underset{G_{V_i \cap V_{i+1}}}{\times} G_{V_{i+1}} \underset{G_{V_{i+1} \cap V_{i+2}}}{\times} \ldots$$

1.3. Topology on $\mathsf{Stab}(\mathfrak{g})$

Choose a covering $(V_i)_{i\in\mathbb{Z}/N\mathbb{Z}}$, stability structure, and its representative given by collections (C_{V_i}) and $(A_i \in G_{C_{V_i}})$. Fundamental system of neighborhoods: vary a bit $Z : \Gamma \to \mathbb{C}$ keeping both (C_{V_i}) and $(A_i \in G_{C_{V_i}})$ *constant*.

In this way $\mathsf{Stab}(\mathfrak{g})$ becomes a Hausdorff complex manifold such that the forgetting map $\mathsf{Stab}(\mathfrak{g}) \to \Gamma^* \otimes \mathbb{C} \simeq \mathbb{C}^n, \quad (Z, (a_\gamma)) \mapsto Z$
is a local analytic isomorphism.

Generalization: replace Γ by a local system of lattices on $\mathbb{C} - \{0\}$, and $\mathfrak{g}$ also by a local system of Lie algebras.

Appears in real life: WKB limits for irregular singularities.

2. Analytic stability structures

From now on we assume $\mathfrak{g}$ = polynomial vector fields on $(\mathbb{C}^\times)^n$:

$$\mathfrak{g} = \left\{ \sum_{\text{finite}} a_{\gamma,j} x^\gamma \cdot \frac{x_j \partial}{\partial x_j} \right\}, \qquad \dim_{\mathbb{C}} \mathfrak{g}_\gamma = n \quad \forall \gamma \in \mathbb{Z}^n$$

Sufficient for 2d-4d wall-crossing.

2.1. Definition: *An **analytic** stability structure is such that* $\exists$ *a representative* $(A_i \in G_{C_{V_i}})$ *such that each transformation* A_i *is analytic:* $\exists\, const > 0$

$$A_i : x_j \mapsto x_j \cdot \left(1 + \sum_{\gamma \in C_{V_i} \cap \Gamma - 0} c^{(i)}_{\gamma,j} x^\gamma \right) \text{ obeys } |c^{(i)}_{\gamma,j}| < const^{|\gamma|}$$

2.2. (Potential) examples and non-examples

If we talk about DT-invariants for 3CY categories, there are several groups of examples:

- Quivers with potentials:
  - derived equivalent to acyclic quivers : yes
  - corresponding to 2CY categories associates with non-oriented graphs (related to Kac polynomials and hyperbolic Kac-Moody algebras): no
  - associated with "Strebel points" in Hitchin bases: yes
- Fukaya categories of certain noncompact 3CY: yes
- Fukaya categories for *compact* 3CY: no.
  Ooguri-Strominger-Vafa conjecture: $|DT(\gamma)| \sim e^{const|\gamma|^2}$

2.3. Main result

Theorem: *the property of stability structure to be analytic is open and closed.*

From the proof we will see that analytic stability structures give rise to a complex analytic object.

Later we will propose another construction, producing a second analytic space which gives (hypothetically) a special class of Écalle resurgence.

What analytic object can be associated with $\exp(O(|\gamma|^2))$ growth?
For a compact 3CY physics predicts that *refined* DT invariants give rise to a quaternionic-Kähler manifold, which gives a holomorphic contact manifold.

2.4. Proof

Openess is obvious. The closedness (and also the independence of the choice of covering (V_i)) is not.

Cones C_{V_i} in $\Gamma \otimes \mathbb{R} = \mathbb{R}^n$ can overlap. We can always modify them such that each C_{V_i} is *rational polyhedral cone*, and $C_{V_i} \cap C_{V_{i+1}}$ is a $(n-1)$-dimensional common face of C_{V_i} and $C_{V_{i+1}}$ for each i.

Get a fan, hence a non-compact toric variety, which contains a wheel of $\mathbb{C}P^1$-s, stratified by toric strata. Formal stability structure $\iff$ deformation of the formal neighborhood of the wheel preserving stratification.

We will see that *analytic* stability structures are *analytic* germs.

Canonical representatives:

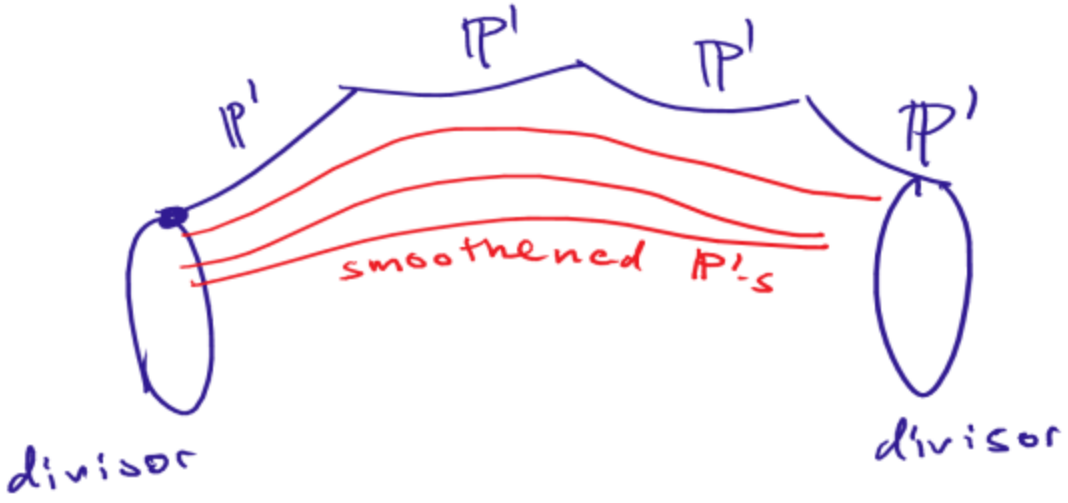
$$\ldots, A_{C^\circ_{V_i}}, A_{C_{V_i} \cap C_{V_{i+1}}}, A_{C^\circ_{V_{i+1}}}, \ldots$$

Claim: *Analyticity* $\iff$ *all canonical representatives are analytic*.

Basic lemma: *For any rational polyhedral cone* C *and its cut by a rational hyperplane* $C = C_- \cup C_0 \cup C_+$ *we have*

$$G^{an}_C = G^{an}_{C_-} \cdot G^{an}_{C_0} \cdot G^{an}_{C_+}$$

Proof of lemma is based on a geometric interpretation of decomposition as above. After additional blow-ups we get two divisors, a chain of $\mathbb{C}P^1$-s which deform by the versal deformation to a $(n-1)$-parameter family of smooth $\mathbb{C}P^1$-s covering locally the space. This is true formally, to go to analytic theory we use Douady spaces.



For Hitchin systems, algebraically it is quite interesting:

a wheel of $\mathbb{C}P^1$-s at infinity of a partial compactification of Betti moduli spaces (character varieties). Families of smooth $\mathbb{C}P^1$-s are representations

$$\pi_1(\mathrm{surface}) \to GL(r, \mathbb{C}[t, t^{-1}])$$

3. Second analytic space and relation to resurgence

3.1. Construction

For an analytic stability structure, draw several cuts (3 is enough) on $\hbar$-plane. Consider the trivial holomorphic bundle

$$(\mathbb{C}^\times)^n_{x_1,\ldots,x_n} \times \mathbb{C}_\hbar \to \mathbb{C}_\hbar$$

and modify it along cuts by

$$x_j \mapsto x_j \cdot \left(1 + \sum_{\gamma \in C_{V_i} \cap \Gamma - 0} e^{-\frac{Z(\gamma)}{\hbar}} c^{(i)}_{\gamma,j} x^\gamma\right)$$

which is A_i conjugated by the multiplicative shift $x_j \mapsto e^{-\frac{Z_j}{\hbar}} x_j$.

Rigorous construction: use the oriented real blow-up at $\hbar = 0$ and Newlander-Nirenberg theorem.

In this way we get a holomorphic submersion $\mathcal{X} \to \mathbb{C}_\hbar$ together with a *formal* trivialization over $\mathbb{C}[[\hbar]]$,

$$\mathcal{X} \times_{\mathbb{C}_\hbar} \mathsf{Specf}\, \mathbb{C}[[\hbar]] \simeq (\mathbb{C}^\times)^n \times \mathsf{Specf}\, \mathbb{C}[[\hbar]]$$

More precisely, $\mathcal{X}$ is only a *germ* at the central fiber $(\mathbb{C}^\times)^n \times \{\hbar = 0\}$.

Real life: consider blow-up of the twistor family at a subvariety of the Hitchin moduli space which is a fiber of Hitchin system (an abelian variety). Then the germ of the total space at the exceptional divisor is our "second analytic space".

3.2 Relation to resurgence.

Master conjecture: *in the above situation, for any germ of a section of* $\mathcal{X} \to \mathbb{C}_\hbar$, *in the formal trivialization written as*

$$x_j(\hbar) = x_j(0) \cdot \exp\left(\sum_{k \geq 1} c_{j,k} \hbar^k\right)$$

each series $\sum_k c_{j,k}\hbar^k$ *is* ***resurgent*** (Borel transform has all nice properties).

$n = 1 \iff$ Écalle-Voronin theory.

General n: work in progress with David Sauzin: sum over trees, estimates.

This will be a geometric formulation of a very common class of resurgent series, including all relevant ones for WKB.