Introduction to the Stokes phenomenon and resurgence

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Donaldson–Thomas Invariants and Resurgence
Simons Collaboration Meeting
Jan. 11-15, 2021

Stokes phenomenon:

- divergent power series can still be useful as asymptotic approximations
- asymptotics jump by exponentially small corrections along rays in $\mathbb{C}$

Borel summation and resurgence (É. Borel, Écalle, . . .)

- "re-sums" the divergent series to get a function
- produces sums that are real-valued and non-perturbatively correct

This lecture: sketch basic ideas in preparation for other talks.

Reference list and (old, sketchy, unedited) lecture notes:

https://www.math.mcgill.ca/bpym/courses/resurgence/

$$\frac{df}{dx} = \frac{1}{x} - \frac{f}{x^2} \qquad \text{Ansatz: } f = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$a_1 = 1, \qquad a_n = -n \cdot a_{n-1} \qquad \Longrightarrow \quad f = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}$$

Problems:

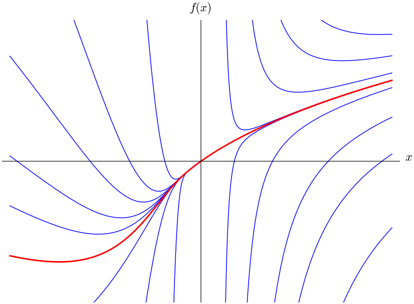
- divergent, i.e. radius of convergence is zero
- constant of integration is absent

Cause: ODE singular at $x = 0$.

NB: Homogenized equation

$$\frac{d\tilde{f}}{dx} = -\frac{\tilde{f}}{x^2} \qquad \rightsquigarrow \qquad \tilde{f} = Ce^{1/x}$$

$$f = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} + Ce^{1/x}$$



Recall standard Laplace transform: $n!x^{n+1} = \int_0^\infty t^n e^{-t/x}\,\mathrm{d}t$

Therefore:

$$\begin{aligned}
f &= \sum_{n=0}^{\infty}(-1)^n n! x^{n+1} \\
&= \sum_{n=0}^{\infty}\int_0^\infty (-1)^n t^n e^{-t/x}\,\mathrm{d}t \\
\text{“}=\text{”} &\quad \int_0^\infty \left(\sum_{n=0}^{\infty}(-1)^n t^n\right) e^{-t/x}\,\mathrm{d}t \\
\text{“}=\text{”} &\quad \int_0^\infty \frac{1}{1+t} e^{-t/x}\,\mathrm{d}t \\
&= e^{-1/x}\mathrm{Ei}(1/x) \qquad \text{for } x > 0 \qquad \text{solves our ODE!} \\
&= \sum_{n=0}^{N}(-1)^n n! x^{n+1} + O(x^{n+1}) \qquad \text{as } x \to 0^+
\end{aligned}$$

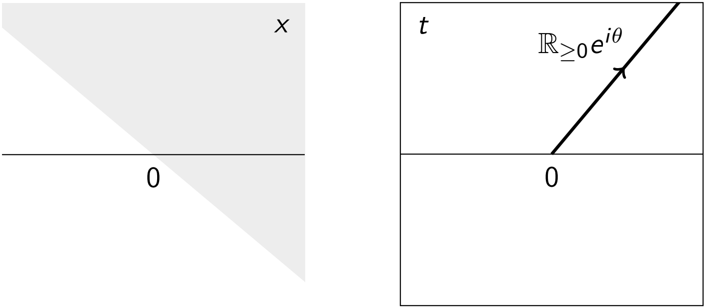
Analytic continuation to $x \in \mathbb{C}$.

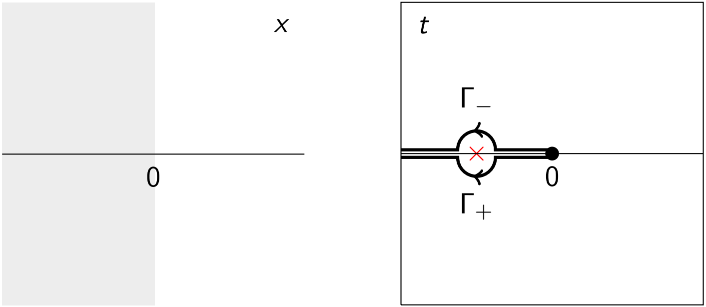
Integral converges as long as $e^{-t/x} \to 0$ as $t \to \infty$, i.e. for $\Re(x) > 0$.

Analytically continue by varying integration contour:

$$f_\theta := \mathscr{L}_\theta(F(t)) := \int_{\mathbb{R}_{\geq 0} e^{i\theta}} F(t) e^{-t/x}\,\mathrm{d}t \qquad F(t) = \frac{1}{1+t}$$

Domain of f_θ:





$$f_{\pi-} - f_{\pi+} = \int_{\Gamma_- - \Gamma_+} \frac{e^{-t/x}}{1+t}\,\mathrm{d}t = 2\pi i \operatorname{Res}\left(\frac{e^{-t/x}}{1+t}; t = -1\right) = 2\pi i \cdot e^{1/x}$$

Solution of the homogeneous equation!

NB: ODE was not used; $e^{1/x}$ is encoded in the divergent series.

Borel summation in general (É. Borel, 1899)

formal Borel transform $\mathscr{B}$

$$x\mathbb{C}[[x]] \rightleftarrows \mathbb{C}[[t]]$$

formal Laplace transform $\mathscr{L}$

$$n!x^{n+1} \longleftrightarrow t^n$$

$$(\mathscr{L}_\theta F)(x) = \int_{\mathbb{R}_{\geq 0} e^{i\theta}} F(t) e^{-t/x}\, \mathrm{d}t$$

Definition

The **Borel sum of** $f = \sum_{n=0}^{\infty} a_n x^{n+1} \in x\mathbb{C}[[x]]$ **in the direction** $\theta \in S^1$ is

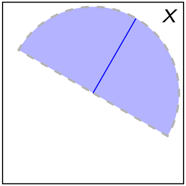
$$f_\theta(x) := (\mathscr{L}_\theta \mathscr{B} f)(x) = \int_{\mathbb{R}_{\geq 0} e^{i\theta}} \underbrace{\left(\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \right)}_{\text{analytically continued to all } t \in \mathbb{R}_{\geq 0} e^{i\theta}} e^{-t/x}\, \mathrm{d}t,$$

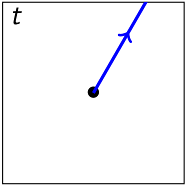
provided the expression converges (i.e. $\mathscr{B}f(t)$ grows at most exponentially)

Check: f converges in a disk $\implies$ $\mathscr{B}f$ is entire and f_θ = ordinary sum

When it exists, the Borel sum f_θ of $f = \sum_{n=0}^{\infty} a_n x^{n+1}$ is:

- defined in a sector centred at $x = 0$ with opening angle π





- asymptotic to f:

$$f_\theta(x) = \sum_{n=0}^{N} a_n x^{n+1} + O(x^{N+1}) \qquad \text{as } x \to 0 \text{ in domain of } f_\theta$$

- compatible with algebraic operations:

$$(f+g)_\theta = f_\theta + g_\theta \qquad (fg)_\theta = f_\theta g_\theta \qquad \frac{df_\theta}{dx} = \left(\frac{df}{dx}\right)_\theta$$

$\therefore$ takes "formal" solutions of polynomial ODEs to "actual" solutions

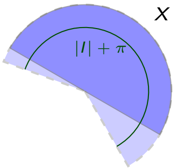
Definition

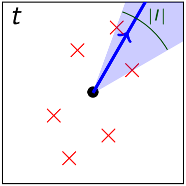
A direction $\theta \in S^1$ is **singular for** $f \in x\mathbb{C}[[x]]$ if f is not Borel summable in the direction θ.

For $I \subset S^1$ open interval containing no singular directions:

$$f_\theta = f_{\theta'} \qquad \forall \theta, \theta' \in I$$

$\therefore$ can glue to Borel sum f_I defined on domain of opening angle $|I| + \pi$



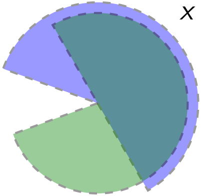


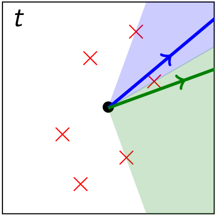
Theorem (Watson)

$I \subset S^1$ nonsingular $\implies$ f_I is the unique function on its domain that is

- *analytic away from $x = 0$*
- *asymptotic to f as $x \to 0$*
- *of "Gevrey class": $\sup |f_I^{(n)}| \leq M^n \cdot (n!)^2$ on compact subsectors*

Stokes phenomenon: sum jumps as we cross a singular $\theta \in S^1$





Example: If $\mathscr{B}(f)$ has a pole at $t_0 \in \mathbb{R}_{\geq 0} e^{i\theta}$, then

$$f_{\theta_-} - f_{\theta_+} = 2\pi i \operatorname{Res}(\mathscr{B}(f); t_0) e^{-t_0/x}$$

Example: If $\mathscr{B}(f)$ has logarithmic branching at $t_0 \in \mathbb{R}_{\geq 0} e^{i\theta}$, then

$$f_{\theta_-} - f_{\theta_+} = \underbrace{\delta_\theta f}_{\substack{\mathscr{L} \text{ of} \\ \text{difference} \\ \text{of branches}}} e^{-t_0/x}$$

Morally: a formal power series is "resurgent" if it is Borel summable in most directions.

Definition ($\exists$ other conventions)

A series $f \in x\mathbb{C}[[x]]$ is **resurgent** if $\mathscr{B}(f) \in \mathbb{C}[[t]]$ extends to an analytic function with only isolated singularities on some ramified covering $M(f) \to \mathbb{C}$, and at most exponential growth as $t \to \infty$ along any sheet.

NB: Stokes phenomenon $\implies$ need to track exponentially small corrections

Definition

A **resurgent trans-series** is a (possibly infinite) formal sum of the form

$$W = \sum_i f_i e^{-t_i/x}$$

where $f_i \in x\mathbb{C}[[x]]$ are resurgent series and $t_i \in \mathbb{C}$.

On the space of resurgent series:

- ring structure, $\frac{d}{dx}$
- Borel sum: integrate over lifts of rays to Riemann surfaces $M(f_i)$
- Stokes phenomenon: unique automorphism S_θ such that

$$(\text{Borel sum for } \theta^-) = (\text{Borel sum for } \theta^+) \circ S_\theta,$$

  Rmk: S_θ unipotent relative to filtration by $|t_i|$
- $S_\theta = \exp(\Delta_\theta)$ for derivation Δ_θ, the "alien derivative"

NB: even if f_i, t_i are real, the Borel sum with $\theta = 0$ need not be, due to singularities when $t \in \mathbb{R}_{\geq 0}$.
Solution: average over all contours $t \to +\infty$, dodging singularities

Theorem (Écalle)

If $f_i \in x\mathbb{R}[[x]]$ and $t_i \in \mathbb{R}$ then the Borel sum of $\sqrt{S_{\theta=0}} \cdot W$ in the direction $\theta = 0^+$ is real-valued for $x > 0$, whenever it converges.

Consider an ODE for a $\mathbb{C}^k$-valued function $y(x)$:

$$\frac{dy}{dx} = \frac{A(x)}{x^2} y$$

with A a $k \times k$ matrix of holomorphic functions.

More invariantly: pair $(\mathcal{E}, \phi)$ of a rank-k holomorphic vector bundle $\mathcal{E}$ on a Riemann surface, and a meromorphic connection ∇ having a second order pole. Locally:

$$\nabla = d - A(x)\frac{dx}{x^2}$$

and the ODE above is $\nabla y = 0$ for a section $y \in \mathcal{E}$.

Goal: classify germ of $(\mathcal{E}, \nabla)$ up to holomorphic gauge equivalence:

$$\nabla \sim \phi \nabla' \phi^{-1}$$

for $\phi(x)$ holomorphic and invertible at $x = 0$.

Simplifying assumption: $A(0)$ is regular semi-simple, so that

$$\nabla = \mathrm{d} - \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_k \end{pmatrix} \frac{\mathrm{d}x}{x^2} - O(\tfrac{1}{x})\,\mathrm{d}x$$

Strategy to solve $\nabla y = 0$: reduce to the simpler diagonal connection

$$\nabla_0 = d - \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_k \end{pmatrix} \frac{\mathrm{d}x}{x^2} - \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \frac{\mathrm{d}x}{x} \qquad \text{w./ sols.} \begin{pmatrix} c_1 e^{-t_1/x_1} x^{\lambda_1} \\ \vdots \\ c_k e^{-t_k/x} x^{\lambda_k} \end{pmatrix}$$

Lemma

There exists a unique formal power series

$$\phi = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} + O(x) \in GL(\mathbb{C}[[x]])$$

such that $\nabla = \phi \nabla_0 \phi^{-1}$.

Fundamental solution of $\nabla y = 0$:

$$Y := \begin{pmatrix} | & & | \\ y^1 & \cdots & y^k \\ | & & | \end{pmatrix} = \phi(x) \cdot \begin{pmatrix} e^{-t_1/x_1} x^{\lambda_1} & & \\ & \ddots & \\ & & e^{-t_k/x} x^{\lambda_k} \end{pmatrix}$$

Singular directions of $\mathscr{B}Y$ in t-plane: $\{\theta_{ij} = \mathrm{Arg}(t_i - t_j)\}$

$$S(\theta_{ij}) := Y_{\theta_{ij}+} \cdot Y_{\theta_{ij}-}^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & \boxed{ij} & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \in U_\pm \subset GL_k(\mathbb{C})$$

Definition

The unipotent matrices $\{S(\theta_{ij})\}_{i,j}$ are the **Stokes data of** ∇.

NB: taking clockwise products of $S(\theta_{ij}) \rightsquigarrow \{\text{Stokes data}\} \cong U_+ \times U_-$

Classification

Theorem (Balser–Jurkat–Lutz)

For any pairwise distinct collection $t_1, \ldots, t_k \in \mathbb{C}$, the procedure of extracting residues $\mathrm{diag}(\lambda_i) \in \mathfrak{h}$ of ∇_0 and Stokes data $S(\theta_{ij})$ of ∇ gives

$$\frac{\{\text{germs } (\mathcal{E}, \nabla) \sim d - \mathrm{diag}(t_i)\frac{dx}{x^2} + \cdots\}}{\text{gauge equivalence}} \cong \frac{U_+ \times U_- \times \mathfrak{h}}{\text{diagonal conjugation}}$$

Generalizations: poles of order > 2, non-semi-simple $A(0)$, replace $GL_k \rightsquigarrow G$ reductive, ...; see especially works of Boalch

NB: RHS is independent of $t_1, \ldots, t_k$

Isomonodromy: how to vary ∇ as function of $t_1, \ldots, t_k$, while keeping Stokes data fixed?

Solution: Take $\nabla(t_1, \ldots, t_k)$ to be restriction of flat connection on total space $\{(x, t_1, \ldots, t_k)\} \rightsquigarrow$ PDE in Bridgeland's lectures

THANK YOU!