Introduction to the Stokes phenomenon and resurgence

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Overview

Stokes phenomenon:
- divergent power series can still be useful as asymptotic approximations
- asymptotics jump by exponentially small corrections along rays in $\mathbb{C}$

Borel summation and resurgence (É. Borel, Écalle, . . . )
- “re-sums” the divergent series to get a function
- produces sums that are real-valued and non-perturbatively correct

This lecture: sketch basic ideas in preparation for other talks.

Reference list and (old, sketchy, unedited) lecture notes:

https://www.math.mcgill.ca/bpym/courses/resurgence/
Euler’s equation

\[
\frac{df}{dx} = \frac{1}{x} - \frac{f}{x^2}
\]

Ansatz: \( f = \sum_{n=0}^{\infty} a_n x^{n+1} \)

\( a_1 = 1, \quad a_n = -n \cdot a_{n-1} \)

\[\Rightarrow f = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} \]

Problems:

- divergent, i.e. radius of convergence is zero
- constant of integration is absent

Cause: ODE singular at \( x = 0 \).

NB: Homogenized equation

\[
\frac{d\tilde{f}}{dx} = -\frac{\tilde{f}}{x^2} \quad \Rightarrow \quad \tilde{f} = Ce^{1/x}
\]
Borel summation of \( f = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} \)

Recall standard Laplace transform: \( n! x^{n+1} = \int_{0}^{\infty} t^n e^{-t/x} \, dt \)

Therefore:

\[
\begin{align*}
f &= \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} \\
&= \sum_{n=0}^{\infty} \int_{0}^{\infty} (-1)^n t^n e^{-t/x} \, dt \\
&= \int_{0}^{\infty} \left( \sum_{n=0}^{\infty} (-1)^n t^n \right) e^{-t/x} \, dt \\
&= \int_{0}^{\infty} \frac{1}{1 + t} \, dt \\
&= e^{-1/x} \text{Ei}(1/x) \quad \text{for } x > 0 \quad \text{solves our ODE!} \\
N \sum_{n=0}^{N} (-1)^n n! x^{n+1} + O(x^{n+1}) & \quad \text{as } x \to 0^+ 
\end{align*}
\]
$f = \int_0^{\infty} \frac{e^{-t/x}}{1+t} \, dt$ for $x > 0$. Case $x < 0$?

Analytic continuation to $x \in \mathbb{C}$.

Integral converges as long as $e^{-t/x} \to 0$ as $t \to \infty$, i.e. for $\Re(x) > 0$.

Analytically continue by varying integration contour:

$$f_\theta := \mathcal{L}_\theta(F(t)) := \int_{\mathbb{R}_0^+} F(t) e^{-t/x} \, dt \quad F(t) = \frac{1}{1 + t}$$

Domain of $f_\theta$: 

![Diagram](image.png)
\[ \int_{\mathbb{R} \geq 0} e^{i\theta} \frac{e^{-x}}{1+t} \, dt, \text{ exceptional case } \theta = \pi \]

\[ f_{\pi^-} - f_{\pi^+} = \int_{\Gamma_- - \Gamma_+} \frac{e^{-x}}{1+t} \, dt = 2\pi i \text{Res} \left( \frac{e^{-x}}{1+t}; t = -1 \right) = 2\pi i \cdot e^{1/x} \]

Solution of the homogeneous equation!

NB: ODE was not used; \( e^{1/x} \) is encoded in the divergent series.
Borel summation in general (É. Borel, 1899)

formal Borel transform $\mathcal{B}$

$\mathbb{C}[[x]] \xrightarrow{\mathcal{B}} \mathbb{C}[[t]]$

formal Laplace transform $\mathcal{L}$

$n!x^{n+1} \xrightarrow{\mathcal{L}^{-1}} t^n$

$(\mathcal{L}_\theta F)(x) = \int_{\mathbb{R}_{\geq 0}e^{i\theta}} F(t)e^{-t/x} \, dt$

Definition

The **Borel sum of** $f = \sum_{n=0}^{\infty} a_n x^{n+1} \in \mathbb{C}[[x]]$ **in the direction** $\theta \in S^1$ is

$$f_\theta(x) := (\mathcal{L}_\theta \mathcal{B}f)(x) = \int_{\mathbb{R}_{\geq 0}e^{i\theta}} \left( \sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \right) e^{-t/x} \, dt,$$

analytically continued to all $t \in \mathbb{R}_{\geq 0}e^{i\theta}$

provided the expression converges (i.e. $\mathcal{B}f(t)$ grows at most exponentially)

**Check:** $f$ converges in a disk $\implies \mathcal{B}f$ is entire and $f_\theta = $ ordinary sum
Properties of Borel summation

When it exists, the Borel sum \( f_\theta \) of \( f = \sum_{n=0}^{\infty} a_n x^{n+1} \) is:

- defined in a sector centred at \( x = 0 \) with opening angle \( \pi \)
- asymptotic to \( f \):
  \[
  f_\theta(x) = \sum_{n=0}^{N} a_n x^{n+1} + O(x^{N+1}) \quad \text{as } x \to 0 \text{ in domain of } f_\theta
  \]
- compatible with algebraic operations:
  \[
  (f + g)_\theta = f_\theta + g_\theta \quad (fg)_\theta = f_\theta g_\theta \quad \frac{df_\theta}{dx} = \left( \frac{df}{dx} \right)_\theta
  \]

\[ \therefore \text{ takes "formal" solutions of polynomial ODEs to "actual" solutions} \]
### Definition

A direction $\theta \in S^1$ is **singular for** $f \in x\mathbb{C}[[x]]$ if $f$ is not Borel summable in the direction $\theta$.

For $I \subset S^1$ open interval containing no singular directions:

$$f_\theta = f_{\theta'}, \quad \forall \theta, \theta' \in I$$

∴ can glue to Borel sum $f_I$ defined on domain of opening angle $|I| + \pi$

### Theorem (Watson)

$I \subset S^1$ nonsingular $\implies f_I$ is the unique function on its domain that is

- analytic away from $x = 0$
- asymptotic to $f$ as $x \to 0$
- of “Gevrey class”: $\sup |f_I^{(n)}| \leq M^n \cdot (n!)^2$ on compact subsectors
Stokes phenomenon (c.f. 1847 study of Airy function)

Stokes phenomenon: sum jumps as we cross a singular $\theta \in S^1$

**Example:** If $\mathcal{B}(f)$ has a pole at $t_0 \in \mathbb{R}_{\geq 0} e^{i\theta}$, then

$$f_{\theta_-} - f_{\theta_+} = 2\pi i \text{Res}(\mathcal{B}(f); t_0) e^{-t_0/x}$$

**Example:** If $\mathcal{B}(f)$ has logarithmic branching at $t_0 \in \mathbb{R}_{\geq 0} e^{i\theta}$, then

$$f_{\theta_-} - f_{\theta_+} = \left[\delta_0 f \right] e^{-t_0/x}$$

$\mathcal{L}$ of difference of branches
**Morally:** a formal power series is “resurgent” if it is Borel summable in most directions.

**Definition (∃ other conventions)**

A series \( f \in \mathbb{C}[[x]] \) is **resurgent** if \( B(f) \in \mathbb{C}[[t]] \) extends to an analytic function with only isolated singularities on some ramified covering \( M(f) \to \mathbb{C} \), and at most exponential growth as \( t \to \infty \) along any sheet.

**NB:** Stokes phenomenon \( \implies \) need to track exponentially small corrections

**Definition**

A **resurgent trans-series** is a (possibly infinite) formal sum of the form

\[
W = \sum_{i} f_i e^{-t_i/x}
\]

where \( f_i \in \mathbb{C}[[x]] \) are resurgent series and \( t_i \in \mathbb{C} \).
Algebra of resurgent series $W = \sum_i f_i e^{-t_i/x}$

On the space of resurgent series:
- ring structure, $\frac{d}{dx}$
- Borel sum: integrate over lifts of rays to Riemann surfaces $M(f_i)$
- Stokes phenomenon: unique automorphism $S_\theta$ such that

$$(\text{Borel sum for } \theta^-) = (\text{Borel sum for } \theta^+) \circ S_\theta,$$

Rmk: $S_\theta$ unipotent relative to filtration by $|t_i|$.
- $S_\theta = \exp(\Delta_\theta)$ for derivation $\Delta_\theta$, the "alien derivative"

**NB:** even if $f_i, t_i$ are real, the Borel sum with $\theta = 0$ need not be, due to singularities when $t \in \mathbb{R}_{\geq 0}$.

**Solution:** average over all contours $t \to +\infty$, dodging singularities

**Theorem (Écalle)**

If $f_i \in x\mathbb{R}[[x]]$ and $t_i \in \mathbb{R}$ then the Borel sum of $\sqrt{S_{\theta=0}} \cdot W$ in the direction $\theta = 0^+$ is real-valued for $x > 0$, whenever it converges.
Consider an ODE for a $\mathbb{C}^k$-valued function $y(x)$:

$$\frac{dy}{dx} = \frac{A(x)}{x^2} y$$

with $A$ a $k \times k$ matrix of holomorphic functions.

More invariantly: pair $(E, \phi)$ of a rank-$k$ holomorphic vector bundle $E$ on a Riemann surface, and a meromorphic connection $\nabla$ having a second order pole. Locally:

$$\nabla = d - A(x)\frac{dx}{x^2}$$

and the ODE above is $\nabla y = 0$ for a section $y \in E$.

Goal: classify germ of $(E, \nabla)$ up to holomorphic gauge equivalence:

$$\nabla \sim \phi \nabla' \phi^{-1}$$

for $\phi(x)$ holomorphic and invertible at $x = 0$. 


Simplifying assumption: $A(0)$ is regular semi-simple, so that

$$\nabla = d - \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \frac{dx}{x^2} - O\left(\frac{1}{x}\right) dx$$

Strategy to solve $\nabla y = 0$: reduce to the simpler diagonal connection

$$\nabla_0 = d - \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \frac{dx}{x^2} - \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} \frac{dx}{x} \quad \text{w./ sols.} \quad \begin{pmatrix} c_1 e^{-t_1/x_1 x^{\lambda_1}} \\ \vdots \\ c_k e^{-t_k/x_k x^{\lambda_k}} \end{pmatrix}$$

Lemma

There exists a unique formal power series

$$\phi = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + O(x) \in GL(\mathbb{C}[[x]])$$

such that $\nabla = \phi \nabla_0 \phi^{-1}$. 
Stokes data for $\nabla = \phi \nabla_0 \phi^{-1}$

Fundamental solution of $\nabla y = 0$:

$$Y := \begin{pmatrix} y^1 & \cdots & y^k \end{pmatrix} = \phi(x) \cdot \begin{pmatrix} e^{-t_1/x_1} x^{\lambda_1} \\ \vdots \\ e^{-t_k/x} x^{\lambda_k} \end{pmatrix}$$

Singular directions of $\mathcal{B}Y$ in $t$-plane: $\{ \theta_{ij} = \text{Arg}(t_i - t_j) \}$

$$S(\theta_{ij}) := Y_{\theta_{ij}^+} \cdot Y_{\theta_{ij}^-}^{-1} = \begin{pmatrix} 1 & 1 & \vdots & ij \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 \\ 1 & 1 \end{pmatrix} \in U_{\pm} \subset GL_k(\mathbb{C})$$

Definition

The unipotent matrices $\{ S(\theta_{ij}) \}_{i,j}$ are the Stokes data of $\nabla$.

NB: taking clockwise products of $S(\theta_{ij}) \leadsto \{ \text{Stokes data} \} \cong U_+ \times U_-$
Theorem (Balser–Jurkat–Lutz)

For any pairwise distinct collection \( t_1, \ldots, t_k \in \mathbb{C} \), the procedure of extracting residues \( \text{diag}(\lambda_i) \in \mathfrak{h} \) of \( \nabla_0 \) and Stokes data \( S(\theta_{ij}) \) of \( \nabla \) gives

\[
\{ \text{germs } (\mathcal{E}, \nabla) \sim d - \text{diag}(t_i) \frac{dx}{x^2} + \cdots \} \xrightarrow{\text{gauge equivalence}} \mathcal{U}_+ \times \mathcal{U}_- \times \mathfrak{h} \xrightarrow{\text{diagonal conjugation}} \]

**Generalizations:** poles of order \( \geq 2 \), non-semi-simple \( A(0) \), replace \( GL_k \leadsto G \) reductive, \( \ldots \); see especially works of Boalch

**NB:** RHS is independent of \( t_1, \ldots, t_k \)

**Isomonodromy:** how to vary \( \nabla \) as function of \( t_1, \ldots, t_k \), while keeping Stokes data fixed?

**Solution:** Take \( \nabla(t_1, \ldots, t_k) \) to be restriction of flat connection on total space \( \{(x, t_1, \ldots, t_k)\} \leadsto \text{PDE in Bridgeland’s lectures} \)

THANK YOU!