FROM RESURGENCE TO TOPOLOGICAL STRINGS

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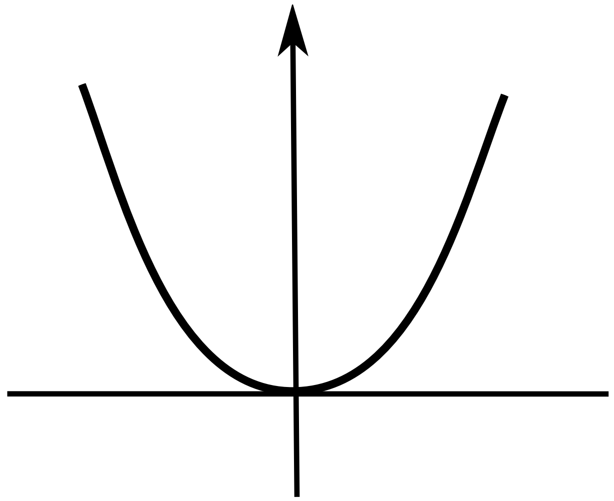
Perturbation theory and its discontents

Perturbation theory in a small parameter remains one of the most fruitful approaches in physics, in the absence of exact solutions.

A simple example is the ground state energy of the quartic oscillator in quantum mechanics



$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{x}^2}{2} + g\mathsf{x}^4$$

$$E_0(g) \sim \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

However, perturbative series must be handled with care. In the above example, the coefficients grow **factorially**

$$E_0(g) \sim \sum_n a_n g^n \qquad a_n \sim n!$$

This turns out to be the norm: perturbative series in quantum theory have zero radius of convergence. Therefore, they do not lead (at least immediately) to functions.

Physicists and mathematicians have developed various tricks to reconstruct the underlying function from the divergent series. After all, most computations in the standard model of elementary particles are based on factorially divergent series!

Making sense of perturbation theory

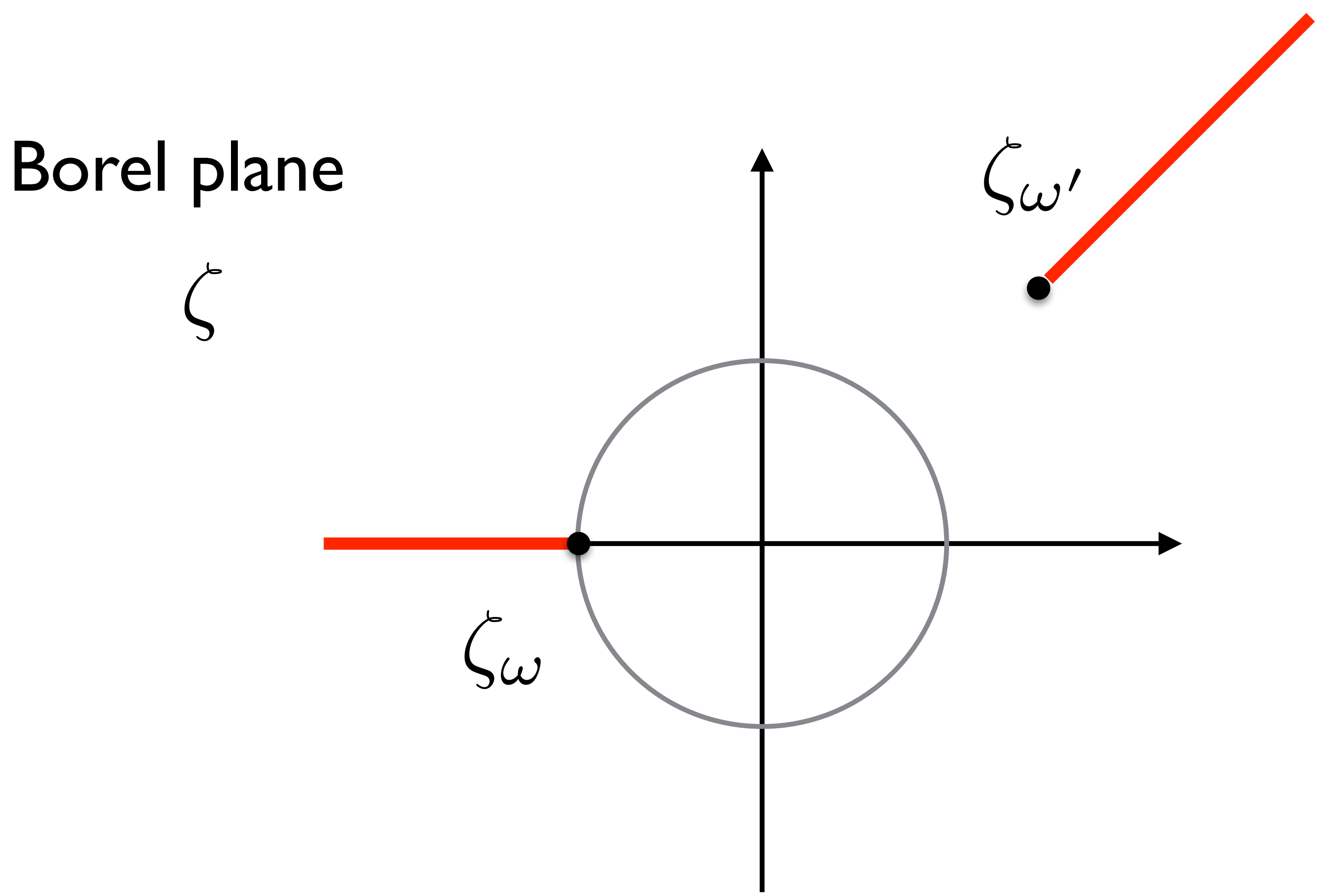
Let us consider a formal power series with factorially growing coefficients

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \qquad\qquad a_n \sim n!$$

These are sometimes called Gevrey-1 series. The first step in resurgence is the **Borel transform**, a deceptively simple way of transforming these series into "nice" functions

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \quad \xrightarrow{\text{Borel transform}} \quad \widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n$$

The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. Very often it can be analytically continued to the complex plane, displaying a set of **singularities** (poles, branch cuts)



The expansion of the Borel transform around each singularity leads to **new formal power series**. For example, for so-called simple resurgent functions, we have only log singularities:

$$\widehat{\varphi}(\zeta) = -\mathsf{S}_\omega \widehat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi} + \cdots$$

Stokes constant

$$\varphi_\omega(z) = \sum_{n \geq 0} a_{n,\omega} z^n$$

These new series are typically associated to **new sectors of the theory,** which are not manifest in the original analysis. In physics, they often correspond to "non-perturbative" sectors.

We can repeat the same analysis for the new functions obtained in this way, and generate further series. At the end, we generate **a set of formal power series**, corresponding to different sectors of the theory, and **a matrix of Stokes constants**

$$\varphi_\omega(z) \qquad \mathsf{S}_{\omega\omega'} \qquad \omega$$

ω labels the sectors

I will refer to these, more general series, as "trans-series." Borel resummation of trans-series makes it possible to reconstruct actual functions, as we saw yesterday in Brent's talk, but I will not focus on that aspect today.

An elementary example: the Airy functions

The formal power series underlying the Airy "Ai" function is

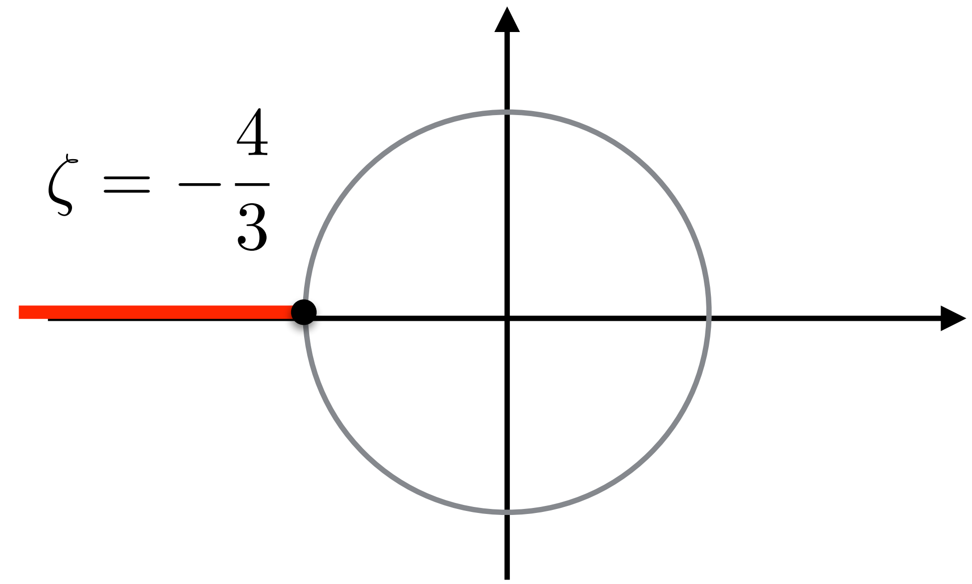
$$\varphi_1(z) = \sum_{n\geq 0} \frac{1}{2\pi}\left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n$$

In this case the resurgent structure can be worked out in detail, since the Borel transform is simply

$$\varphi_1(\zeta) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{3\zeta}{4}\right)$$

This a simple resurgent function, with a log singularity along the negative real axis:

Borel plane:



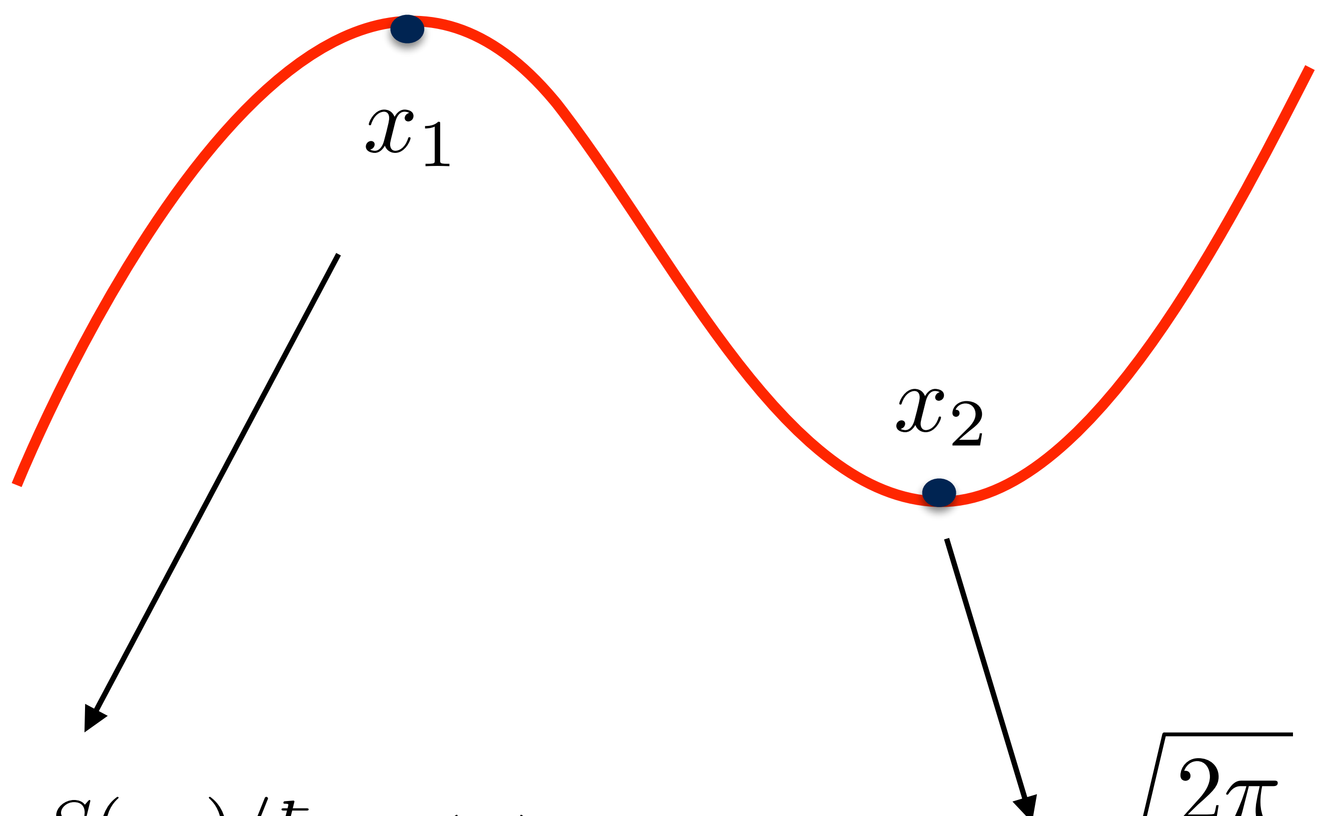
By studying the expansion around the singularity, one finds the other formal power series involved in the game, underlying the "Bi" Airy function

$$\varphi_2(z) = \sum_{n\geq 0} \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n$$

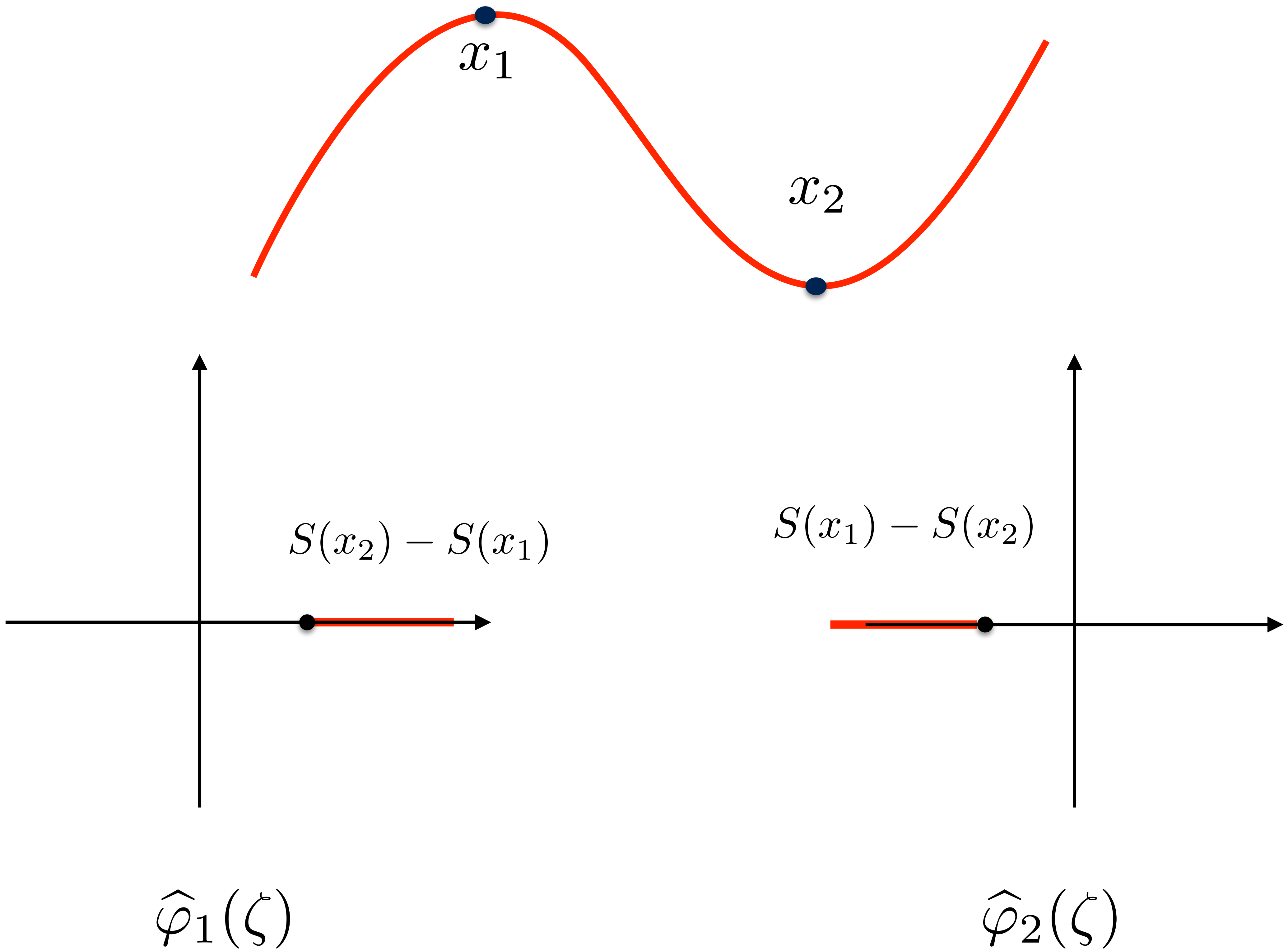
The Stokes constants are $\mathsf{S}_{12} = \mathsf{S}_{21} = 1$

As the Airy example shows, saddle-point analysis of one-dimensional integrals is a rich source of formal power (trans) series:

$$\mathcal{I}(\hbar) = \int \mathrm{d}x \, \mathrm{e}^{-S(x)/\hbar}$$



$$\mathrm{i}\sqrt{\frac{2\pi}{\hbar}}\mathrm{e}^{-S(x_1)/\hbar}\varphi_1(\hbar) \qquad \sqrt{\frac{2\pi}{\hbar}}\mathrm{e}^{-S(x_2)/\hbar}\varphi_2(\hbar)$$


x_1
x_2
$S(x_2) - S(x_1)$
$S(x_1) - S(x_2)$
$\widehat{\varphi}_1(\zeta)$
$\widehat{\varphi}_2(\zeta)$

Resurgence in QFT

One way to approach resurgence in physics is to think about the path integral as a generalized one-dimensional integral. In this case, different formal power series or trans-series can be obtained by asymptotic expansions around different critical points (or instantons)

$$Z(\hbar) = \int \mathcal{D}\phi(x)\, \mathrm{e}^{-S(\phi(x))/\hbar}$$

Sometimes, path integrals reduce to ordinary integrals, and this leads to some of the best studied examples of resurgence in quantum field theory.

(Complex) Chern-Simons theory

A relevant example for mathematics is (complex) Chern-Simons (CS) theory, which defines invariants of three-manifolds through a path integral with CS action [Witten, Gukov, …]

$$Z_M(\tau) = \int \mathcal{D}A \, \exp\left(\frac{\mathrm{i}}{\tau} S_{\mathrm{CS}}(A)\right)$$

τ → coupling constant $\qquad \tau \in \mathbb{C}\backslash(-\infty, 0]$

I will focus on the gauge group *SL(2,C)*

When M is the complement of a hyperbolic knot K in the three-sphere, it has been argued that the partition function of complex CS theory can be reduced to a finite-dimensional **"state integral"** [Kashaev, Hikami, Dimofte et al., Andersen-Kashaev]

$$Z_K(\tau) = \int \mathrm{e}^{-W(\mathbf{x};\tau)/\tau} \mathrm{d}\,\mathbf{x}$$

Note for the experts: This is a slightly simplified version of the theory. The full version depends in addition on a complex modulus, which we set here to zero

Saddle-points correspond to flat complex connections σ on *M*. The expansions of the state integral around these connections have the form

$$\exp\left(\frac{V_\sigma + \mathrm{i}\mathcal{C}_\sigma}{2\pi\tau}\right)\varphi_\sigma(\tau)$$

$$\varphi_\sigma(\tau) = \sum_{n\geq 0} a_n^\sigma \tau^n$$

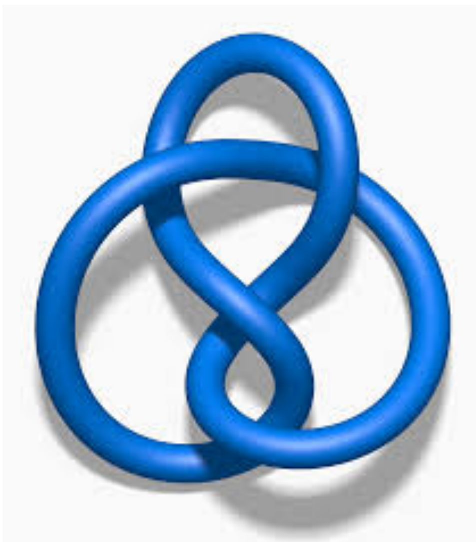
factorially divergent series!

$$a_n^\sigma \sim n!$$

Among these connections there is always the "geometric connection" *g* (corresponding to the geodesically complete hyperbolic metric on *M*), and its conjugate *c*, with

$$V_{g,c} = \pm V$$

hyperbolic volume



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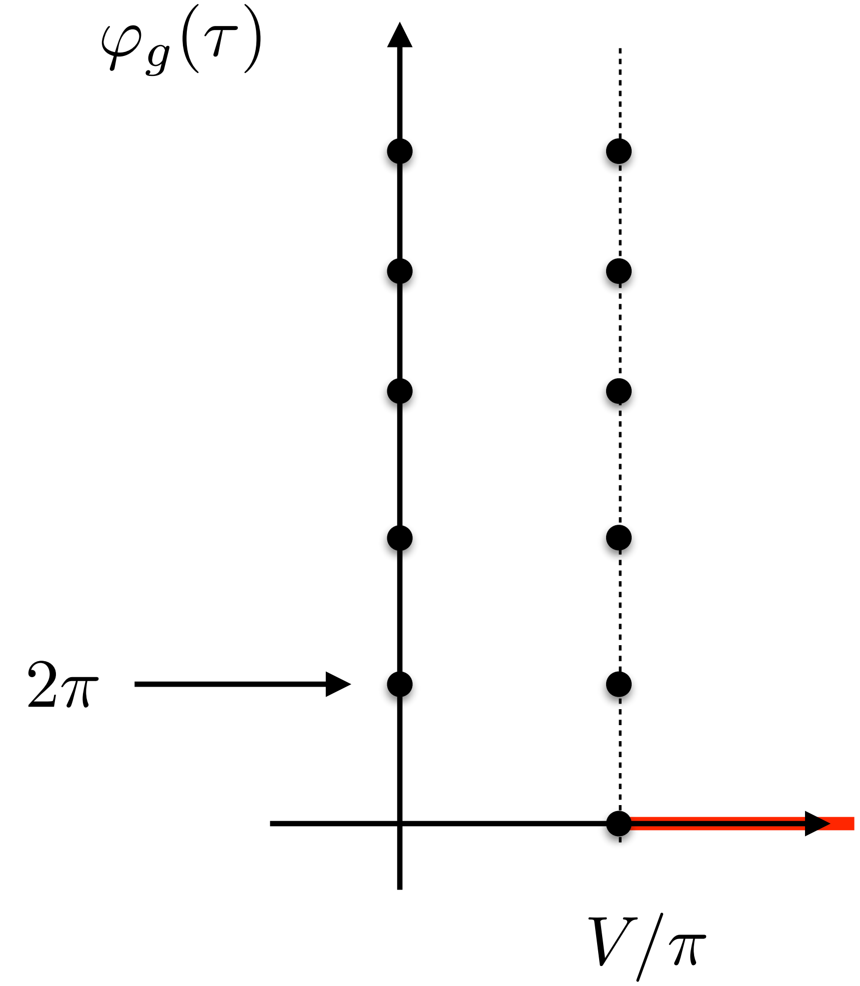
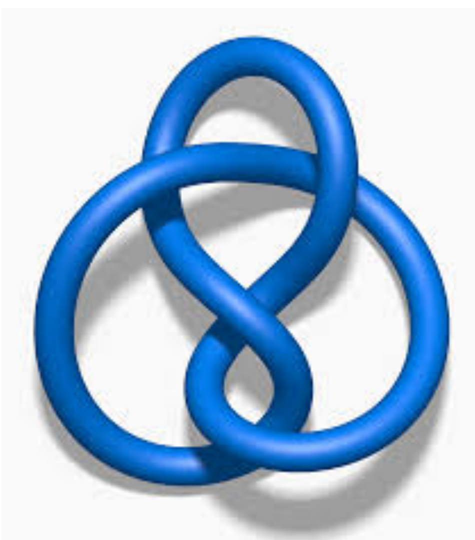
The simplest example of such a structure is the "figure-eight knot" where there are only these two saddle points *g*, *c*

It turns out that the Borel transform of each perturbative series appearing in the saddle-point expansion has an **infinite number of singularities**, due to multi-valuedness of the Chern-Simons action. The trans-series are:

$$\varphi_{g,c}(\tau)\mathrm{e}^{-\frac{2\pi \mathrm{i} n}{\tau}} \qquad n \in \mathbb{Z}$$



In particular, there is an infinite number of Stokes constants, which turn out to be **integer numbers!** [Garoufalidis-Gu-M.M.]. We find in this way a new set of **integer invariants of a hyperbolic knot**



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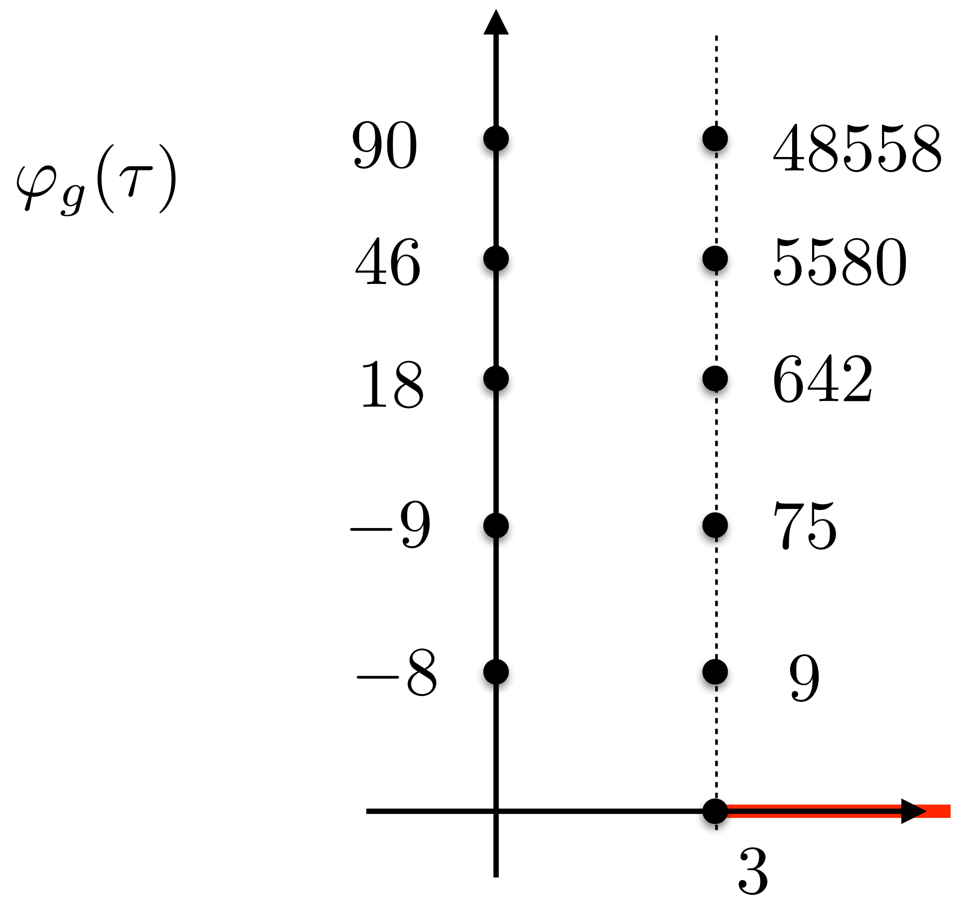
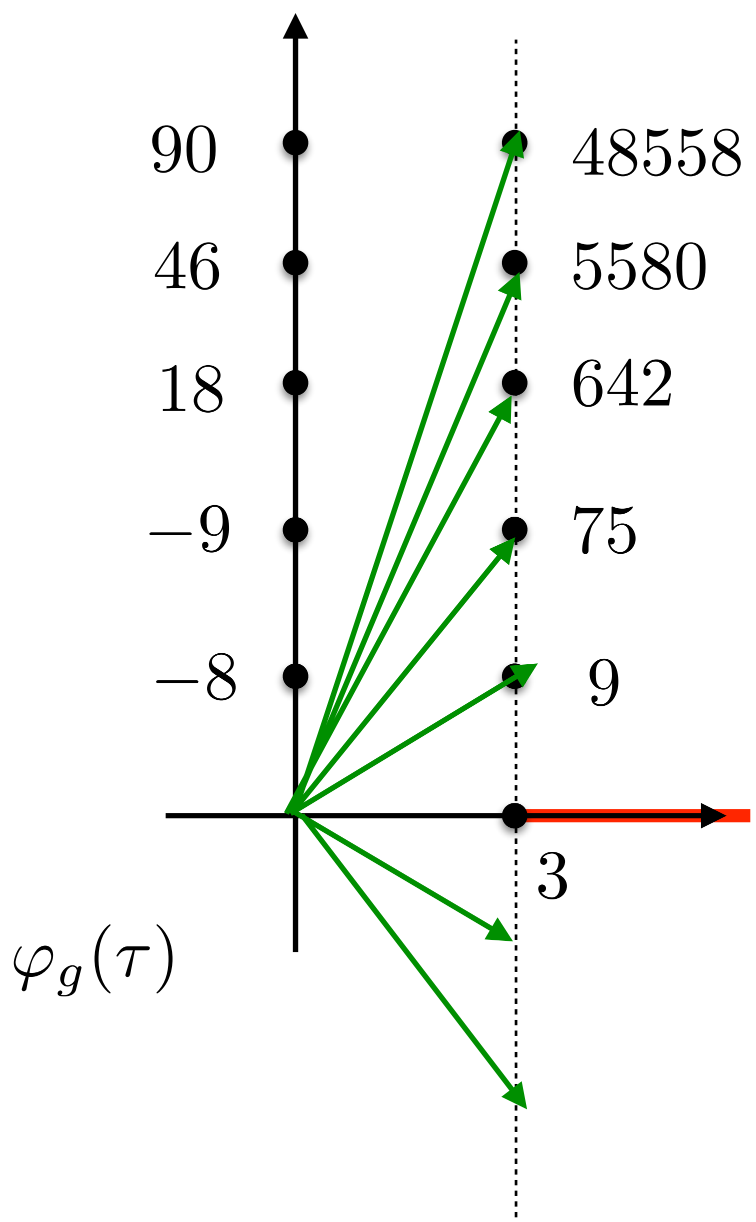


We have developed a theory of these invariants, involving q-series and deformations thereof. In some cases, they are related to known BPS invariants through the 3d/3d correspondence [Dimofte-Gaiotto-Gukov]

We call these singularity structures "peacock patterns" and they will reappear later.



There are two important caveats for the study of resurgence in field/string theory.

In **quantum field theory**, not all relevant formal power series come from instantons, i.e. from saddle-points of the path integral. This is the problem of **renormalons**.

In **string theory**, the situation is in some ways worst, since we only know how to construct perturbative series around the "trivial" sector. We do not know in general what is the well-defined function underlying this series, and we have no first-principles method of constructing additional power series solutions for "non-perturbative" sectors.

Topological string theory

Let *X* be a Calabi-Yau threefold, and let us consider the generating function of Gromov-Witten invariants of genus g:

$$F_g(t) = \sum_{d \geq 1} N_{g,d} \, \mathrm{e}^{-dt}$$

genus g free energy

For simplicity, I assume that there is a single Kahler parameter *t*. This is a convergent series, and the radius of convergence is independent of *g*.

As functions of the modulus *t*, they are generalized hypergeometric functions, with a complicated branch cut structure.

The **total free energy** is a formal power series in the string coupling constant

$$F(t, g_s) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

However, for fixed *t* inside the convergence region, this is a **factorially divergent** series

$$F_g(t) \sim (2g)!, \qquad g \gg 1$$

1) Is there a well-defined function leading to the above asymptotic expansion?

2) What is the resurgent structure of this series?

One could think that re-arranging this free energy in terms of GV/DT invariants improves things. Indeed, this leads to a partial resummation of the perturbative series:

$$F(t, g_s) = \sum_{d \geq 1} c_d(q) \mathrm{e}^{-dt} \qquad q = \mathrm{e}^{\mathrm{i} g_s}$$

However, this has its problems too. For example, even as a formal power series it is not well-defined when *q* is in the unit circle. This is similar to the behavior of the compact dilogarithm, and it suggests that we are missing a full sector of the theory.

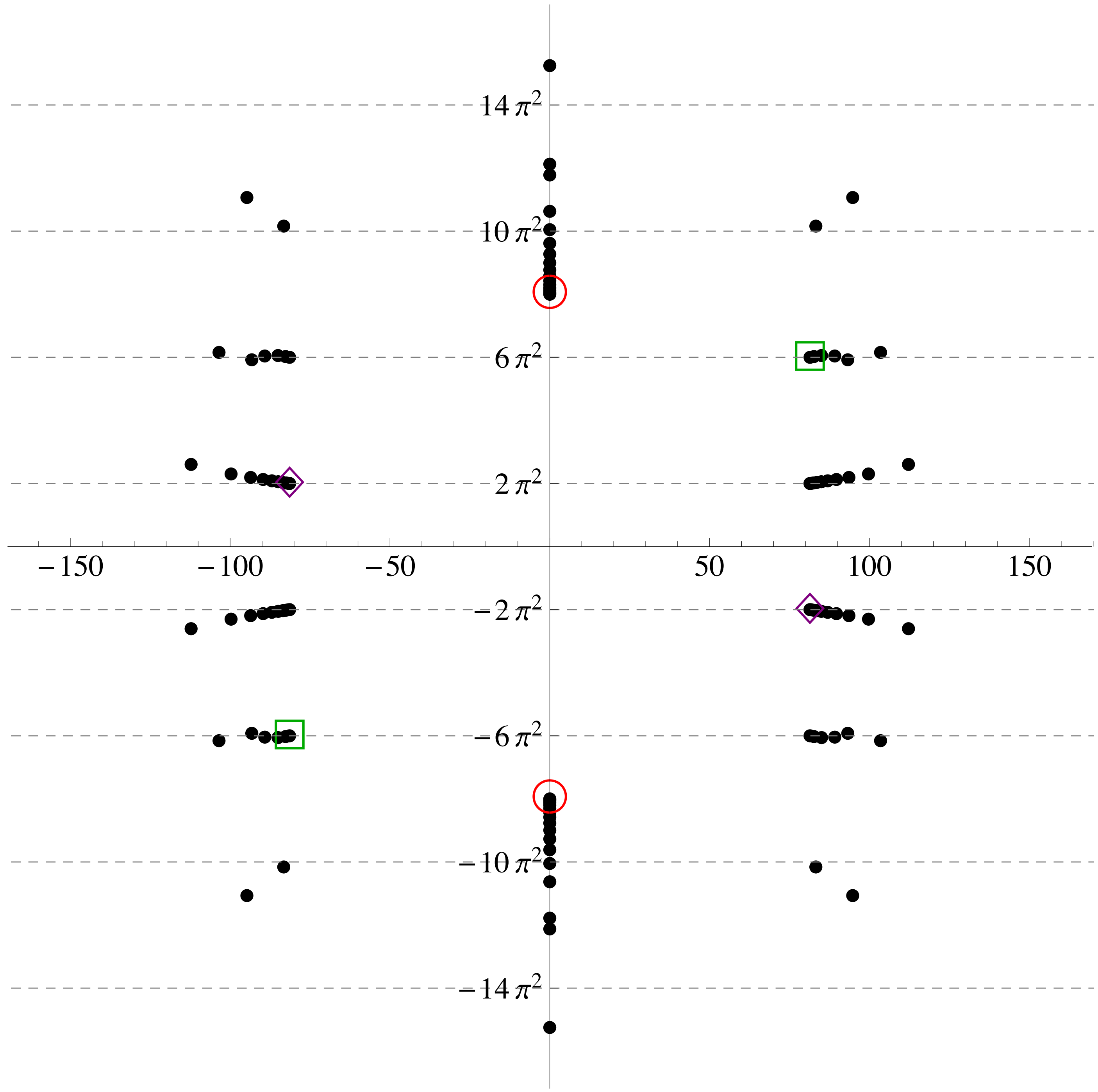
The resurgent structure of topological strings

The first question to ask is: where are the singularities of the Borel transform?

$$\widehat{F}(t,\zeta) = \sum_{g\geq 0} \frac{F_g(t)}{(2g)!}\zeta^{2g}$$

There is evidence that they are located at the lattice of periods of the mirror CY X (although only the case in which X is toric has been studied explicitly).

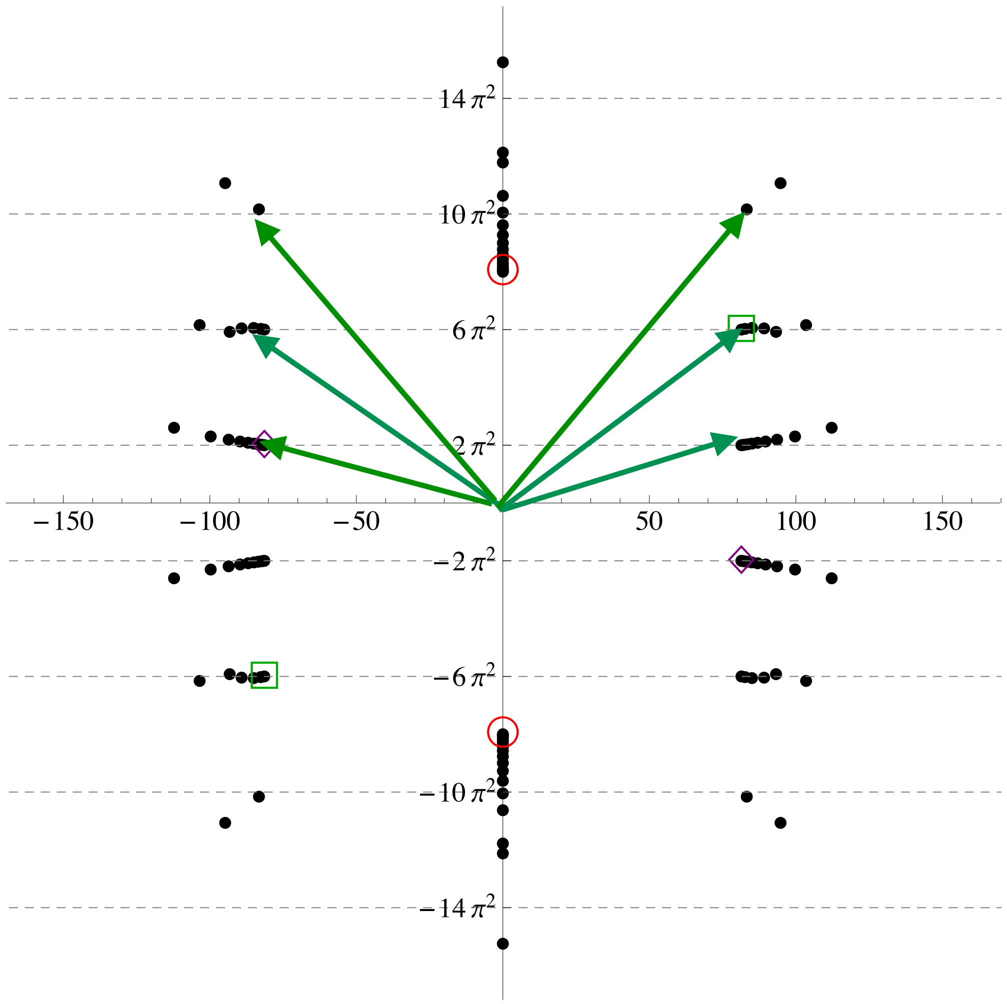
$$X = \mathcal{O}(-3) \to \mathbb{P}^2$$



$$\diamond = t \qquad \bigcirc = \frac{\partial F_0}{\partial t}$$

[Couso-Santamaria, M.M. and Schiappa]

Note the presence of towers of singularities, separated by a constant $4\pi^2$, and leading again to "peacock patterns"



A method to calculate the corresponding trans-series has been proposed by [Couso-Santamaria et al.]
It is based on finding trans-series solutions to the holomorphic anomaly equations of BCOV, similar to what does in the case of non-linear ODEs.

The take home message is that **the generating functionals appearing in enumerative geometry are the tip of the iceberg in a resurgent structure with (potentially) very rich information**.

We are far from a full understanding this structure, in particular the values of the Stokes constants.

We also do not know what is the actual function behind the factorially divergent series!

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There is however a general conjecture in the case of **toric** CYs, in which the total topological string free energy appears as the asymptotic expansion of an **entire function** on the CY moduli space [Hatsuda-Grassi-M.M.]:

$$\Xi(\kappa, g_s) \sim \exp\left(\sum_{g\geq 0} F_g(t) g_s^{2g-2}\right) \qquad \begin{matrix} g_s \to 0 \\ t = g_s \log(\kappa) \end{matrix}$$

This function is the Fredholm determinant of a trace-class operator on $L^2(\mathbb{R})$ which can be obtained by quantization of the mirror curve. In particular, it is well-defined for real values of the string coupling constant.

Conclusions and open questions

Resurgence indicates the existence of a **universal, and rich mathematical structure** in quantum theories, characterized by a (typically infinite) collection of trans-series and Stokes constants. This might provide a **unifying language for quantum physics and for quantum geometry/topology**.

There are some examples where this structure has been understood in some detail, like complex Chern-Simons theory for hyperbolic knots. We find there infinite series of integer Stokes constants. When does this happen, more generally?

In the case of topological strings, there has been some progres in understanding this structure.

The existing evidence indicates strongly that current approaches to the enumerative geometry of CY threefolds are just approximations to a deeper formulation.

In this formulation, generating functionals of enumerative invariants are asymptotic expansions of an underlying entire function, and one needs to include additional non-perturbative sectors.

The resulting story is similar (but much more complicated) to what one finds in complex Chern-Simons theory. This is being explored in work in progress with Jie Gu.

Thank you for your attention!