4

Spectral Networks

&

Nonabelianization Map

But we also have other BPS degeneracies:

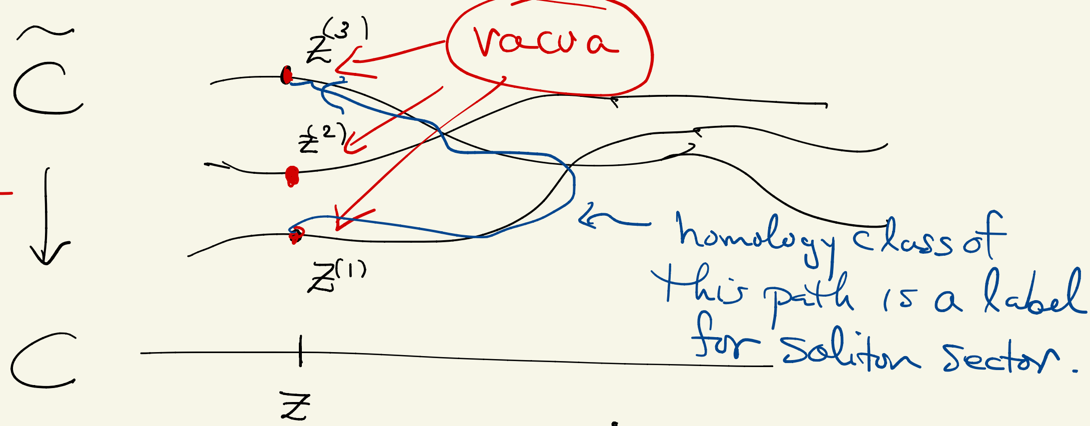
① $\boxed{\overline{\Omega}(L_{\mathfrak{R},\wp,\underline{\zeta}}\,;\,\underline{u})}$ (framed BPS degeneracies)

W.C. of $\overline{\Omega}$ can be deduced by a simple physical argument and consistency then implies KS WCF for $\Omega(\gamma;u)$

2d "QFT" $\quad$ $\boxed{\phi_i \text{ of } W}$

② $\mathcal{D}_z$ has soliton degeneracies

How to label solitons?



$$\pi^{-1}(z) = \{ z^{(i)} \}$$

Soliton sectors of $\mathcal{S}_z$ are labeled by $\gamma \in \Gamma(z,z) := \bigcup_{i,j} \Gamma_{ij}(z,z)$ ← sheets

$$\Gamma_{ij}(z,z) = \left\{ \text{Chains } c \text{ in } \tilde{C} \text{ s.t. } \partial c = z^{(i)} - z^{(j)} \right\} \Big/ \sim$$

homology

Central charges

$\partial \gamma = z^{(i)} - z^{(j)}$

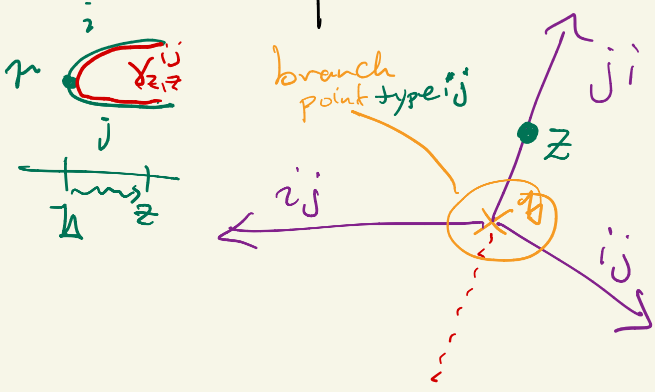
$$Z(\gamma) = \oint_\gamma \lambda$$

$\exists$ Soliton degeneracies: $\mu(\gamma)$

Physical def$\underline{^n}$ of Spectral Network

$$\mathcal{W}_\zeta = \left\{ z \in C \;\middle|\; \begin{array}{l} \exists\ \gamma \in \Gamma(z,z) \\ Z(\gamma) = \zeta \cdot |Z(\gamma)| \\ \mu(\gamma) \neq 0 \end{array} \right\}$$

There is an algorithm for constructing $\underline{\mu(\gamma)}$ by evaluating the spectral network from the branch points of $\pi: \tilde{C} \to C$

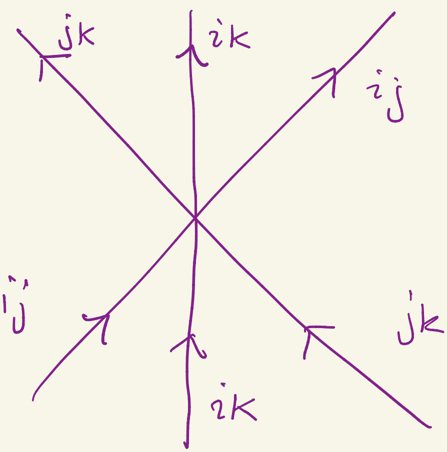


WKB curves

$$\text{phase}\left[\int_{z_1}^{z} \left(\underline{\lambda^{(i)} - \lambda^{(j)}}\right)\right] = \underline{\zeta}$$

$$\boxed{\mu\left(\gamma^{ij}_{z,z}\right) = 1}$$

Then apply simple wall-crossing formulae when lines cross

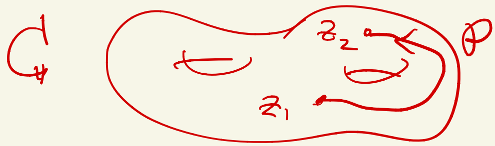


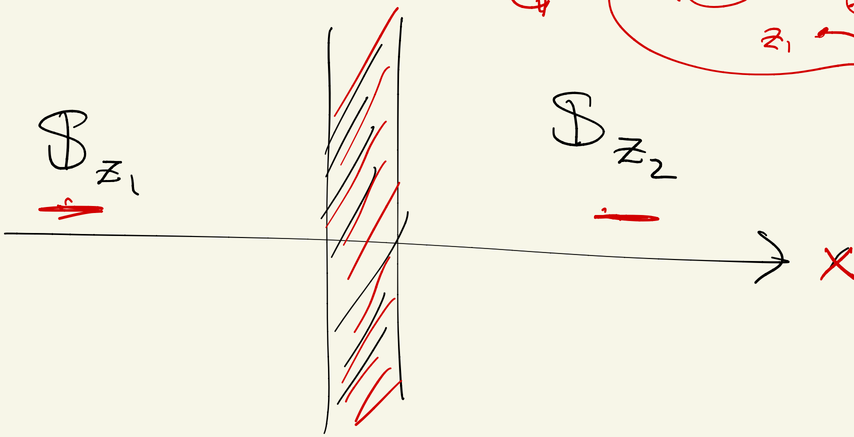
$\boxed{\text{3 or more sheets}}$

This can even be implemented by computer:

1. Andy Neitzke homepage
2. LOOM : software written by Pietro Longhi & Chan Park

(3) We can also have domain walls between $\mathfrak{B}_{z_1}$ and $\mathfrak{B}_{z_2}$ for $z_1, z_2 \in \mathbb{C}$, just like the LG case

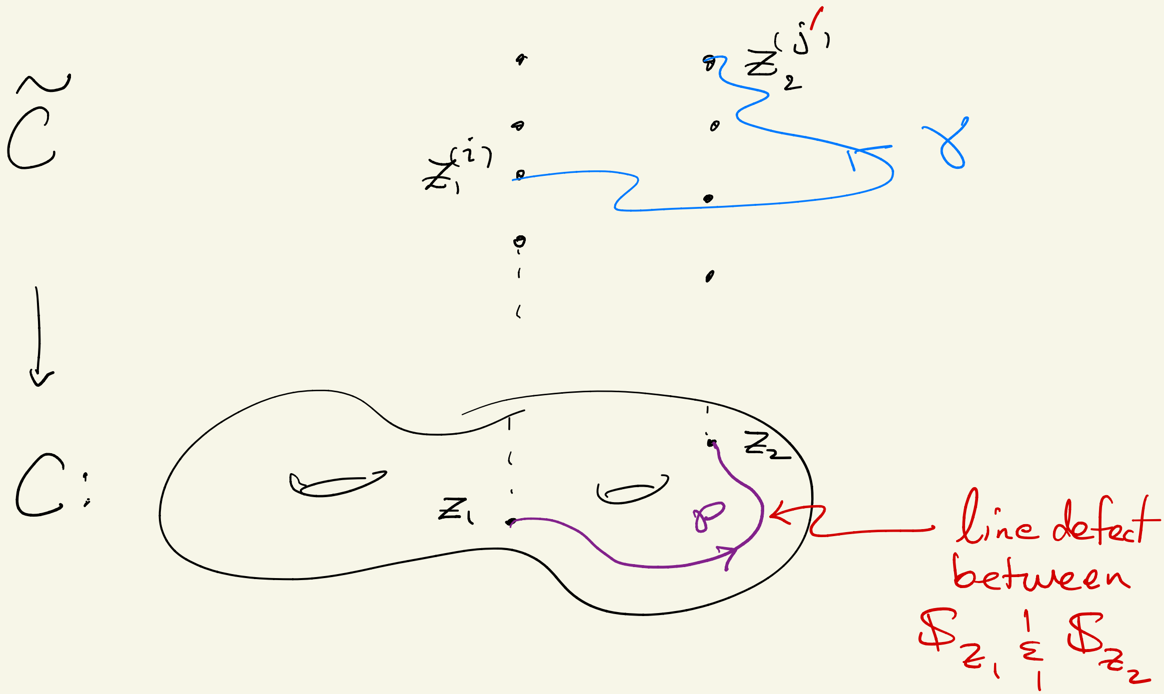




and these are labeled by isotopy classes of paths

$$\wp : z_1 \rightsquigarrow z_2 \quad \text{in } \mathbb{C}$$

$\Rightarrow$ Framed BPS solitons



Soliton sectors:

$$\gamma \in \Gamma(z_1, z_2) := \bigcup_{i,j'} \Gamma_{ij'}(z_1^{(i)}, z_2^{(j')})$$

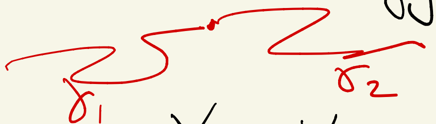
$$\Gamma(z_1^{(i)}, z_2^{(j')}) = \left\{ \text{Chains in } \tilde{C} \text{ with } \partial c = z_1^{(i)} - z_2^{(j')} \right\} / \sim$$

Central charge

$$Z(\gamma) = \int_\gamma \lambda$$

The BPS degeneracies $\overline{\Omega}(\wp, \zeta, \gamma)$ are determined as follows:

→ Homology Path Algebra: $\pi_{\leq 1}(\tilde{C}) \cong H_{\leq 1}(\tilde{C})$



$$X_{\gamma_1} X_{\gamma_2} = \begin{cases} X_{\gamma_1 + \gamma_2} & \text{if comp. makes sense} \\ 0 & \text{else} \end{cases}$$

$$\boxed{\mathbb{F}(\wp, \zeta) := \sum_{\gamma \in \Gamma(z_1, z_2)} \overline{\Omega}(\wp, \zeta, \gamma) X_\gamma}$$

"Formal parallel transport"

$\wp : z_1 \rightsquigarrow z_2$ path in C

Claim [GMN, 2012]: $\exists!$ degeneracies

1.) $\overline{\Omega}(\wp, \zeta, \gamma)$ $\quad \forall \wp, \ \forall \gamma \in \Gamma(z_1, z_2)$

2.) $\mu(\gamma)$ $\quad \gamma \in \Gamma(z, z) \ \forall z \in C$

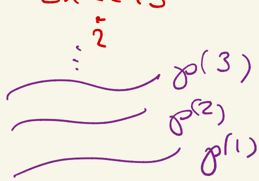
such that

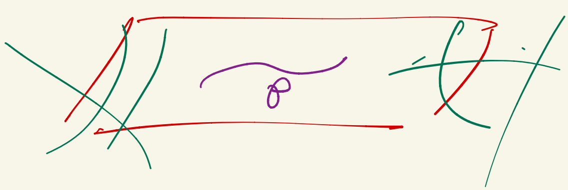
A.) Homotopy invariance: $\mathbb{F}(\wp_1, \zeta) = \mathbb{F}(\wp_2, \zeta)$ if $\wp_1 \sim \wp_2$ (fixed endpoints)

B.) Gluing: $\mathbb{F}(\wp_1, \zeta)\,\mathbb{F}(\wp_2, \zeta) = \mathbb{F}(\wp_1 \circ \wp_2, \zeta)$

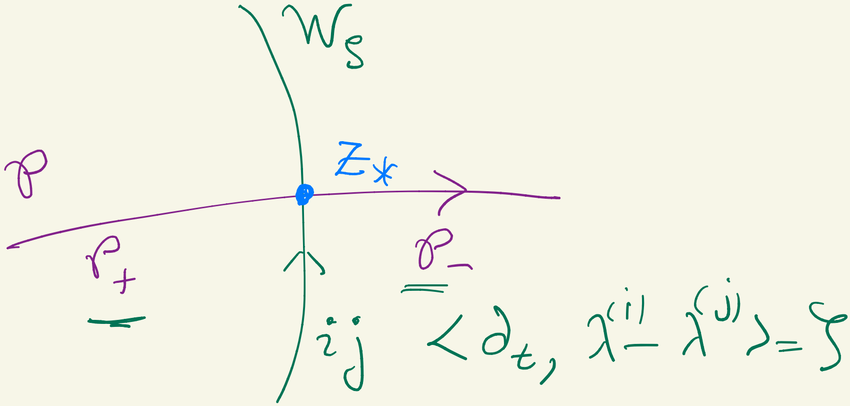
C.) If $\wp \cap \mathcal{W}_\zeta = \phi$.

$$F(\wp, \zeta) = \sum_{\text{sheets}} X_{\wp^{(i)}} := D(\wp)$$



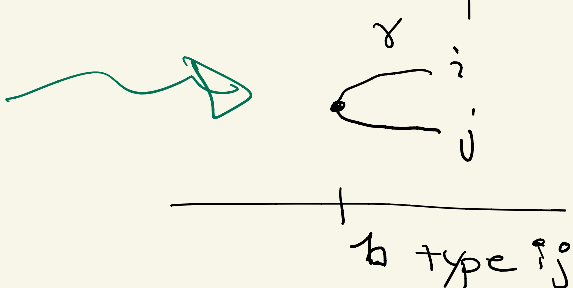


D.) Detour Rule



$$\boxed{\mathbb{F}(\wp,\zeta) = \mathbb{D}(\wp_+) \prod_{\gamma \in \Gamma_{ij}(z_*,z_*)} \left(1+\mu(\gamma) X_\gamma\right) \mathbb{D}(\wp_-)}$$

Idea of Proof: Build up the $\mu(\gamma)$ and $\overline{\Omega}(\wp,\zeta,\gamma)$ starting from the branch points:



$\mu(\gamma) = 1$

Remarks

1.) Replacing $X_\gamma \Rightarrow$ $\boxed{\text{Parallel transport by flat } GL(1,\mathbb{C}) \text{ connection } \nabla^{ab} \text{ on } \tilde{C}}$

$$\mathbb{F}(p,\zeta) = \sum \underline{\overline{\Omega}}(p,\zeta,\gamma) \exp \int_\gamma \nabla^{ab}$$

is the $\parallel$-transport for a nonabelian flat connection on C

- gives the <u>nonabelianization map</u>

$$\Psi_{W_\zeta} : \mathcal{M}(\tilde{C}, GL(1)) \dashrightarrow \mathcal{M}_{flat}(C, GL(k))$$

$k = \#$ sheets

<u>Claim:</u> $\mathbb{F}$ is holomorphic symplectic.

Only provides coordinates on a chart in $\mathcal{M}_{flat}(C, GL(k))$ determined by W_ζ

2.) Fei & Andy – following some earlier work have sought to generalize the homology path algebra to a noncommutative Heisenberg algebra (for $\wp$ closed)

3.) The 2d theory $\mathfrak{S}_z$ has an A_∞ category of branes (generalizing the Fukaya-Seidel category)

There should be categorical analogs of the above where $\boxed{\mathbb{F}(\wp)}$ is a functorial analog of flat parallel transport.

$$\mathbb{F}(\wp) : \mathcal{B}r(\mathfrak{S}_{z_1}) \longrightarrow \mathcal{B}r(\mathfrak{S}_{z_2})$$

For the case of L.G. models this was actually constructed in Gaiotto-Moore-Witten, 2015.

It should lead to a categorified version of Stokes phenomenon.

S_{ij} walls were associated with ("S-wall") crossing functors $\mathfrak{S}_{ij}$ in GMW. These are categorical generalizations of Stokes factors.

Example of 2d4d

4d $SU(2)$ pure.

$C = \mathbb{P}^1$ with 2 punctures

$$\lambda^2 = \left(\frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z} \right) dz^{\otimes 2}$$

Couple this to $\mathbb{CP}^1$ model. It has global $SU(2)$ symmetry.

for z in certain regions

Can identify w/ S_z!

See GMN 1103.2598 section 8.3

Two math references related to this:

- A_∞ category of branes for $\mathfrak{S}_z$:

  G. Kerr + Y. Soibelman 1711.03695

- Math formulation/construction of functor $\mathbb{F}(\rho)$ giving categorical Stokes factors:

  M. Kapranov, Y. Soibelman, L. Soukhanov, 2011.00845