Informal Remarks Preparatory For/Complementary To Fei Yan’s Talk

Simons Collaboration On Special Holonomy
Jan. 13, 2021
Arbeitstagung

Useful
Comments On:

1. Physics Background
2. Defects And Their BPS States
3. Class S
4. Spectral Networks + Nonabelianization Map
5. RH Problems, Integral Equations and Hyperkähler Geometry
1. Physics Background

- Physicists assume many things and have intuitions and examples in mind that they take for granted, but which are not obvious to anyone else.

- Hamburg School On Higgs Bundles, Sept. 2018: Talk # 84 on my homepage goes back to the beginning:

  * Branes + Geometrization of Higgs Mechanism
  * M5 Branes
  * 6d (2,0) Theory
  * Geometrical pictures for class $\Sigma$ and their BPS states

Goal:

Explain the physics intuition behind theory of "spectral networks"
2. Defects

* There exists a rigorous theory in the context of extended TQFT ("cobordism hypothesis").

* In the SUSY context: New BPS deg's

4d $N=2$ SQFT: $\Omega(\mathcal{Y}) \rightarrow$ DT

- Line defects $\mapsto \Omega(L, \mathcal{Y})$ "framed"
- Surface defects $\mapsto \mathcal{M}(\mathcal{Y}_{ij})$ "soliton"
- Line defects in surface $\mapsto \Omega(\mathcal{L}_p, \mathcal{Y}_{ij})$ "framed soliton"

* $\Omega(\mathcal{Y}) \Rightarrow$ Well-known RH problem useful for construction of HK metrics on moduli $\mathcal{M}$: space of solutions to Hitchin equations on a R.S. $\mathcal{G}'$ (with singularities)
other BPS

* deg's \Rightarrow\text{ Similar RH problems }

Constructing:

a.) Hyper-holo connections on certain vector bundles over $M$

b.) Explicit construction of solutions to Hitchin equations on $\mathbb{R}$-Surface $C$.

Now give some examples of defects.
Example 1: Soliton \( \frac{1}{i} \) framed soliton deg's:

\( X \) exact Kähler \( \omega = d\lambda \)

\( W: X \to \mathbb{C} \) superpotential (hale, Morse)

\((X,W) \to 1+1 \) diml massive LG model. (Phys)

\( \rightarrow \) Fukaya–Seidel category (Math)

Critical points of \( W: \{ \phi_i \} \to \)

\[ \mathcal{E}_{\phi_i} = \{ \phi: \mathbb{R} \to X, \phi \to \phi_i \}_{\phi \to +\infty} \]

\[ h = \int_{\mathbb{R}} [\phi^*(\lambda) - \text{Re} (\mathcal{E} W(\phi(x))) \, dx] \]

\( Sh = 0 \) soliton eq. \( \frac{d\phi}{dx} = S \nabla W \)

\[ \mu_{ij} = X (\text{Morse Complex}) \]

"Soliton degeneracies"
Consider a manifold $G'$ of Morse superpotentials:

$$\overline{W}(\phi; z), \ z \in G'$$

$\mapsto \phi(x): z_1 \mapsto z_2$ path in $G'$

\[ h = \int [\phi^*(\alpha) - \text{Re}(\overline{s} W(\phi(x), z(x)))dx] \]

$sh=0$ soliton-like $q$ gives a new Morse complex.

Physical picture:

- Line defect

$$M_{ij}(\phi) = \chi(\text{Morse Complex})$$
Example 2: Line Defects In 4d Gauge Theory w/ gauge group G

4d spacetime: \( TR^3 \times S^1 \)

Wilson:
\[
\varphi \in \mathcal{Y} \otimes \mathbb{C}
\]

\[
L(R, s) = \text{Tr} \text{ Pexp} \int_R \left( 5^\varphi + A + 5\varphi^+ \right)
\]

\[
Q \in \Lambda \text{char}(G) \quad \text{highest wt of } R.
\]

\`t Hooft:

In path integral, put bc:

\[
\tilde{x} \rightarrow \bar{x}_0 \in \mathbb{R} \quad F^x \sim P \quad \sin \theta \delta \phi + \ldots
\]

\[
\rightarrow \text{Re}(S^{-1} \varphi) \sim \frac{P}{r} + \ldots
\]

\[
P \in \Lambda \text{cochar}(G) \subset \mathbb{C}
\]
Put them together

\[ \bf{\Lambda}_{\text{char}} \oplus \Lambda_{\text{cochar}} \]

\[ \Rightarrow \text{Wilson - 't Hooft lines: } L_{P,Q,S} \]

These are "UV descriptions of the line defects" because they tell us how to modify the path integral of the nonabelian field theory.

At \( x \to \infty \) we have b.c.

\[ \Rightarrow F(x) \sim \sum_{m} \sin \theta d\theta d\phi \quad \forall \in \Lambda_{\text{coroot}} \]

This is a long-distance/IR condition

\[ \varphi \sim \langle \varphi \rangle = \psi \in B = \text{base of Hitchin fibration} \]
Without line defects (Smooth monopoles)

\[ \Omega (Y_{m, u}) = \dim_c (\ker D) \text{ monopole mod.sp.} \]

With line defects:

Use singular monopoles w/ singularity E

\[ \Omega (L_{p_s}, Y_m) = \dim_c (\ker D) \text{ sing.mon. mod.sp.} \]

The presence of the line defect at \( \bar{X} = X_0 \times 1 \times R_t \times S^1_t \) has modified the Hilbert space as a representation of \( N=2 \) super-Painleve algebra (which contains Hamiltonian). So spectrum of BPS - or groundstates - has changed.
Example 3: Surface Defect.

General idea:

- 4d spacetime $\mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$

Coordinates $(x, y, z, t)$

- $\mathbb{R}_y^2 \subset \mathbb{R}_{x,y,z,t}^4$: Subspace with fixed $y_0, z_0$.

- 4d QFT $E_4$ w/ gauge group $G$

- 2d QFT $E_2$ w/ global symmetries $G$

2d-4d system:

Couple $E_4$ to $E_2$ supported on $\mathbb{R}_y^2$ supported on $\mathbb{R}_{y, z_0}^2$ by adding to action

$$\int_{\mathbb{R}_y^2} \left< z^*(A^\mu), j^\mu \right> \, d\text{vol}(\mathbb{R}_{y, z_0}^2)$$
The BPS particles of $E_8$ can bind to the surface defect

$$y = y_0, \quad z = z_0$$

$$\Rightarrow$$ Nontrivial interplay between 2d + 4d BPS degeneracies.

If we put a line defect in the surface defect we get "framed 2d4d degeneracies".
③ Class $S$

$\mathfrak{g} = \text{s.s. Lie algebra w/ ADE summands}$

$\Rightarrow$ 6d QFT $S(\mathfrak{g})$

See below for comment on the definition of $S(\mathfrak{g})$.

- $C = \text{Riemann surface}$
- $D = \text{"defect data"}$
  - divisor $\subset C$ support $\{p_{\alpha}\}$
  - Choice of orbits in $\mathfrak{g}_0$ at $p_{\alpha}$

$\Rightarrow$ 4d QFT $S[\mathfrak{g}, C, D]$

Proof: $6 = 4 + 2$ $M_4 \times C$

Partial topological twist $\Rightarrow$

independence of some quantities on Kähler class of $C'$, $\text{area}(C) \to 0$
Example:

→ $\mathfrak{g} \otimes \mathfrak{su}(2) \cdot 6d$

→ $C$: genus $g$

→ $D$: $n$ punctures, orbit at $p$: $\begin{pmatrix} m_2^+ & 0 \\ -m_2 \end{pmatrix}$

4d gauge theory has gauge group $G$

- Lie $(G) = \mathfrak{su}(2)^{3g-3+n} = \mathfrak{g}_{4d}$

- Coupling constants $\tau_i$ parametrize conformal structure of $C_{g,n}$

(Many different descriptions based on pants decomposition: Gaiotto)

\[ \tau \]

- $p_x$, $n=0$ makes sense, but is qualitatively different.
There are 't Hooft–Wilson line defects \( L_{p+q} \),

\[ P+Q \in \Lambda_{\text{cochar} (\mathfrak{g}_{4d})} \oplus \Lambda_{\text{char} (\mathfrak{g}_{4d})} \]

Drukker–Morrison–Okuda: These are Dehn–Thurston coordinates for isotopy class of closed 1-dimensional submanifold \( P < C \).

(at least ... when \( p \) is connected ...)

So we label line defects by

\( P = \text{isotopy class of closed curve in } C \).
Two "facts"

about 6d theory $S[\mathcal{g}]$

N.B. No definition of $S[\mathcal{g}]$ exists, even by physical standards where it is considered "obvious" that four-dimensional (nonanomalous) gauge theories exist.

An attempt to write a list of working rules ("axioms") which physicists use to produce mathematically well-defined statements and conjectures can be found in
My Felix Klein lecture notes in section 6.6 pp. 78-80. See talk #47 on my homepage.
$S[\gamma]$ has **surface defects**

In 6d spacetime $\mathbb{R}^3 \times S^1 \times \mathbb{C}$

(A) $\text{Supp}(S) = \{ \vec{x}_0 \} \times S^1 \times \mathbb{R} \quad \forall \, \theta \in \mathbb{C}$

$\Rightarrow$ Line defect in 4d theory on $\{ \vec{x}_0 \} \times S^1$

Isotopy class of $P$ is a "UV label" generalizing the labels of 't Hooft-Wilson lines

(B) $\text{Supp}(S) = \mathbb{R}^3 \times S^1 \times \{ \gamma \} \quad \forall \gamma \in \mathbb{C}$

$\Rightarrow$ Surface defect in 4d theory, $S[\gamma]$,

More careful analysis: $L_p$ also labeled by rep $R$ of $\gamma$ and phase $S$. 
2. When $S[\phi]$ is compactified on a circle, the LEET is

\[ \text{5D SYM} \Rightarrow \]

- $\phi$ gauge connection $A$
- $\phi_c$ adjoint scalar $\psi$

$S[\phi]$ on

\[ TR^3 \times S^1_R \times C \]

$S^1 \rightarrow \text{area}(C)$

$E \ll \frac{1}{R}$

$\text{HK } \sigma\text{-model}$

\[ TR^3 \rightarrow M_{SW} \]

$E \ll \frac{1}{\sqrt{\text{area}(C)}}$

$\text{HK } \sigma\text{-model}$

\[ TR^3 \rightarrow M_{\text{Hitchin}} \]
Answer to question: "How did you get the Hitchin equations?"

10D SYM (others are reductions/ trunc's)

\[ A^M, \lambda \quad M = 0, \ldots, 9 \]

\[ Q^\lambda = 0 \quad \Rightarrow \quad \gamma^{MN} F_{MN} A^\lambda = 0. \]

For a suitable spinor \( \xi \)

One example: \( F^+ = 0 \).

Another example: Hitchin eqs.

We did not follow Hitchin's route of reducing \( F^+ = 0 \) to two dimensions.
**IR description**

\[ \langle \varphi \rangle = \lim_{x \to \infty} \varphi(x) \]

- **Ω** = Coulomb branch = base of Hitchin fibration

- \[ \hat{C} \subset \tilde{T}^* \mathcal{C} \]

  \[ \int_{\hat{C}} \mathcal{C} \]

  Abelian gauge theory \[ U(1)^{\nu} \]

  Electro-magnetic charge lattice of IR AbThy

  \[ \Gamma \] is a subquotient of \( H_1(\Sigma, \mathbb{Z}) \)

  \[ Z = \phi_{\lambda} : \Gamma \to \mathcal{C} \] determines IR LEFT

  \[ \implies \Omega(\gamma; u) \text{ etc.} \]

  T.B. talk \( \nu = A_1 \), \( \Gamma = H_1(\Sigma, \mathbb{Z}) \)