Informal Remarks Preparatory For/ Complementary To Fei Yan's Talk

Simons Collaboration On Special Holonomy

Jan. 13, 2021

* Arbeitstagung

* Useful

Comments On:

1. Physics Background

2. Defects And Their BPS States

3. Class $\mathcal{S}$

4. Spectral Networks + Nonabelianization Map

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5. RH Problems, Integral Equations and Hyperkähler Geometry

① Physics Background

— Physicists assume many things and have intuitions and examples in mind that they take for granted, but which are not obvious to anyone else.

— Hamburg School On Higgs Bundles, Sept. 2018: Talk #84 on my homepage goes back to the beginning:

* Branes + Geometrization of Higgs Mechanism
* M5 Branes
* 6d (2,0) Theory
* Geometrical pictures for class S & their BPS states

Goal:

Explain the physics intuition behind theory of "Spectral networks"

(2) Defects

* $\exists$ rigorous theory in the context of extended TQFT ("cobordism hypothesis")

* In susy context: <u>New</u> BPS deg's

4d $N=2$ SQFT: $\Omega(\gamma) \longleftrightarrow$ DT

line defects $\rightsquigarrow \underline{\overline{\Omega}}(L, \gamma)$ "framed"

surface defects $\rightsquigarrow \mu(\gamma_{ij})$ "soliton"

line defects in surface $\rightsquigarrow \underline{\overline{\Omega}}(L_\rho, \gamma_{ij})$ "framed soliton"

* $\Omega(\gamma) \Longrightarrow$ Well-known RH problem useful for construction of HK metrics on moduli

$\mathcal{M}$: space of solutions to Hitchin equations on a R.S. C (with singularities) other BPS deg's

$*$ other BPS deg's $\Longrightarrow$ Similar RH problems

Constructing:

a.) Hyper-holo connections on certain vector bundles over $\mathcal{M}$

b.) Explicit construction of solutions to Hitchin equations on R.Surface C.

Now give some examples of defects.

- Example 1: Soliton & framed soliton deg's:

X exact Kähler $\omega = d\lambda$

✓ $W: X \longrightarrow \mathbb{C}$ superpotential (holo, Morse)

(X, W) $\longrightarrow$ 1+1 dim'l massive LG model. (Phys)

$\longrightarrow$ Fukaya-Seidel category (Math)

Critical points of W: $\{\phi_i\}$ $\leftarrow$

$$\mathcal{X}_{ij} := \left\{ \phi : \mathbb{R} \to X \,,\ \phi \begin{array}{l} \longrightarrow \phi_i \quad x \to -\infty \\ \longrightarrow \phi_j \quad x \to +\infty \end{array} \right\}$$

$$h = \int_{\mathbb{R}} \left[\phi^*(\lambda) - \mathrm{Re}\left(\zeta W(\phi(x))\right) dx \right]$$

phase

$\delta h = 0$ soliton eq. $\dfrac{d\phi}{dx} = \zeta \overline{\nabla W}$

$$\boxed{\mu_{ij} = \chi(\text{Morse Complex})}$$

"Soliton degeneracies"

Consider a manifold C' of Morse superpotentials

$$W(\phi ; z), \quad z \in C'$$

$\rightsquigarrow$ $\wp(x): z_1 \rightsquigarrow z_2$ path in C'

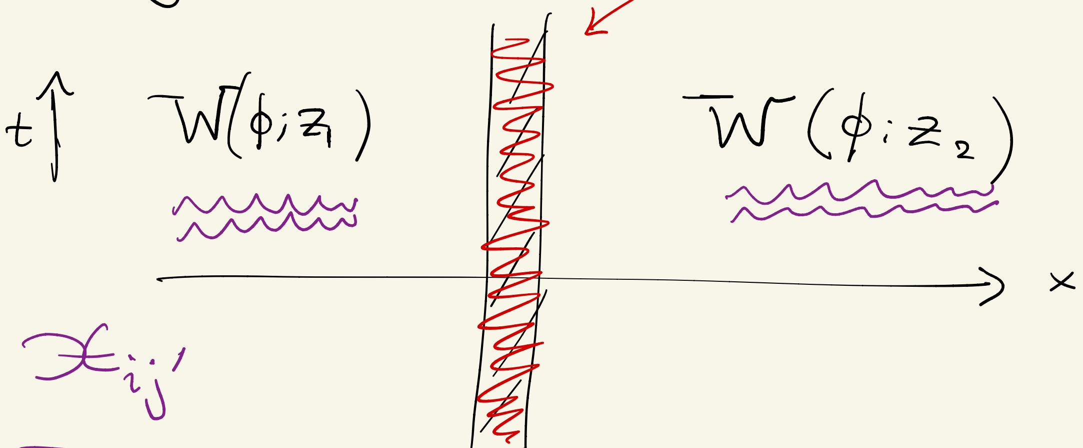
- $h = \int \left[\phi^*(\lambda) - \mathrm{Re}\left(\zeta W(\phi(x), z(x))\right) dx \right]$

$\delta h = 0$ soliton-like eq.

gives a new Morse complex.

Physical picture

defect: Line defect



$$\boxed{\mu_{ij'}(\wp) = \chi(\text{Morse Complex})}$$

"framed BPS deg"

- Example 2: Line Defects In 4d Gauge Theory w/ gauge group G

$$\boxed{\text{4d spacetime: } \quad \mathbb{R}^3 \times S^1}$$

$\varphi \in \mathfrak{g} \otimes \mathbb{C}$

Wilson:

$$L(R,\zeta) = \mathrm{Tr}_R \, P\exp \int_{\{\vec{x}_0\}\times S^1} (\zeta^{-1}\varphi + A + \zeta\varphi^{\dagger})$$

$Q \in \Lambda_{char}(G)$ highest wt of R.

't Hooft:

In path integral, put b.c.:

$$\vec{x} \to \vec{x}_0 \in \mathbb{R} \qquad F(\vec{x}) \sim P \sin\theta \, d\theta \, d\phi + \cdots$$

$$\longrightarrow \mathrm{Re}(\zeta^{-1}\varphi) \sim \frac{P}{r} + \cdots$$

$$\boxed{P \in \Lambda_{cochar}(G) \subset \mathfrak{g}}$$

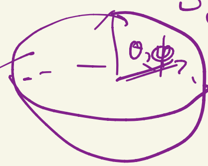
Put them together

$$P \oplus Q \in \Lambda_{char} \oplus \Lambda_{cochar}$$

$\Rightarrow$ Wilson-'t Hooft lines: $L_{P,Q,\zeta}$

These are "UV descriptions of the line defects" because they tell us how to modify the path integral of the nonabelian field theory.

S^2_∞



At $\vec{x} \longrightarrow \infty$ we have b.c.

- $\vec{F}(\vec{x}) \sim \gamma_m \sin\theta \, d\theta \, d\phi \qquad \gamma_m \in \Lambda_{coroot}$

This is a long-distance/IR condition

- $\varphi \sim \langle \varphi \rangle = u \in \mathcal{B} =$ base of Hitchin fibration

Without line defects (smooth monopoles)

$$\Omega(\gamma_m, u) = \dim_{\mathbb{C}} \left(\ker_{L^2} \not{D}_{\text{monopole mod. sp.}}^{\gamma} \right)$$

dim (Monopole Moduli) $\sim 4|\gamma_m|$

- With line defects:

Use singular monopoles w/ singularity P.

$F \longrightarrow P \sin\theta\, d\theta\, d\phi$
$\varphi \longrightarrow P/r$

$$\boxed{\underline{\Omega}(L_{P,\zeta}, \gamma_m) = \dim_{\mathbb{C}} \left(\ker_{L^2} \not{D}_{\text{sing. mon. mod sp}} \right)}$$

The presence of the line defect at $\{\vec{x} = \vec{x}_0\} \times \mathbb{R}_t$ $(\times S^1_t)$ has modified the Hilbert space as a representation of $N=2$ super-Poinc. algebra (which contains Hamiltonian). So spectrum of BPS — or groundstates — has changed.

- Example 3: Surface Defect.

General idea:

- 4d spacetime $\underline{\underline{\mathbb{R}^4}}$ or $\underline{\underline{\mathbb{R}^3 \times S^1_t}}$

  Coordinates $(\underline{x, y, z, t})$

- $\mathbb{R}^2_{y_0, z_0} \subset \mathbb{R}^4_{xyz,t}$: Subspace with fixed $\underline{y_0}, \underline{z_0}$.

- $\underline{\text{4d QFT}}\ \mathcal{C}_4$ w/ $\underline{\text{gauge group } G}$

- $\underline{\text{2d QFT } \mathcal{C}_2}$ w/ $\underline{\text{global symmetries } G}$

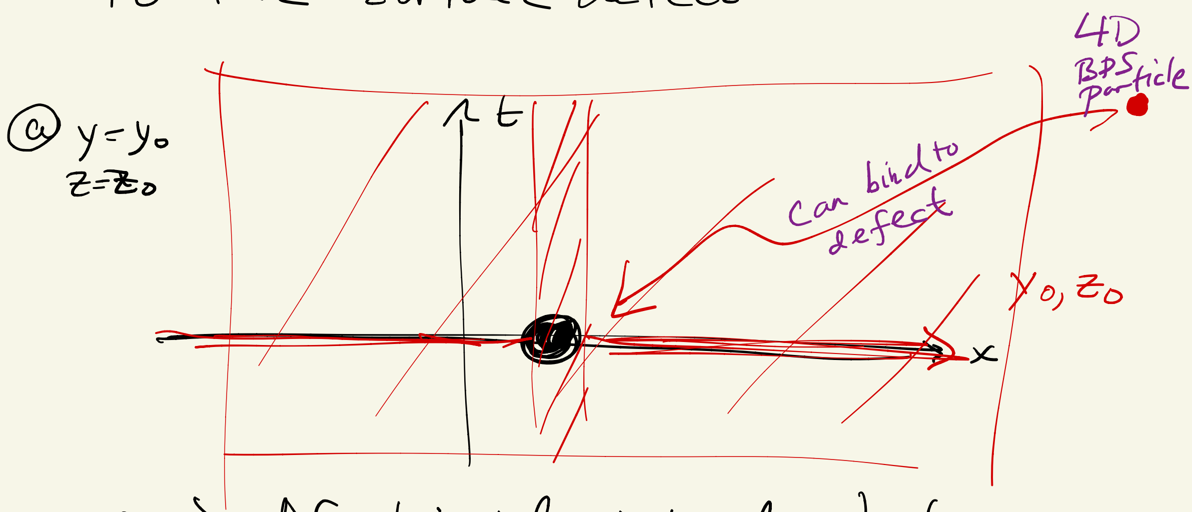
$\boxed{\text{2d-4d system:}}$

Couple $\underline{\mathcal{C}_4}$ to $\underline{\mathcal{C}_2}$ supported on $\mathbb{R}^2_{y_0 z_0}$

by adding to action

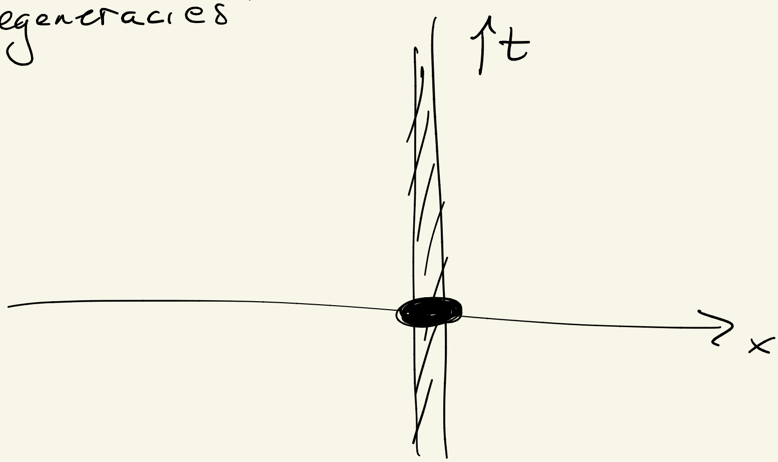
$$\int_{\mathbb{R}^2_{y_0 z_0}} \langle \underline{z^*(A_\mu)}, \underline{j^\mu} \rangle \ \underline{d(\text{vol})}$$

The BPS particles of $\mathcal{C}_4$ can bind to the surface defect



$\rightsquigarrow$ Nontrivial interplay between 2d + 4d BPS degeneracies.

If we put a line defect in the surface defect we get "framed 2d4d degeneracies"



③ Class $\mathcal{S}$

$\mathfrak{g}$ = s.s. Lie algebra w/ ADE summands

$\Longrightarrow$ 6d QFT $\mathcal{S}(\mathfrak{g})$

See below for comment on the definition of $\mathcal{S}(\mathfrak{g})$

- C = Riemann surface
- D = "defect data"
  - divisor $\subset \underline{C}$ support $\underline{\{P_\alpha\}}$
  - $\underline{\text{Choice of orbits}}$ in $\underline{\mathfrak{g}_{\mathbb{C}}}$ at $\underline{P_\alpha}$

$\Longrightarrow$ 4d QFT $\mathcal{S}[\underline{\mathfrak{g}, C, D}]$

Proof: $6 = 4 + 2$ $\quad M_4 \times C$

$\underline{\text{partial topological twist}} \Rightarrow$ independence of some quantities on $\underline{\text{Kähler class of } C}$. $\quad \text{area}(C) \to 0$

Example:

→ $\mathfrak{g}$: $su(2)$. 6d

→ C : genus g

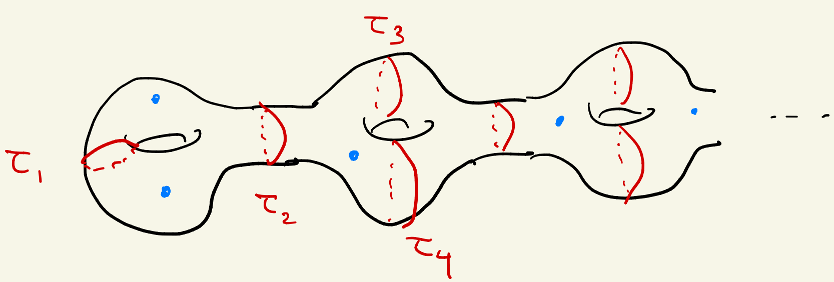
→ D : n punctures, orbit at p_α: $\begin{pmatrix} m_\alpha^{\neq 0} & \\ & -m_\alpha \end{pmatrix}$

4d gauge theory has gauge group G

- $\mathrm{Lie}(G) = su(2)^{\boxed{3g-3+n}} = \mathfrak{g}_{4d}$

- Coupling constants $\tau_i = \theta_i + \sqrt{-1}/g_i^2$ parametrize conformal structure of $\boxed{C_{g,n}}$

(Many different descriptions based on pants decomposition: Gaiotto)



• p_α

$n=0$ makes sense, but is qualitatively different.

There are 't Hooft-Wilson line defects $L_{P \oplus Q, \wp}$

$$P \oplus Q \in \Lambda_{\text{cochar}}(\mathfrak{g}_{4d}) \oplus \Lambda_{\text{char}}(\mathfrak{g}_{4d})$$

Drukker-Morrison-Okuda: These are Dehn-Thurston coordinates for isotopy class of closed 1-dimensional submanifold $\wp \subset C$

(at least .... when $\wp$ is connected ....)

So we label line defects by

$$\boxed{\wp = \text{isotopy class of closed curve in } C}$$

Two "facts"

about 6d theory $S[\mathfrak{g}]$

N.B. No definition of $S[\mathfrak{g}]$ exists, even by physical standards where it is considered "obvious" that four-dimensional (nonanomalous) gauge theories exist.

An attempt to write a list of working rules ("axioms") which physicists use to produce mathematically well-defined statements and conjectures can be found in

my Felix Klein lecture notes in
Section 6.6 pp. 78-80.
See talk #47 on my homepage.

1. $\mathcal{S}[\mathfrak{g}]$ has surface defects

In 6d spacetime $\mathbb{R}^3 \times S^1 \times C$

(A) $\mathrm{Supp}(\mathbb{S}) = \{\vec{x}_0\} \times S^1 \times \wp \qquad \wp \subset C$

$\Longrightarrow$ Line defect in 4d theory on $\{\vec{x}_0\} \times S^1$ $\qquad \boxed{L_{\wp, \zeta, \mathcal{R}}}$

Isotopy class of $\wp$ is a "UV label" generalizing the labels of 't Hooft - Wilson lines

(B) $\mathrm{Supp}(\mathbb{S}) = \boxed{\mathbb{R}_{y_0, z_0} \times S^1}_{\subset \mathbb{R}^3} \times \{z\}$, $z \in C$

$\Longrightarrow$ Surface defect in 4d theory, $\mathbb{S}_z$

More careful analysis: $L_\wp$ also labeled by rep $\mathcal{R}$ of $\mathfrak{g}$ and phase ζ.

② When $S[\mathfrak{g}]$ is compactified on a circle, the LEET is

5D SYM $\Longrightarrow$

- $\mathfrak{g}$ gauge connection A
- $\mathfrak{g}_{\mathbb{C}}$ adjoint scalar φ

$$QA = \psi$$

$$0 = Q\Psi = \underbrace{F^{+} + [\varphi, \varphi^{\dagger}]}_{=0}$$

$S[\mathfrak{g}]$ on

$$\begin{tikzcd}[column sep=small]
 & \boxed{\mathbb{R}^3 \times S^1_R \times C} \arrow[dl, "C"', "\mathrm{area}(C) \ll R^2"'] \arrow[dr, "S^1", "R^2 \ll \mathrm{area}(C)"] & \\
\boxed{\begin{array}{c} S[\mathfrak{g}, C, D] \\ \text{on } \mathbb{R}^3 \times S^1 \end{array}} \arrow[d, "E \ll \frac{1}{R}"'] & & \boxed{\begin{array}{c} \mathfrak{g} \text{ SYM on} \\ \boxed{\mathbb{R}^3 \times C} \end{array}} \arrow[d, "E \ll \frac{1}{\sqrt{\mathrm{area}(C)}}"] \\
\boxed{\text{HK } \sigma\text{-model}} \arrow[rr, phantom, "="] & & \text{HK } \sigma\text{-model} \\
\mathbb{R}^3 \to \mathcal{M}_{SW} & & \boxed{\mathbb{R}^3 \to \mathcal{M}_{Hitchin}}
\end{tikzcd}$$

Answer to question: "How did you get the Hitchin equations"

10D SYM (others are reductions/trunc's)

$$A_M, \lambda \qquad M=0,\dots,9$$

$$Q\lambda = 0 \implies \gamma^{MN} F_{MN}\, s = 0.$$

for a suitable spinor s

one example: $F^+ = 0$.

another example: Hitchin eqs.

We did not follow Hitchin's route of reducing $F^+ = 0$ to two dimensions.

IR description

$\langle \varphi \rangle = \lim \varphi(\vec{x})$

$\mathcal{B}$ = Coulomb branch = base of Hitchin fibration

$$\begin{array}{c} \tilde{C} \subset T^*C \\ \downarrow \pi \\ C \end{array}$$

spectral curve = Seiberg-Witten curve

Abelian gauge theory $U(1)^r$

λ = Liouville form = S-W differential

Electro-magnetic charge lattice of IR Ab. Thy

$\Rightarrow$ Γ is a subquotient of $H_1(\Sigma, \mathbb{Z})$

$Z = \oint \lambda : \Gamma \longrightarrow \mathbb{C}$ determines IR LEEFT

$$\Longrightarrow \boxed{\Omega(\gamma; u) \text{ etc.}}$$

T.B. talk $\mathfrak{g} = A_1$ $\Gamma = H_1^-(\Sigma, \mathbb{Z})$