From Donaldson-Thomas invariants to hyperkähler structures

Tom Bridgeland

University of Sheffield

Lecture 3

Riemann-Hilbert problems

Recap: irregular singularity

Consider an equation for $Y\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$ of the form

$$\frac{d}{d\epsilon} Y(\epsilon) = \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon} \right) Y(\epsilon)$$

with constant matrices $U, V \in \mathfrak{g} = \mathfrak{g}l_n(\mathbb{C})$, such that

(I) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}^{\mathrm{reg}}$ is diagonal with $u_i \neq u_j$,

(II) $V \in \mathfrak{g}^{\mathrm{od}}$ has zeroes on the diagonal.

The Stokes rays of the equation at $\epsilon = 0$ are defined to be the rays

$$\mathbb{R}_{>0} \cdot (u_i - u_j) = \mathbb{R}_{>0} \cdot U(\alpha).$$

Recap: Stokes factors

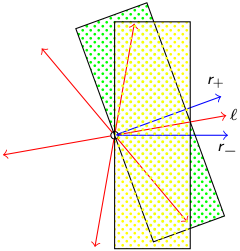
Theorem (Balser, Jurkat, Lutz)

For any half-plane $\mathbb{H}_r \subset \mathbb{C}^*$ centered on a non-Stokes ray $r \subset \mathbb{C}^*$, there is a unique solution $Y_r \colon \mathbb{H}_r \to G$ such that

$$Y_r(\epsilon) \cdot \exp(U/\epsilon) \to 1 \text{ as } \epsilon \to 0.$$

Stokes factors $\mathbb{S}(\ell) \in G$ defined by

$$Y_{r+}(\epsilon) = Y_{r_-}(\epsilon) \cdot \mathbb{S}(\ell)$$



RH problem: case $G = \mathrm{GL}_n(\mathbb{C})$

Suppose given $U \in \mathfrak{h}^{\mathrm{reg}}$ and the Stokes factors $\mathbb{S}(\ell)$. To reconstruct the connection we first try to find the half-plane solutions $Y_r(t)$.

Riemann-Hilbert problem

For each non-Stokes ray $r \subset \mathbb{C}^*$ find $Y_r \colon \mathbb{H}_r \to G$ such that

$$Y_r(\epsilon) \cdot \exp(U/\epsilon) \to 1 \text{ as } \epsilon \to 0 \text{ in } \mathbb{H}_r,$$

$$|\epsilon|^{-k} < \|Y_r(\epsilon)\| < |\epsilon|^k \text{ as } \epsilon \to \infty \text{ in } \mathbb{H}_r,$$

and if $\Delta \subset \mathbb{C}^*$ is a convex sector with $\partial\Delta = \{r_+\} \cup \{r_-\}$ then

$$Y_{r+}(\epsilon) = Y_{r_-}(\epsilon) \cdot \mathbb{S}(\Delta) \text{ for } \epsilon \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

Even in this finite-dimensional setting, existence and uniqueness of solutions is subtle, cf. Hilbert's 21st problem.

Recap: wall-crossing formula

Let $\mathcal{D}$ be a CY_3 Δ-category, with a stability condition $\sigma = (Z, \mathcal{P})$.

$$\Gamma = K_0(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}, \qquad \mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n.$$

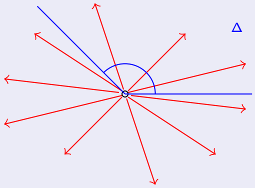
For each ray $\ell \subset \mathbb{C}^*$, define an element $\mathbb{S}(\ell) \in G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$:

$$\mathbb{S}(\ell)^*(x_\beta) = x_\beta \cdot \prod_{Z(\gamma)\in\ell} (1 - x_\gamma)^{\Omega(\gamma)\cdot\langle\beta,\gamma\rangle}$$

The wall-crossing formula:

$$\mathbb{S}_\sigma(\Delta) = \overset{\curvearrowright}{\prod_{\ell\in\Delta}} \mathbb{S}_\sigma(\ell) \in G$$

is constant as σ varies.



Recap: DT invariants as Stokes data

The wall-crossing formula is the isomonodromy condition for a family of meromorphic connections on the trivial G-bundle over $\mathbb{P}^1$:

$$\nabla_\sigma = d - \left(\frac{Z}{\epsilon^2} + \frac{\mathsf{Ham}_F}{\epsilon}\right) d\epsilon,$$

parameterised by the points of $M = \mathsf{Stab}(\mathcal{D})$, where

- $Z \in \mathfrak{h}$ is the central charge $Z \colon \Gamma \to \mathbb{C}$,
- $F = \sum_{\gamma \in \Gamma^\times} F_\gamma \cdot x_\gamma \in \mathfrak{g}^{\mathrm{od}}$ is a function on $\mathbb{T}$.

$$W(Z, \theta) = \sum_{\gamma \in \Gamma^\times} F_\gamma(Z) \cdot \frac{e^{\theta(\gamma)}}{Z(\gamma)} \colon \mathcal{T}\,\mathsf{Stab}(\mathcal{D}) \to \mathbb{C}.$$

Isomonodromy equation $\implies$ Plebański's second heavenly equation.

The DT Riemann-Hilbert problem

Fix a stability condition $\sigma = (Z, \mathcal{P})$ and a point $\xi = \exp(\theta) \in \mathbb{T}$.

We are taking $G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$, and composing $Y\colon \mathbb{C}^* \to G$ with the evaluation map $\mathrm{ev}_\xi : G \to \mathbb{T}$.

For each non-Stokes ray $r \subset \mathbb{C}^*$ find $X_r\colon \mathbb{H}_r \to \mathbb{T}$ such that

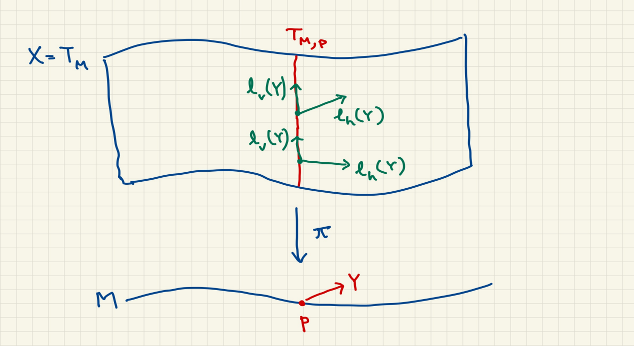
$$X_r(\epsilon) \cdot \exp(Z/\epsilon) \to \xi \in \mathbb{T} \text{ as } t \to 0 \text{ in } \mathbb{H}_r,$$

$$|\epsilon|^{-k} < \|X_r(\epsilon)\| < |\epsilon|^k \text{ as } \epsilon \to \infty \text{ in } \mathbb{H}_r,$$

and if $\Delta \subset \mathbb{C}^*$ is a convex sector with $\partial\Delta = \{r_+\} \cup \{r_-\}$ then

$$X_{r+}(\epsilon) = X_{r_-}(\epsilon) \cdot \mathbb{S}(\Delta) \text{ for } \epsilon \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

Recap: Joyce structures



- Expect a Joyce structure on $M = \mathrm{Stab}(\mathcal{D})$.
- Pencil of non-linear connections on $\pi\colon\ \mathrm{Tot}(\mathcal{T}_M) \to M$.

Strategy

- Start with the DT invariants and try to solve the DT RH problem describing a function $X\colon \mathbb{C}^* \to \mathbb{T}$.
- The DT RH problem depends on a point of M with co-ordinates z_i, and a point $\xi = \exp(\theta) \in \mathbb{T}$ with co-ordinates θ_j.
- Substitute the components of the solution $x_k = \log X_k(z_i, \theta_j, \epsilon)$ into the above equation to get the Plebański function W.
- Use the function W to define a Joyce structure on M.
- Describe solutions using generating functions.

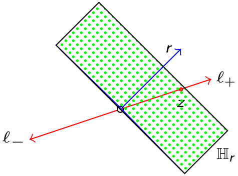
Solutions to the RH problem describe twistor lines in the twistor space of the complex hyperkähler manifold.

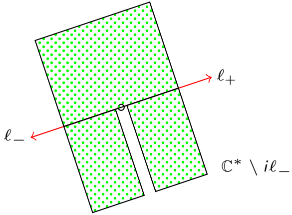
Can we go the other way: start with a Joyce structure and define associated DT invariants?

Is there a more direct way to define the Joyce structure?!

Example: doubled A_1

- $\Gamma_D = \Gamma \oplus \Gamma^\vee = \mathbb{Z}\gamma \oplus \mathbb{Z}\gamma^\vee$ with $\langle \gamma^\vee, \gamma \rangle = 1$,
- $Z(\gamma) = z \in \mathbb{C}^*$ and $Z(\gamma^\vee) = z^\vee \in \mathbb{C}$,
- $\Omega(\pm\gamma) = 1$ with all other BPS invariants 0.





Note that this example is uncoupled:

$$\Omega(\alpha) \neq 0 \text{ and } \Omega(\beta) \neq 0 \implies \langle \alpha, \beta \rangle = 0.$$

In general this condition ensures that the wall-crossing automorphisms for different rays commute.

DT RH problem with $\theta = 0$

Find functions $X_\pm, \hat{X}_\pm \colon \mathbb{C}^* \setminus i\ell_\pm \to \mathbb{C}^*$ such that:

$$X_-(\epsilon) = X_+(\epsilon) \qquad \hat{X}_-(\epsilon) = \begin{cases} \hat{X}_-(\epsilon)\cdot(1 - X(\epsilon)^{+1}) & \mathrm{Re}(\epsilon/z) > 0 \\ \hat{X}_-(\epsilon)\cdot(1 - X(\epsilon)^{-1}) & \mathrm{Re}(\epsilon/z) < 0 \end{cases}$$

As $\epsilon \to 0$:

$$X_\pm(\epsilon)\cdot e^{z/\epsilon} \to 1 \text{ and } \hat{X}_\pm(\epsilon)\cdot e^{\hat{z}/\epsilon} \to 1,$$

As $t \to \infty$:

$$|\epsilon|^{-k} < |X_\pm(\epsilon)|, |\hat{X}_\pm(t)| < |\epsilon|^k.$$

Modified gamma function

The unique solution to the problem is

$$X_{\pm}(\epsilon) = e^{-z/\epsilon}, \qquad \hat{X}_{\pm}(\epsilon) = e^{-\hat{z}/\epsilon} \cdot \Lambda\Big(\frac{\pm z}{2\pi i \epsilon}\Big)^{\pm 1},$$

where $\Lambda(w)$ is the modified gamma function

$$\Lambda(w) = \frac{e^w \cdot \Gamma(w)}{\sqrt{2\pi} \cdot w^{w-\frac{1}{2}}} \sim \exp\Big(\sum_{g=1}^{\infty} \frac{B_{2g} \cdot w^{1-2g}}{2g(2g-1)}\Big).$$

There is a similar solution for arbitrary θ but it has poles where $X_{\pm}(\epsilon) = 1$.

The Laplace transform of $\log X(\epsilon)$ is

$$\frac{z}{2\pi i s}\Big(\frac{1}{1-e^{-2\pi i s/z}} - \frac{z}{2\pi i s} - \frac{1}{2}\Big)$$

The Laplace transform has poles at the points $Z(\alpha)$ for which $\mathrm{DT}(\alpha) \neq 0$.

Joyce structure

The Joyce or Plebański function is

$$W(z,\theta) = \frac{\theta^3}{6z}.$$

The resulting complex hyperkähler structure is flat.

The only non-trivial data is the linear Joyce connection, given by

$$\nabla^J_{\frac{\partial}{\partial z_i}}\left(\frac{\partial}{\partial z_j}\right) = \sum_{p,q} \eta^{pq} \frac{\partial^3 \mathcal{F}}{\partial z_i \partial z_j \partial z_p} \frac{\partial}{\partial z_q}.$$

In this case the prepotential $\mathcal{F}$ is

$$\mathcal{F}(z) = \frac{1}{2} z^2 \log(z).$$

Generating function for solutions

Setting $\hat{Y}(\epsilon) = \exp(\hat{z}/\epsilon) \cdot \hat{X}(\epsilon)$ we define $\tau = \tau(z, \epsilon)$ by

$$\frac{\partial \log \tau}{\partial z} = \frac{1}{2\pi i} \cdot \frac{\partial \log Y}{\partial \epsilon}.$$

Describes the image of the locus $\theta = 0$. Only valid in the uncoupled case?

Then we can write $\tau(z, \epsilon) = \Upsilon(z/2\pi i\epsilon)$ with

$$\Upsilon(w) = \frac{e^{-\zeta'(-1)} \cdot e^{\frac{3}{4}w^2} \cdot G(w+1)}{(2\pi)^{\frac{w}{2}} \cdot w^{\frac{w^2}{2}}} \sim \sum_{g \geq 2} \frac{B_{2g} \cdot w^{2-2g}}{2g \cdot (2g-2)}$$

a modified Barnes G-function.

Name for this in physics?

Small Calabi-Yau threefolds

We can solve the RH problems for the DT theory associated to coherent sheaves on a non-compact CY_3 without compact divisors.

For simplicity we just consider the resolved conifold. Then

$$\Gamma = \mathbb{Z}^{\oplus 2}, \qquad \langle -, - \rangle = 0.$$

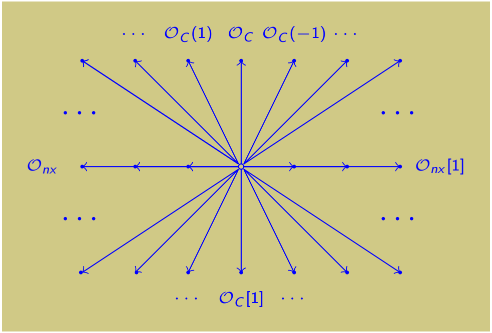
The space of stability conditions is a cover of

$$M = \left\{ (v, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } v + dw \neq 0 \text{ for all } d \in \mathbb{Z} \right\} \subset \mathbb{C}^2$$

with central charge $Z(r, d) = rv + dw$.

Again this is uncoupled. We must consider $\Gamma_D = \Gamma \oplus \Gamma^\vee$.

The ~~ray~~ peacock diagram



$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm(1, d) \text{ for some } d \in \mathbb{Z}, \\ -2 & \text{if } \gamma = (0, d) \text{ for some } 0 \neq d \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Solution and prepotential

There is a solution to the RH problem (unique when $\theta = 0$) given explicitly using Barnes double and triple gamma functions.

The Joyce or Plebański function is

$$W(v, w, \theta, \phi) = \frac{1}{6w^4} \cdot \frac{(v\phi - w\theta)^3}{1 - e^{-2\pi i v/w}}.$$

Again, the resulting complex hyperkähler metric is flat.

The prepotential is

$$\mathcal{F}(v, w) = \frac{w^2}{(2\pi i)^3} \cdot \mathrm{Li}_3(e^{2\pi i v/w}),$$

and coincides with the $g = 0$ GW generating function.

A non-perturbative partition function

Defining τ as before we get

$$\tau(v,w,\epsilon) = \exp(H+R) \sim \text{sporadic } g=0,1 \text{ terms } +$$
$$+\sum_{g\geq 2} \frac{B_{2g}}{2g\,(2g-2)!}\Big(\operatorname{Li}_{3-2g}(e^{2\pi i v/w}) + \frac{B_{2g-2}}{2g-2}\Big)\Big(\frac{2\pi i\epsilon}{w}\Big)^{2g-2}$$

$$H(v,w,\epsilon) = \int_{\mathbb{R}+i\epsilon} \frac{e^{vs}-1}{e^{ws}-1}\cdot\frac{-e^{\epsilon s}}{(e^{\epsilon s}-1)^2}\cdot\frac{ds}{s}$$

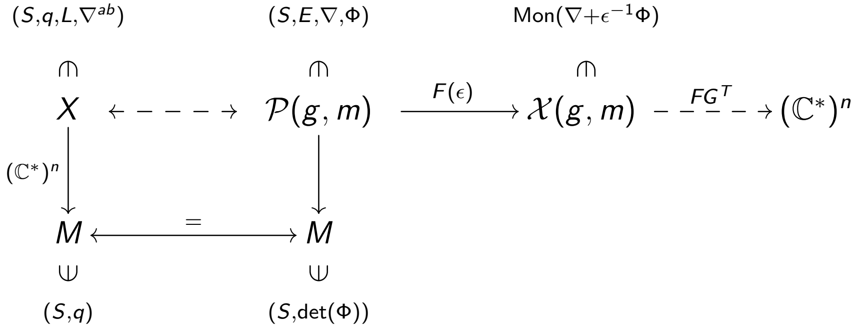
$$R(v,w,\epsilon) = \Big(\frac{w}{2\pi i\epsilon}\Big)^2\Big(\operatorname{Li}_3(e^{2\pi i v/w}) - \zeta(3)\Big) + \frac{\pi i v}{12 w}$$

For $g\geq 2$ the expansion coincides with the GW generating function.

Note that the Joyce structure is determined by just the genus 0 data, but implicitly determines the higher genus data.

Quadratic differential examples

Take $\mathcal{D} = \mathcal{D}(g, m)$ so that $M = \mathrm{Quad}(g, m)$.



Conjecture

If we take T to be the WKB triangulation of the differential $\epsilon^{-1} \cdot q$, the composite map gives a solution to the DT RH problem.

Quadratic differential examples

- Very closely related to work of Gaiotto, Moore, Neitzke.
- Jumping ensured by properties of Fock-Goncharov co-ordinates.
- WKB analysis for $\epsilon \to 0$.
- Proofs by Allegretti in many cases for $\theta = 0$ (the oper locus).
- Completely checked in the A_2 case with Masoero.

Further questions

How best to describe the solutions to the DT RH problem?

- On the zero section $\theta = 0$ the solution is given by the Fock-Goncharov co-ordinates of the monodromy of opers.

  Relation to the generating function for the oper locus and work of Nekrasov-Rosly-Shatashvili?

- Asking for the solution to be constant is the isomonodromy condition for the connections $\nabla + \epsilon^{-1}\Phi$ parameterised by X.

  Relation to Painlevé and isomonodromy τ-functions.

  Relation to the GW generating function of the mirror threefold? Coman-Longhi-Pomoni-Teschner

- Relation to topological recursion partition function.

  Iwaki-Kidwai: the Borel sum coincides with the DT RH τ-function for hypergeometric spectral curves (uncoupled).

Refined DT theory defines a q-DT RH problem.

The A_1 case solved by Barbieri-TB-Stoppa. The resolved conifold case should be do-able.

Can the quadratic differential case be approached using skein algebras? Fei Yan-Neitzke.

Twistor space of a complex HK

For each point q of

$$Q = \{[a : b : c] \in \mathbb{P}^2 : a^2 + b^2 + c^2 = 0\} = \mathbb{P}^1$$

there is a foliation $\mathcal{F}(q)$ of X with leaves tangent to

$$\mathcal{H}(q) = \ker(aI + bJ + cK) \subset \mathcal{T}_X.$$

Introduce the twistor space

$$Z = \{(q, \mathcal{L}) : q \in Q, \mathcal{L} \text{ is a leaf of } \mathcal{F}(q)\} \xrightarrow{\pi} Q = \mathbb{P}^1.$$

Each point $x \in X$ defines a section $s_x \colon Q \to Z$ by

$$s_x(q) = (q, \text{leaf of } \mathcal{F}(q) \text{ through } x).$$

In general it seems unlikely that Z is Hausdorff.

Twistor space of a Joyce structure

Consider again the twistor space

$$\pi\colon Z \to Q = \mathbb{P}^1.$$

We can take an affine co-ordinate ϵ on Q so that

$$\mathcal{H}(\epsilon) = \ker\big((J - iK) + 2i\epsilon I + \epsilon^2(J + iK)\big).$$

Consider the fibres $Z_\epsilon := \pi^{-1}(\epsilon)$.

- For an affine symplectic fibration $\pi\colon X \to M$ the vertical sub-bundle is $\mathcal{H}(0) = \ker(J - iK)$. Thus $Z_0 = M$.
- In the case of a Joyce structure, the Euler vector field ensures that $Z_\epsilon = Z_1$ for all $\epsilon \in \mathbb{C}^*$.

In the quadratic differential examples $Z_0 = \mathcal{Q}(g)$ and Z_1 is a cover of $\mathcal{X}(g)$. Can we construct Z directly?

Twistor perspective on RH problem

- There is a projection $X \times \mathbb{P}^1 \to Z$.
- In the case of a Joyce structure the homogeneity gives

$$p^{-1}(\mathbb{C}^*) \cong Z_1 \times \mathbb{C}^*.$$

- Take a system of Darboux co-ordinates $x \colon Z_1 \to \mathbb{C}^n$.
- The components of the composite

$$X \times \mathbb{C}^* \to p^{-1}(\mathbb{C}^*) \to Z_1 \to \mathbb{C}^n$$

define functions $x_k = x_k(z_i, \theta_j, \epsilon)$.
- These functions x_k are killed by the vector field

$$\frac{\partial}{\partial \epsilon} - \frac{Z}{\epsilon^2} - \frac{\mathrm{Ham}_F}{\epsilon}$$

where $Z = \sum_i z_i \cdot \frac{\partial}{\partial \theta_i}$ and $F = \sum_i z_i \cdot \frac{\partial W}{\partial \theta_i}$.

Canonical solutions in half-planes?

Analogue of Balser-Jurkat-Lutz property

Can we choose the co-ordinates x_k so that as $\epsilon \to 0$ in a half-plane

$$x_k(z_i, \theta_j, \epsilon) \sim \epsilon^{-1} z_k + \theta_k \ ?$$

- Comparing BJL co-ordinates $x \colon Z_1 \to \mathbb{C}^n$ for different half-planes should give symplectic automorphisms of $\mathbb{C}^n$, or rather $(\mathbb{C}^*)^n$, whose generating functions give the "DT invariants" of the Joyce structure.
- The BJL property holds for the Joyce structures we found using the DT RH problem (by construction). The associated "DT invariants" are the ones we started with.