From Donaldson-Thomas invariants
to hyperkähler structures

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Lecture 2
Joyce structures

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Recap: irregular singularity

Consider an equation for $Y : \mathbb{C}^* \to G = \text{GL}_n(\mathbb{C})$ of the form

$$\frac{d}{d\epsilon} Y(\epsilon) = \left( \frac{U}{\epsilon^2} + \frac{V}{\epsilon} \right) Y(\epsilon)$$

with constant matrices $U, V \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, such that

(I) $U = \text{diag}(u_1, \cdots, u_n) \in \mathfrak{h}^{\text{reg}}$ is diagonal with $u_i \neq u_j$,

(II) $V \in \mathfrak{g}^{\text{od}}$ has zeroes on the diagonal.

The Laplace transform of the equation has regular singularities at the points $u_i$.

Isomonodromic deformations are governed by the p.d.e.

$$dV_\gamma(U) = \sum_{\alpha + \beta = \gamma} [V_\alpha, V_\beta] \cdot d \log U(\beta), \quad V = \sum_{\gamma \in \Phi} V_\gamma.$$
Recap: DT invariants as Stokes data

The WCF is the isomonodromy condition for a family of meromorphic connections on the trivial $G = \text{Aut}_{\{\_,-\}}(\mathbb{T})$ bundle over $\mathbb{P}^1$:

$$\nabla_\sigma = d - \left( \frac{Z}{\epsilon^2} + \frac{\text{Ham}_F}{\epsilon} \right) d\epsilon,$$

parameterised by the points of $M = \text{Stab}(\mathcal{D})$, where

- $Z \in \mathfrak{h}$ is the central charge $Z : \Gamma \to \mathbb{C}$,
- $F = \sum_{\gamma \in \Gamma \times} F_\gamma \cdot x_\gamma \in g^{od}$ is a function on $\mathbb{T}$.

By analogy expect the Borel transform of solutions to have poles at the points $Z(\gamma)$.

$$W(Z, \theta) = \sum_{\gamma \in \Gamma \times} F_\gamma(Z) \cdot \frac{e^{\theta(\gamma)}}{Z(\gamma)} : \mathcal{T} \text{ Stab}(\mathcal{D}) \to \mathbb{C}.$$
The wall-crossing formula shows that DT invariants can be naturally interpreted as defining Stokes data.

The relevant group is \( G = \text{Aut}_{\{-,-\}}(\mathbb{T}) \) with \( \mathbb{T} = (\mathbb{C}^*)^n \).

Should assume growth condition on the DT invariants.

The space \( M = \text{Stab}(\mathcal{D}) \) parameterizes an isomonodromic family of meromorphic connections on the trivial \( G \)-bundle over \( \mathbb{P}^1 \).

There is a close analogy with the theory of semisimple Frobenius manifolds, which is obtained by taking \( G = \text{GL}_n(\mathbb{C}) \).

We call the analogous gadget on the DT side a Joyce structure.

It can be reformulated in terms of a complex hyperkähler structure on the total space of the tangent bundle \( \mathcal{T}_M \).

Double dimension: if \( \mathcal{D} = \mathcal{D}^b \text{Fuk}(X) \) with \( X \) a compact CY_3

\[
dim_{\mathbb{C}} \text{Tot}(\mathcal{T}_M) = 4(h^{2,1} + 1) = 4 \dim_{\mathbb{C}} \hat{\mathcal{M}}_{\mathbb{C}}(X).
\]
**Complex hyperkähler structure**

Let $X$ be a complex manifold with holomorphic tangent bundle $\mathcal{T}_X$.

**Definition**

A complex hyperkähler structure on $X$ consists of a non-degenerate, symmetric bilinear form $g : \mathcal{T}_X \otimes \mathcal{T}_X \to \mathcal{O}_X$, and endomorphisms

$$I, J, K \in \text{End} (\mathcal{T}_X), \quad I^2 = J^2 = K^2 = IJK = -1,$$

which preserve $g$, and are parallel for the Levi-Civita connection.

Define holomorphic symplectic forms

$$\Omega_\pm (v, w) := g(v, (J \pm iK)w).$$

Note that $\ker(I \pm i) = \ker(\Omega_\pm) \subset \mathcal{T}_X$ is half-dimensional.
Affine symplectic fibration

Let $X$ be a complex hyperkähler manifold.

**Definition**

A holomorphic map $\pi: X \to M$ is an affine symplectic fibration if $\Omega_- = \pi^*(\eta)$ for some holomorphic symplectic form $\eta$ on $M$.

- $\ker(I - i) = \ker(\Omega_-)$ is the vertical sub-bundle $\ker \pi_* \subset T_X$.
- $\dim_{\mathbb{C}}(X) = 2 \dim_{\mathbb{C}}(M)$.
- $\Omega_+$ restricts to a holomorphic symplectic form on each fibre of $\pi$.
- the inverse of $\pi_*$ gives an isomorphism $\ell_h: \pi^*(\mathcal{T}_M) \to \ker(I + i)$.
- $\ell_v := J \circ \ell_h$ gives an isomorphism $\ell_v: \pi^*(\mathcal{T}_M) \to \ker(\pi_*)$.
- the lifting maps $\ell_h + \epsilon^{-1}\ell_v$ define a pencil of flat, symplectic Ehresmann connections on $\pi: X \to M$. 
Total space of the tangent bundle

From now on we assume that

- \( X = \text{Tot}(\mathcal{T}_M) \) with the projection map \( \pi: X \to M \).
- \( \ell_v: \pi^*(\mathcal{T}_M) \to \ker(\pi_*) \) is the obvious canonical map.
Definition of a Joyce structure

We impose extra conditions on our affine symplectic fibration

$$\pi : X := \text{Tot}(\mathcal{T}_M) \longrightarrow M.$$ 

- Homogeneity. There is a vector field $E$ on $X$ such that
  $$\mathcal{L}_E(g) = g, \quad \mathcal{L}_E(I) = 0, \quad \mathcal{L}_E(J \pm iK) = \mp (J \pm iK).$$

- Involution. The action of $-1$ on the fibres of $\pi$ satisfies
  $$\iota^*(g) = -g, \quad \iota^*(I) = I, \quad \iota^*(J \pm iK) = -(J \pm iK).$$

- Periodicity. The structure is invariant under translation by a bundle of lattices $\mathbb{Z}^n \subset \mathbb{C}^n$ in the fibres of $\pi$.
  - the structure descends to the quotient $(\mathbb{C}^*)^n$ bundle over $M$,
  - the bundle of lattices gives an affine structure on $M$. 
Local description: co-ordinates

Take a Darboux co-ordinate system \((z_i)^n_{i=1}\) for the holomorphic symplectic form \(\eta\) on \(M\)

\[
\eta = \sum_{i,j} \eta_{ij} \cdot dz_i \wedge dz_j.
\]

This gives co-ordinates \((z_i, \theta_j)^n_{i,j=1}\) on the total space \(X = \text{Tot}(\mathcal{T}_M)\) by writing a tangent vector in the form

\[
v = \sum_{i=1}^{n} \theta_i \cdot \frac{\partial}{\partial z_i} \in \mathcal{T}_{M,p}.
\]
Define vector fields on $X$

$$v_i = \frac{\partial}{\partial \theta_i} = \ell_v\left(\frac{\partial}{\partial z_i}\right), \quad h_i = \ell_h\left(\frac{\partial}{\partial z_i}\right).$$

Then the complex hyperkähler structure is

$$g(v_i, v_j) = 0, \quad g(v_i, h_j) = \omega_{ij}, \quad g(h_i, h_j) = 0.$$

$$I(v_i) = i \cdot v_i, \quad J(v_i) = h_i, \quad K(v_i) = -ih_i,$$

$$I(h_i) = -i \cdot h_i, \quad J(h_i) = -v_i, \quad K(h_i) = -iv_i.$$

The interesting data is the horizontal vector fields $h_i$.

The Lax equation $[h_i + \epsilon^{-1}v_i, h_j + \epsilon^{-1}v_j] = 0$ holds for all $\epsilon \in \mathbb{C}^*$. 
SECOND PLEBANSKI FUNCTION

There is a holomorphic function $W : X \to \mathbb{C}$ such that

$$h_i = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial}{\partial \theta_q}.$$

Then $W$ must satisfy the second heavenly equation

$$\frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} - \sum_{p,q} \eta^{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q} = 0.$$

In the case of a Joyce structure we can choose $(z_i)_{i=1}^n$ so that:

- $W$ is homogeneous of degree $-1$ in the $z_i$,
- $W$ is an odd function of the $\theta_i$,
- The derivatives $\frac{\partial^2 W}{\partial \theta_i \partial \theta_j}$ are periodic in $\theta_i$. 
Linear Joyce connection

The bundle of lattices in the definition of a Joyce structure induces a flat, torsion-free connection $\nabla$ on the tangent bundle $T_M$. We always assume the co-ordinates $z_i$ are flat for $\nabla$.

The Levi-Civita connection restricted to the zero-section $M \subset X$ induces another flat, torsion-free metric connection on $T_M$:

$$\nabla^J \frac{\partial}{\partial z_i} \left( \frac{\partial}{\partial z_j} \right) = \sum_{p,q} \eta^{pq} \frac{\partial^3 W}{\partial \theta_i \partial \theta_j \partial \theta_p} \bigg|_{\theta=0} \frac{\partial}{\partial z_q}.$$

Warning: in general we must allow $W : X \to \mathbb{C}$ to have poles, in which case it is not clear that $\nabla^J$ is well-defined!

In examples it seems that the so-called “physicist’s slices” of $M = \text{Stab}(\mathcal{D})$ are $\nabla^J$-flat Lagrangian submanifolds of $M$. 
**Example: Quadratic Differentials**

Let $M = Q(g)$ be the space of pairs $(S, q)$ with

- $S$ is a Riemann surface of genus $g \geq 2$,
- $q \in H^0(S, \omega_S^\otimes 2)$ has simple zeroes.

We are also interested in the meromorphic analogues $Q(g, m)$ where $q$ is required to have poles of given fixed orders $m = (m_1, \cdots, m_d)$.

Each pair $(S, q)$ determines a smooth double cover

$$\hat{S} = \{ y^2 - q(x) \} \subset \text{Tot}(T_S^*)$$

with a projection $p: \hat{S} \to S$ and a covering involution $\iota$.

There is a bundle of lattices $H_1(\hat{S}, \mathbb{Z})^- \cong \mathbb{Z}^\oplus n$ over $Q(g)$.
Period map and tangent bundle

There is a local bi-holomorphism

\[ \varpi : Q(g) \to H^1(\hat{S}, \mathbb{C})^- \] \hspace{1cm} \[ (S, q) \mapsto [\sqrt{q}] \].

Let \( X \) be the space of quadruples \((S, q, L, \nabla^{ab})\) where

- \( L \) is a line bundle on \( \hat{S} \) satisfying \( L \otimes \iota^*(L) \cong p^*(\omega_S) \),
- \( \nabla^{ab} \) is an anti-invariant holomorphic connection on \( L \otimes \iota^*(L)^\vee \).

There is an obvious projection \( \pi : X \to M \).

- The fibres of \( \pi : \text{Tot}(T_M) \to M \) are the groups \( H^1(\hat{S}, \mathbb{C})^- \).
- The fibres of \( \pi : X \to M \) are the groups \( H^1(\hat{S}, \mathbb{C}^*)^- \).

So \( X \) is the quotient of \( \text{Tot}(T_M) \) by the lattices \( H_1(\hat{S}, \mathbb{Z})^- \). In fact not quite: need to quotient \( X \) by finite group \( \{ \pm 1 \}^{2g} \).

Try to construct a pencil of non-linear connections on \( \pi : X \to M \).
Doubled moduli space

Let $\mathcal{P}(g)$ be the space of quadruples $(S, E, \nabla, \Phi)$ where

- $S$ is a Riemann surface of genus $g$,
- $E$ is a rank 2 bundle on $S$ with trivial determinant,
- $\nabla$ is a flat, holomorphic connection on $E$,
- $\Phi: E \to E \otimes \omega_S$ is a trace-free Higgs field on $E$.

There is a map $\pi: \mathcal{P}(g) \to \mathcal{Q}(g)$ defined by $(S, q = \det(\Phi))$.

**Main claim.** There is a birational equivalence

$$b: X \dashrightarrow \mathcal{P}(g)$$

sending $(S, q, L, \nabla^{ab})$ to the bundle $E = p_*(L)$ on $S$ with its canonical Higgs field $\Phi$. The new point is that for generic $L$, the connection $\nabla^{ab}$ induces a flat connection $\nabla$ on $E$. 
**Isomonodromy connections**

Take $G = \text{PGL}_2(\mathbb{C})$ and $\mathcal{X}(g) = \text{Hom}(\pi_1(S_g), G)/G$.

For each $\epsilon \in \mathbb{C}^*$, pulling back the trivial connection on the right gives a non-linear symplectic connection on $\pi : X \to M$.

**Conjecture**

*This construction defines a Joyce structure on $M = \text{Quad}(g)$.*
Example: the $A_2$ case

This is the case $\mathcal{D} = \mathcal{D}(g, m)$ with $g = 0$ and $m = \{7\}$. The space

$$M = \text{Stab}(\mathcal{D})/\text{Aut}(\mathcal{D}) = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 \neq 0\}/\mu_5$$

parameterizes quadratic differentials on $\mathbb{P}^1$ of the form

$$q(x) = (x^3 + ax + b)dx^2$$

with simple zeroes and a single pole of order 7. The central charge co-ordinates are of the form

$$z_i = \int_{\gamma_i} \sqrt{x^3 + ax + b} \, dx.$$

The linear Joyce connection has $(a, b)$ as flat co-ordinates.
JOYCE STRUCTURE IN THE $A_2$ CASE

An affine piece of the fibres of $\pi : X \to M$ are parameterised by

$$\{(q, p) \in \mathbb{C}^2 : p^2 = q^3 + aq + b\}, \quad r \in \mathbb{C}.$$

The corresponding connection and Higgs field are

$$\nabla = d + \frac{1}{2p} \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} \, dx, \quad \Phi = \begin{pmatrix} p & x^2 + xq + q^2 + a \\ x - q & -p \end{pmatrix} \, dx.$$

The isomonodromy connection for $r = 0$ is described by Painlevé I. The Plebański function is

$$W = \frac{2ap^2 + 3p(3b - 2aq)r + (6aq^2 - 9bq + 4a^2)r^2 - 2apr^3}{4(4a^3 + 27b^2)p}.$$