From Donaldson-Thomas invariants to hyperkähler structures

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Lecture 2

Joyce structures

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Recap: irregular singularity

Consider an equation for $Y\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$ of the form

$$\frac{d}{d\epsilon} Y(\epsilon) = \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon}\right) Y(\epsilon)$$

with constant matrices $U, V \in \mathfrak{g} = \mathfrak{g}l_n(\mathbb{C})$, such that

(I) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}^{\mathrm{reg}}$ is diagonal with $u_i \neq u_j$,

(II) $V \in \mathfrak{g}^{\mathrm{od}}$ has zeroes on the diagonal.

The Laplace transform of the equation has regular singularities at the points u_i.

Isomonodromic deformations are governed by the p.d.e.

$$dV_\gamma(U) = \sum_{\alpha+\beta=\gamma} [V_\alpha, V_\beta] \cdot d\log U(\beta), \qquad V = \sum_{\gamma \in \Phi} V_\gamma.$$

Recap: DT invariants as Stokes data

The WCF is the isomonodromy condition for a family of meromorphic connections on the trivial $G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$ bundle over $\mathbb{P}^1$:

$$\nabla_\sigma = d - \left(\frac{Z}{\epsilon^2} + \frac{\mathrm{Ham}_F}{\epsilon}\right) d\epsilon,$$

parameterised by the points of $M = \mathrm{Stab}(\mathcal{D})$, where

- $Z \in \mathfrak{h}$ is the central charge $Z \colon \Gamma \to \mathbb{C}$,
- $F = \sum_{\gamma \in \Gamma^\times} F_\gamma \cdot x_\gamma \in \mathfrak{g}^{\mathrm{od}}$ is a function on $\mathbb{T}$.

By analogy expect the Borel transform of solutions to have poles at the points $Z(\gamma)$.

$$W(Z, \theta) = \sum_{\gamma \in \Gamma^\times} F_\gamma(Z) \cdot \frac{e^{\theta(\gamma)}}{Z(\gamma)} \colon \mathcal{T}\,\mathrm{Stab}(\mathcal{D}) \to \mathbb{C}.$$

Isomonodromy equation $\implies$ Plebański's second heavenly equation.

Summary

- The wall-crossing formula shows that DT invariants can be naturally interpreted as defining Stokes data.
- The relevant group is $G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$ with $\mathbb{T} = (\mathbb{C}^*)^n$.
- Should assume growth condition on the DT invariants.
- The space $M = \mathrm{Stab}(\mathcal{D})$ parameterizes an isomonodromic family of meromorphic connections on the trivial G-bundle over $\mathbb{P}^1$.
- There is a close analogy with the theory of semisimple Frobenius manifolds, which is obtained by taking $G = \mathrm{GL}_n(\mathbb{C})$.
- We call the analogous gadget on the DT side a Joyce structure.
- It can be reformulated in terms of a complex hyperkähler structure on the total space of the tangent bundle $\mathcal{T}_M$.
- Double dimension: if $\mathcal{D} = \mathcal{D}^b\,\mathrm{Fuk}(X)$ with X a compact CY_3

$$\dim_{\mathbb{C}} \mathrm{Tot}(\mathcal{T}_M) = 4(h^{2,1}+1) = 4\dim_{\mathbb{C}} \hat{\mathcal{M}}_{\mathbb{C}}(X).$$

Complex hyperkahler structure

Let X be a complex manifold with holomorphic tangent bundle $\mathcal{T}_X$.

Definition

A complex hyperkähler structure on X consists of a non-degenerate, symmetric bilinear form $g\colon \mathcal{T}_X \otimes \mathcal{T}_X \to \mathcal{O}_X$, and endomorphisms

$$I, J, K \in \mathrm{End}(\mathcal{T}_X), \qquad I^2 = J^2 = K^2 = IJK = -1,$$

which preserve g, and are parallel for the Levi-Civita connection.

Define holomorphic symplectic forms

$$\Omega_{\pm}(v, w) := g(v, (J \pm iK)w).$$

Note that $\ker(I \pm i) = \ker(\Omega_{\pm}) \subset \mathcal{T}_X$ is half-dimensional.

Affine symplectic fibration

Let X be a complex hyperkähler manifold.

Definition

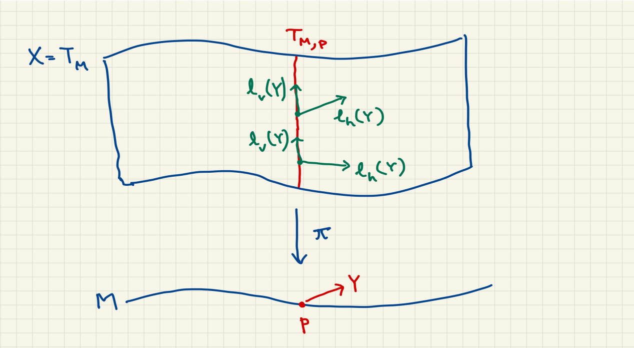
A holomorphic map $\pi\colon X \to M$ is an affine symplectic fibration if $\Omega_- = \pi^*(\eta)$ for some holomorphic symplectic form η on M.

- $\ker(I - i) = \ker(\Omega_-)$ is the vertical sub-bundle $\ker \pi_* \subset \mathcal{T}_X$.
- $\dim_{\mathbb{C}}(X) = 2\dim_{\mathbb{C}}(M)$.
- Ω_+ restricts to a holomorphic symplectic form on each fibre of π.
- the inverse of π_* gives an isomorphism $\ell_h\colon \pi^*(\mathcal{T}_M) \to \ker(I + i)$.
- $\ell_v := J \circ \ell_h$ gives an isomorphism $\ell_v\colon \pi^*(\mathcal{T}_M) \to \ker(\pi_*)$.
- the lifting maps $\ell_h + \epsilon^{-1}\ell_v$ define a pencil of flat, symplectic Ehresmann connections on $\pi\colon X \to M$.

Total space of the tangent bundle

From now on we assume that

- $X = \mathrm{Tot}(\mathcal{T}_M)$ with the projection map $\pi\colon X \to M$.
- $\ell_v\colon \pi^*(\mathcal{T}_M) \to \ker(\pi_*)$ is the obvious canonical map.



Definition of a Joyce structure

We impose extra conditions on our affine symplectic fibration

$$\pi\colon X := \mathrm{Tot}(\mathcal{T}_M) \longrightarrow M.$$

- Homogeneity. There is a vector field E on X such that
$$\mathcal{L}_E(g) = g, \quad \mathcal{L}_E(I) = 0, \quad \mathcal{L}_E(J \pm iK) = \mp(J \pm iK).$$
- Involution. The action of -1 on the fibres of π satisfies
$$\iota^*(g) = -g, \quad \iota^*(I) = I, \quad \iota^*(J \pm iK) = -(J \pm iK).$$
- Periodicity. The structure is invariant under translation by a bundle of lattices $\mathbb{Z}^{\oplus n} \subset \mathbb{C}^{\oplus n}$ in the fibres of π.
  - the structure descends to the quotient $(\mathbb{C}^*)^n$ bundle over M,
  - the bundle of lattices gives an affine structure on M.

Local description: co-ordinates

Take a Darboux co-ordinate system $(z_i)_{i=1}^n$ for the holomorphic symplectic form η on M

$$\eta = \sum_{i,j} \eta_{ij} \cdot dz_i \wedge dz_j.$$

This gives co-ordinates $(z_i, \theta_j)_{i,j=1}^n$ on the total space $X = \mathrm{Tot}(\mathcal{T}_M)$ by writing a tangent vector in the form

$$v = \sum_{i=1}^n \theta_i \cdot \frac{\partial}{\partial z_i} \in \mathcal{T}_{M,p}.$$

Local description

Define vector fields on X

$$v_i = \frac{\partial}{\partial \theta_i} = \ell_v\Big(\frac{\partial}{\partial z_i}\Big), \qquad h_i = \ell_h\Big(\frac{\partial}{\partial z_i}\Big).$$

Then the complex hyperkähler structure is

$$g(v_i, v_j) = 0, \qquad g(v_i, h_j) = \omega_{ij}, \qquad g(h_i, h_j) = 0.$$

$$I(v_i) = i \cdot v_i, \qquad J(v_i) = h_i, \qquad K(v_i) = -ih_i,$$
$$I(h_i) = -i \cdot h_i, \qquad J(h_i) = -v_i, \qquad K(h_i) = -iv_i.$$

The interesting data is the horizontal vector fields h_i.

The Lax equation $\big[h_i + \epsilon^{-1} v_i, h_j + \epsilon^{-1} v_j\big] = 0$ holds for all $\epsilon \in \mathbb{C}^*$.

Second Plebanski function

There is a holomorphic function $W\colon X \to \mathbb{C}$ such that

$$h_i = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial}{\partial \theta_q}.$$

Then W must satsify the second heavenly equation

$$\frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} - \sum_{p,q} \eta^{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q} = 0.$$

In the case of a Joyce structure we can choose $(z_i)_{i=1}^n$ so that:

- W is homogeneous of degree -1 in the z_i,
- W is an odd function of the θ_i,
- The derivatives $\frac{\partial^2 W}{\partial \theta_i \partial \theta_j}$ are periodic in θ_i.

Linear Joyce connection

The bundle of lattices in the definition of a Joyce structure induces a flat, torsion-free connection ∇ on the tangent bundle $\mathcal{T}_M$.

We always assume the co-ordinates z_i are flat for ∇.

The Levi-Civita connection restricted to the zero-section $M \subset X$ induces another flat, torsion-free metric connection on $\mathcal{T}_M$:

$$\nabla^J{}_{\frac{\partial}{\partial z_i}}\Big(\frac{\partial}{\partial z_j}\Big) = \sum_{p,q} \eta^{pq} \, \frac{\partial^3 W}{\partial\theta_i \partial\theta_j \partial\theta_p}\Big|_{\theta=0} \frac{\partial}{\partial z_q}.$$

Warning: in general we must allow $W \colon X \to \mathbb{C}$ to have poles, in which case it is not clear that ∇^J is well-defined!

In examples it seems that the so-called "physicist's slices" of $M = \mathrm{Stab}(\mathcal{D})$ are ∇^J-flat Lagrangian submanifolds of M.

Example: quadratic differentials

Let $M = \mathcal{Q}(g)$ be the space of pairs (S, q) with

- S is a Riemann surface of genus $g \geq 2$,
- $q \in H^0(S, \omega_S^{\otimes 2})$ has simple zeroes.

We are also interested in the meromorphic analogues $\mathcal{Q}(g, m)$ where q is required to have poles of given fixed orders $m = (m_1, \cdots, m_d)$.

Each pair (S, q) determines a smooth double cover

$$\hat{S} = \{y^2 - q(x)\} \subset \mathrm{Tot}(\mathcal{T}_S^*)$$

with a projection $p \colon \hat{S} \to S$ and a covering involution ι.

There is a bundle of lattices $H_1(\hat{S}, \mathbb{Z})^- \cong \mathbb{Z}^{\oplus n}$ over $\mathcal{Q}(g)$.

Period map and tangent bundle

There is a local bi-holomorphism

$$\varpi\colon \widetilde{\mathcal{Q}(g)} \longrightarrow H^1(\hat{S},\mathbb{C})^-, \qquad (S,q) \mapsto [\sqrt{q}\,].$$

Let X be the space of quadruples (S,q,L,∇^{ab}) where

- L is a line bundle on $\hat{S}$ satisfiying $L\otimes\iota^*(L)\cong p^*(\omega_S)$,
- ∇^{ab} is an anti-invariant holomorphic connection on $L\otimes\iota^*(L)^\vee$.

There is an obvious projection $\pi\colon X\to M$.

- The fibres of $\pi\colon\ \mathrm{Tot}(\mathcal{T}_M)\to M$ are the groups $H^1(\hat{S},\mathbb{C})^-$.
- The fibres of $\pi\colon X\to M$ are the groups $H^1(\hat{S},\mathbb{C}^*)^-$.

So X is the quotient of $\mathrm{Tot}(\mathcal{T}_M)$ by the lattices $H_1(\hat{S},\mathbb{Z})^-$. In fact not quite: need to quotient X by finite group $\{\pm 1\}^{2g}$.

Try to construct a pencil of non-linear connections on $\pi\colon X\to M$.

Doubled moduli space

Let $\mathcal{P}(g)$ be the space of quadruples (S, E, ∇, Φ) where

- S is a Riemann surface of genus g,
- E is a rank 2 bundle on S with trivial determinant,
- ∇ is a flat, holomorphic connection on E,
- $\Phi\colon E \to E \otimes \omega_S$ is a trace-free Higgs field on E.

There is a map $\pi\colon \mathcal{P}(g) \to \mathcal{Q}(g)$ defined by $(S, q = \det(\Phi))$.

Main claim. There is a birational equivalence

$$b\colon X \dashrightarrow \mathcal{P}(g)$$

sending (S, q, L, ∇^{ab}) to the bundle $E = p_*(L)$ on S with its canonical Higgs field Φ. The new point is that for generic L, the connection ∇^{ab} induces a flat connection ∇ on E.

Isomonodromy connections

Take $G = \mathrm{PGL}_2(\mathbb{C})$ and $\mathcal{X}(g) = \mathrm{Hom}(\pi_1(S_g), G)/G$.

$$
\begin{array}{ccccc}
(S,q,L,\nabla^{ab}) & & (S,E,\nabla,\Phi) & & \mathrm{Mon}(\nabla + \epsilon^{-1}\Phi) \\
\pitchfork & & \pitchfork & & \pitchfork \\
X & \overset{b}{\longleftarrow - - \rightarrow} & \mathcal{P}(g) & \xrightarrow{F(\epsilon)} & \mathcal{X}(g) \times M \\
{\scriptstyle (\mathbb{C}^*)^n}\downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{=} & M & \xrightarrow{=} & M
\end{array}
$$

For each $\epsilon \in \mathbb{C}^*$, pulling back the trivial connection on the right gives a non-linear symplectic connection on $\pi \colon X \to M$.

Conjecture

This construction defines a Joyce structure on $M = \mathrm{Quad}(g)$.

Example: the A_2 case

This is the case $\mathcal{D} = \mathcal{D}(g, m)$ with $g = 0$ and $m = \{7\}$. The space

$$M = \mathrm{Stab}(\mathcal{D}) / \mathrm{Aut}(\mathcal{D}) = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 \neq 0\} / \mu_5$$

parameterizes quadratic differentials on $\mathbb{P}^1$ of the form

$$q(x) = (x^3 + ax + b)dx^2$$

with simple zeroes and a single pole of order 7.

The central charge co-ordinates are of the form

$$z_i = \int_{\gamma_i} \sqrt{x^3 + ax + b}\, dx.$$

The linear Joyce connection has (a, b) as flat co-ordinates.

JOYCE STRUCTURE IN THE A_2 CASE

An affine piece of the fibres of $\pi\colon X \to M$ are parameterised by

$$\left\{(q,p) \in \mathbb{C}^2 : p^2 = q^3 + aq + b\right\}, \qquad r \in \mathbb{C}.$$

The corresponding connection and Higgs field are

$$\nabla = d + \frac{1}{2p}\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} dx, \quad \Phi = \begin{pmatrix} p & x^2 + xq + q^2 + a \\ x - q & -p \end{pmatrix} dx.$$

The isomonodromy connection for $r = 0$ is described by Painlevé I.

The Plebański function is

$$W = \frac{2ap^2 + 3p(3b - 2aq)r + (6aq^2 - 9bq + 4a^2)r^2 - 2apr^3}{4(4a^3 + 27b^2)p}.$$