From Donaldson-Thomas invariants to hyperkähler structures

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Introduction

- The mathematical framework for studying wall-crossing of DT invariants is the stability space of a CY_3 triangulated category.
- The stability space is a complex manifold. We want to encode the DT invariants in a geometric structure on it.
- We expect to find a complex hyperkähler structure on the total space of the tangent bundle (joint work with Ian Strachan).
- The key idea is to view DT invariants as defining non-linear Stokes data for an isomonodromic family of connections over stability space taking values in a group of symplectomorphisms.
- This connects with the much broader subject of resurgence in quantum field theory.

Introduction

- Constructing the hyperkähler structure from the DT invariants involves solving a class of Riemann-Hilbert problems.
- This is very closely related to work of Gaiotto, Moore and Neitzke from a decade ago: it is the "conformal limit".
- A large class of examples is obtained from Fukaya categories of threefolds of the form $u^2 + v^2 + w^2 = q(x)$. These relate to
  - Theories of class S with gauge group $SU(2)$.
  - Trajectory structures of quadratic differentials.
  - Exact WKB analysis.
  - The Hitchin system.
- Some of this still a bit speculative. Need to compute more general examples. Much work still to do!

Plan of the talks

1. Stability space, DT invariants, wall-crossing formula, Stokes matrices, iso-Stokes deformations, DT invariants as Stokes data.

   Reineke, Joyce, Kontsevich and Soibelman, joint work with Toledano Laredo.

2. Complex hyperkähler manifolds with affine symplectic fibrations, local description, twistor spaces, main class of examples.

   old work of Plebański (1970s), joint work with Strachan.

3. Riemann-Hilbert problems, solutions in simple examples, generating functions for solutions, a non-perturbative topological string partition function for the resolved conifold.

   close to work of Gaiotto, Moore and Neitzke.

Lecture 1

DT invariants as Stokes data

Please skip if you've heard it before!

Recap on stability space

Let $\mathcal{D}$ be a Δ-category and assume that $\Gamma := K_0(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$.

The space of stability conditions $\mathrm{Stab}(\mathcal{D})$ parameterises pairs

- a group homomorphism $Z \colon \Gamma \to \mathbb{C}$,
- an $\mathbb{R}$-graded full subcategory $\mathcal{P} = \cup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset \mathcal{D}$,

satisfying some axioms.

The map Z is called the central charge, and the objects of $\mathcal{P}(\phi)$ are the semistables of phase ϕ.

Theorem

The forgetful map $\sigma = (Z, \mathcal{P}) \mapsto Z$ defines a local isomorphism

$$\pi \colon \mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \cong \mathbb{C}^n.$$

Choosing a basis $(\gamma_1, \cdots, \gamma_n)$ for Γ gives local co-ordinates $z_i = Z(\gamma_i) \in \mathbb{C}$ on the complex manifold $\mathrm{Stab}(\mathcal{D})$.

Examples of stability conditions

- Let $X = X(I, \omega)$ be a compact CY threefold. We expect a stability condition $\sigma(I) = (Z, \mathcal{P})$ on $\mathcal{D}_\omega = \mathcal{D}^b \operatorname{Fuk}(X, \omega)$ with
  - $Z(L) = \int_L \Omega^{3,0} \in \mathbb{C}$,
  - $\mathcal{P}(\phi) = \{\text{special Lagrangians of phase } \phi\} \subset \mathcal{D}_\omega$.
- The space $\operatorname{Stab}(\mathcal{D})$ carries an action of the group of autoequivalences $\operatorname{Aut}(\mathcal{D})$. We expect a Lagrangian embedding

$$\hat{\mathcal{M}}_{\mathbb{C}}(X, \omega) \hookrightarrow \operatorname{Stab}(\mathcal{D}_\omega)/\operatorname{Aut}(\mathcal{D}_\omega),$$

  where the space $\hat{\mathcal{M}}_{\mathbb{C}}(X, \omega)$ parameterizes complex structures together with a choice of nonzero holomorphic 3-form $\Omega^{3,0}$.

  Hard to see these half-dimensional "physicist's slices" in the abstract set-up.

- Chunyi Li recently constructed open subsets in $\operatorname{Stab}(\mathcal{D})$ for $\mathcal{D} = \mathcal{D}^b \operatorname{Coh}(X)$ with X a quintic CY threefold.

Example: quadratic differentials

Theorem (Ivan Smith, tb)

Fix $g \geq 0$, and $m = (m_1, \cdots, m_d)$, with $d \geq 1$ and $m_i \geq 3$. Then there is a CY_3 triangulated category $\mathcal{D} = \mathcal{D}(g, m)$ such that

$$\operatorname{Stab} \mathcal{D}(g, m) / \operatorname{Aut} \mathcal{D}(g, m) \cong \operatorname{Quad}(g, m),$$

where $\operatorname{Quad}(g, m)$ parameterizes pairs (S, q), with S a compact Riemann surface of genus g, and $q = q(x) dx^{\otimes 2}$ a quadratic differential on S having d poles of order m_i, and simple zeroes.

Related to theories of class S with gauge group SU(2). Physicist's slice (the Coulomb branch) is given by fixing the surface S.

The category $\mathcal{D}(g, m)$ embeds in the derived Fukaya category of a non-compact CY threefold $\pi \colon X \to S$ defined locally by

$$X = \{u^2 + v^2 + w^2 = q(x)\} \subset \mathbb{C}^3 \times S.$$

DT invariants

Assume now that $\mathcal{D}$ has the CY_3 property

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(B, A[3])^{\vee}.$$

Given $\sigma \in \mathrm{Stab}(\mathcal{D})$ one can define DT invariants

$$\mathrm{DT}_{\sigma}(\gamma) = \text{``}\, e(\mathcal{M}^{\sigma-ss}(\gamma)) \text{''} \in \mathbb{Q}, \qquad \gamma \in \Gamma.$$

There are equivalent BPS invariants $\Omega_{\sigma}(\gamma)$ defined by

$$\mathrm{DT}_{\sigma}(\alpha) = \sum_{\alpha = k \cdot \beta} \frac{1}{k^2} \cdot \Omega_{\sigma}(\beta).$$

The variation of $\mathrm{DT}_{\sigma}(\gamma)$ with $\sigma \in \mathrm{Stab}(\mathcal{D})$ is controlled by the wall-crossing formula: knowledge of all the invariants $\mathrm{DT}_{\sigma}(\gamma)$ at one stability condition determines them at all nearby stability conditions.

Poisson torus

Introduce the algebraic torus

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n, \qquad \mathbb{C}[\mathbb{T}] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma.$$

The CY_3 condition ensures that the Euler form

$$\langle [E], [F] \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}^i_{\mathcal{D}}(E, F[i]) \colon \Gamma \times \Gamma \to \mathbb{Z}$$

is skew-symmetric. It induces an invariant Poisson structure on $\mathbb{T}$:

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_{\alpha+\beta}.$$

Poisson vector fields on $\mathbb{T}$

Consider the Lie algebra $\mathsf{vect}_{\{-,-\}}(\mathbb{T})$ of algebraic vector fields on $\mathbb{T}$ whose flows preserve the Poisson bracket $\{-,-\}$.

There is a root decomposition

$$\mathfrak{g} = \mathsf{vect}_{\{-,-\}}(\mathbb{T}) = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}.$$

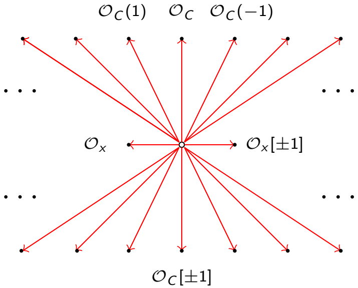
The subalgebra $\mathfrak{h}$ consists of translation-invariant vector fields on $\mathbb{T}$, and can be identified with $T_e(\mathbb{T}) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$.

The subalgebra $\mathfrak{g}^{\mathrm{od}}$ consists of Hamiltonian vector fields, and is the Poisson algebra of non-constant algebraic functions on $\mathbb{T}$

$$\mathfrak{g}^{\mathrm{od}} = \bigoplus_{\gamma \in \Gamma^\times} \mathfrak{g}_\alpha = \bigoplus_{\gamma \in \Gamma^\times} \mathbb{C} \cdot x_\gamma .$$

Ray diagram

A stability condition $(Z, \mathcal{P})$ has an associated ray diagram: the set of rays in $\mathbb{C}^*$ spanned by the central charges $Z(E)$ of stable objects.



The above example is for $\mathcal{D}^b \operatorname{Coh}(X)$ with X the resolved conifold: a CY threefold containing a single rational curve C.

DT AUTOMORPHISMS

Try to define, for each ray $\ell \subset \mathbb{C}^*$, an automorphism $\mathbb{S}(\ell)$ of $\mathbb{T}$:

$$\mathbb{S}(\ell)^*(x_\beta) = \exp\left\{ \sum_{Z(\gamma)\in\ell} \mathsf{DT}_\sigma(\gamma) \cdot x_\gamma, - \right\}(x_\beta) = x_\beta \cdot \prod_{Z(\gamma)\in\ell} (1 - x_\gamma)^{\Omega(\gamma)\cdot\langle\beta,\gamma\rangle}$$

Work required to make rigorous sense of this. Possible approaches:

- formal series: replace $\operatorname{Aut}\mathbb{C}[x_i^{\pm 1}]$ with $\operatorname{Aut}\mathbb{C}[[x_i]]$,
- work with birational automorphisms of $\mathbb{T}$,
- use automorphisms defined on analytic open subsets of $\mathbb{T}$.

We are ignoring quadratic refinements: really we should replace $\mathbb{T}$ by a torsor over it which introduces some signs.

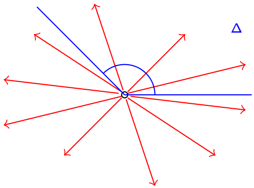
The wall-crossing formula

Controls the change in the DT invariants as $\sigma \in \mathsf{Stab}(\mathcal{D})$ varies.

For any convex sector $\Delta \subset \mathbb{C}^*$, the clockwise ordered product

$$\mathbb{S}_\sigma(\Delta) = \prod_{\ell \in \Delta}^{\curvearrowright} \mathbb{S}_\sigma(\ell) \in \mathsf{Aut}(\mathbb{T})$$

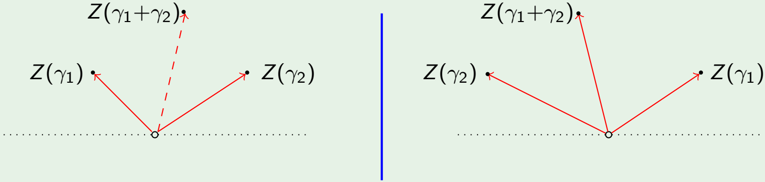
is constant, providing no "active ray" crosses $\partial\Delta$.



The formula makes sense in an appropriate completed torus algebra.

Example: the case of the A_2 quiver

Here $\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ with $\langle \gamma_1, \gamma_2 \rangle = 1$, and $\mathbb{C}[\mathbb{T}] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$.



$\Omega(\gamma_1) = \Omega(\gamma_2) = 1$ $\qquad$ $\Omega(\gamma_2) = \Omega(\gamma_1 + \gamma_2) = \Omega(\gamma_1) = 1$

The wall-crossing formula is the cluster pentagon identity

$$C_{\gamma_1} \circ C_{\gamma_2} = C_{\gamma_2} \circ C_{\gamma_1+\gamma_2} \circ C_{\gamma_1},$$

$$C_\alpha \colon x_\beta \mapsto x_\beta \cdot (1 + x_\alpha)^{\langle \alpha, \beta \rangle}.$$

Introduction to Stokes data

We first consider Stokes data in the finite-dimensional setting

$$\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}), \qquad G = \mathrm{GL}_n(\mathbb{C}).$$

We use the standard root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}, \qquad \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

Later we will replace these with the infinite-dimensional objects

$$\mathfrak{g} = \mathsf{vect}_{\{-,-\}}(\mathbb{T}), \qquad G = \mathsf{Aut}_{\{-,-\}}(\mathbb{T}),$$

with the analogous root decompositon

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}, \qquad \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\gamma \in \Gamma^\times} \mathbb{C} \cdot x_\gamma.$$

An irregular singularity

Consider an equation for $Y\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$ of the form

$$\frac{d}{d\epsilon} Y(\epsilon) = \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon} \right) Y(\epsilon)$$

with constant matrices $U, V \in \mathfrak{g} = \mathfrak{g}l_n(\mathbb{C})$, such that

(I) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}^{\mathrm{reg}}$ is diagonal with $u_i \neq u_j$,

(II) $V \in \mathfrak{g}^{\mathrm{od}}$ has zeroes on the diagonal.

This is the simplest example of an irregular singularity. When $V = 0$ the solution is $Y(\epsilon) = \exp(-U/\epsilon)$.

The Stokes rays of the equation at $\epsilon = 0$ are defined to be the rays

$$\mathbb{R}_{>0} \cdot (u_i - u_j) = \mathbb{R}_{>0} \cdot U(\alpha).$$

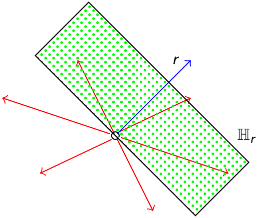
The Laplace transform of the equation has regular singularities at the points u_i.

Canonical solutions in half-planes

Theorem (Balser, Jurkat, Lutz)

For any half-plane $\mathbb{H}_r \subset \mathbb{C}^*$ centered on a non-Stokes ray $r \subset \mathbb{C}^*$, there is a unique solution $Y_r \colon \mathbb{H}_r \to G$ such that

$$Y_r(\epsilon) \cdot \exp(U/\epsilon) \to 1 \text{ as } \epsilon \to 0.$$

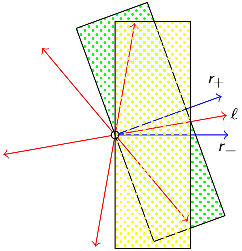


Stokes factors

The Stokes factor $\mathbb{S}(\ell)$ associated to a Stokes ray ℓ is defined by

$$Y_{r+}(\epsilon) = Y_{r_-}(\epsilon) \cdot \mathbb{S}(\ell), \qquad \mathbb{S}(\ell) \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha \Big) \subset G,$$

where $r_\pm$ are small perturbations of ℓ.



Isomonodromic deformations

Recall our matrix differential equation

$$\frac{d}{d\epsilon}Y(\epsilon) = \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon}\right)Y(\epsilon), \qquad U \in \mathfrak{h}^{\mathrm{reg}},\ V \in \mathfrak{g}^{\mathrm{od}}.$$

If we vary the leading term $U \in \mathfrak{h}^{\mathrm{reg}}$, we can uniquely deform the matrix $V = V(U)$ so that the Stokes factors $\mathbb{S}(\ell)$ remain constant.

This isomonodromy condition is equivalent to the p.d.e.

$$dV_\gamma(U) = \sum_{\alpha+\beta=\gamma} [V_\alpha, V_\beta] \cdot d\log U(\beta), \qquad V = \sum_{\gamma\in\Phi} V_\gamma.$$

This is the irregular version of the Schlesinger equations. Closely related to semi-simple Frobenius structures.

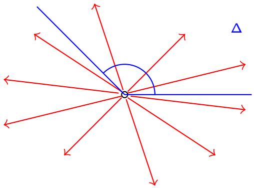
Precise statement of isomonodromy

Note that as $U \in \mathfrak{h}^{reg}$ varies, the Stokes rays $\mathbb{R}_{>0} \cdot U(\alpha)$ may cross.

For any convex sector $\Delta \subset \mathbb{C}^*$, the clockwise product over rays

$$\mathbb{S}(\Delta) = \prod_{\ell \in \Delta}^{\curvearrowright} \mathbb{S}(\ell) \in G,$$

should be constant, providing no Stokes ray crosses $\partial\Delta$.



DT invariants as Stokes data

Let $\mathcal{D}$ be a CY_3 Δ-category, and take as before

$$\mathfrak{g} = \mathsf{vect}_{\{-,-\}}(\mathbb{T}), \qquad G = \mathsf{Aut}_{\{-,-\}}(\mathbb{T}).$$

The wall-crossing formula is the isomonodromy condition for a family of meromorphic connections on the trivial G-bundle over $\mathbb{P}^1$:

$$\nabla_\sigma = d - \left(\frac{Z}{\epsilon^2} + \frac{\mathsf{Ham}_F}{\epsilon}\right) d\epsilon,$$

parameterised by the points of $\mathsf{Stab}(\mathcal{D})$, where

- $Z \in \mathfrak{h}$ is the central charge $Z \colon \Gamma \to \mathbb{C}$,
- $F = \sum_{\gamma \in \Gamma^\times} F_\gamma \cdot x_\gamma \in \mathfrak{g}^{\mathrm{od}}$ is a function on $\mathbb{T}$.

The Fourier coefficients $F_\gamma = F_\gamma(Z)$ are holomorphic functions on $\mathsf{Stab}(\mathcal{D})$ defined implicitly by the DT invariants.

Isomonodromy equation

The isomonodromy p.d.e. is

$$dF_\gamma(Z) = \sum_{\alpha+\beta=\gamma} \langle \alpha, \beta \rangle \cdot F_\alpha F_\beta \cdot d\log Z(\beta).$$

Define the Joyce function

$$W(Z,\theta) = \sum_{\gamma \in \Gamma^\times} F_\gamma(Z) \cdot \frac{e^{\theta(\gamma)}}{Z(\gamma)} \colon \mathcal{T}\,\mathrm{Stab}(\mathcal{D}) \to \mathbb{C}.$$

Note that tangent vectors to $\mathrm{Stab}(\mathcal{D})$ are deformations of the central charge, i.e. linear maps $\theta \colon \Gamma \to \mathbb{C}$.

The isomonodromy p.d.e. becomes

$$\frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta^{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q}.$$

We took a basis $(\gamma_1, \cdots, \gamma_n) \subset \Gamma$ and set $z_i = Z(\gamma_i)$, $\theta_i = \theta(\gamma_i)$ and $\eta^{ij} = \langle \gamma_i, \gamma_j \rangle$.

This is Plebanski's second heavenly equation (1975). Relation to isomondromy goes back to Mason and Newman (1989).