Geometric description of topological string partition functions from quantum curves and DT invariants

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15 January 2021

Based on [arXiv:1811.01978] with J. Teschner and E. Pomoni; [arXiv:2004.04585] with J. Teschner and P. Longhi

Donaldson-Thomas invariants and Resurgence

Simons Collaboration Meeting 11-15 January 2021

Topological string partition functions

Topological string partition functions are objects of interest

- *Physics*
  - *instanton partition functions*
  - *index counting BPS states*

- *Mathematics*
  - *enumerative invariants* (Gromov-Witten, Donaldson-Thomas, Gopakumar-Vafa)

$$\log Z_{top} = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g$$

Approaches towards computation

holomorphic anomaly, topological vertex, topological recursion, spectral theory

[Bershadsky, Cecotti, Ooguri, Vafa '94] [Aganagic, Klemm, Marino, Vafa '05] [Chekhov, Eynard, Orantin '06] [Grassi, Hatsuda, Marino '14]

Topological string partition functions

Aim: geometric characterisation of the topological string partition function Z_{top}

… through its relation to quantum curves and DT invariants

Topological strings come in two variants related by mirror symmetry

- *A model* ◦ *type IIA string theory on a Calabi-Yau threefold X*

  ↪ Kähler moduli **t** $\qquad \log Z_{top}(\mathbf{t},\lambda) = \sum_{g=0}^{\infty} \lambda^{2g-2}\mathscr{F}_g(\mathbf{t})$

- *B model* ◦ *type IIB string theory on a Calabi-Yau threefold Y*

  ↪ Complex moduli **m** $\qquad$ mirror map **t**(**m**)

Are $Z_{top}(\mathbf{t})$ locally defined functions on $\mathscr{M}_{Kähler}(X) \leftrightarrow \mathscr{M}_{Cplx}(Y)$?

Geometric setup

Type IIB string theory on a local CY3 Y_Σ

$$Y_\Sigma \;: vw - P(x,y) = 0 \quad \text{where} \quad P(x,y) = y^2 + q(x)\ , \quad x \in C$$

Riemann surface $C_{0,n}$

$$q(x) = \sum_{r=1}^{n} \frac{E_r}{x - z_r} + \frac{\delta_r}{(x - z_r)^2} + \dots$$

$q(x)dx^2$ quadratic differential on C

irregular singularities $\longleftrightarrow$ poles of order >2

SW curve $\Sigma = \{(x,y), P(x,y) = 0\} \subset T^*C$

Moduli space $\mathscr{M}_{\rm cplx}(Y)$ of pairs (C,q)

complex structure moduli **m**=$\{E_r, \delta_r, z_r\}$

Geometric engineering of 4d, $\mathcal{N}$= 2 theories of class S

[Katz, Klemm, Vafa '96] [Gaiotto '09]

Examples

- $C_{0,2}$ $\longleftrightarrow$ pure SU(2) super Yang-Mills theory
- $C_{0,4}$ $\longleftrightarrow$ SU(2) 4-flavour super Yang-Mills

Geometric setup

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Geometric engineering of 4d, $\mathcal{N}$= 2 theories of class S

[Katz, Klemm, Vafa '96] [Gaiotto '09]

Special geometry $\mathscr{B} := \mathscr{M}_{\rm cplx}(Y)$ homological coordinates $a_r = \int_{\alpha_r} y dx, \ \check{a}^r = \int_{\beta_r} y dx = \frac{\partial \mathscr{F}}{\partial a_r}$

Hitchin moduli space $\mathscr{M}_{\rm H}(Y)$ of Higgs pairs $(\mathscr{E}, \varphi)$

$\{\alpha_r, \beta_r\}$ basis for $H_1(\Sigma, \mathbb{Z})$

torus fibration $\mathscr{M}_H(Y) \longrightarrow \mathscr{B}$ where $[\mathscr{E}, \varphi] \ \mapsto \ {\rm tr}(\varphi^2) \sim q(x)$

Topological string partition functions

Guiding idea

close relation between Z_{top} and a quantisation of the SW curve

Broad questions

How are the variables of Z_{top} related to the parameters of the quantum curve ?

Exact WKB - how does this approach work to define these parameters?

Topological string partition functions

Aim: *geometric characterisation of the topological string partition function* Z_{top}

… through its relation to quantum curves and DT invariants

Topological string partition functions come in two variants

A model *type IIA string theory on a Calabi-Yau threefold X*

Kähler moduli $\mathbf{t}$

$$\log Z_{top}(\mathbf{t}) = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g(\mathbf{t})$$

B model *type IIB string theory on a Calabi-Yau threefold Y*

Complex moduli $\mathbf{m}$ mirror map $\mathbf{t}(\mathbf{m})$

$Z_{top}(\mathbf{t})$ are locally defined functions on $\mathcal{M}_{Kähler}(X) \longleftrightarrow \mathcal{M}_{Cplx}(Y)$

Physics predicts that higher-genus corrections are encoded in quantum SW curves

[Aganagic, Dijkgraaf, Klemm, Marino, Vafa '03], [Dijkgraaf, Hollands, Sulkowski, Vafa '08] …

Emerging geometric picture

Preview

- The moduli space of quantum curves $\mathscr{X}$

- Cover $\mathscr{X}$ with local patches $\mathscr{U}_\imath$ and Darboux coordinates $(\mathbf{x}_\imath, \check{\mathbf{x}}^\imath)$

- Normalised isomonodromic tau-functions $\mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}^\imath)$

[IC, Pomoni, Teschner '18]

[IC, Longhi, Teschner '20]

$$\hookrightarrow \mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}(\mathbf{n},\check{\mathbf{x}}_\imath)} Z_{\text{top}}(\mathbf{x}_\imath + \mathbf{n}) \quad ♣ \qquad \mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = F_{\imath\jmath}(\mathbf{x}_\imath, \mathbf{x}_\jmath)\mathscr{T}_\jmath(\mathbf{x}_\jmath, \check{\mathbf{x}}_\jmath)$$

Emerging geometric picture

- The moduli space of quantum curves $\mathscr{Z} = \mathscr{M}_H(Y) \times \mathbb{C}^\times$ torus fibration $\mathscr{M}_H(C) \longrightarrow \mathscr{B}$

  ~ space of $\{[\hbar, \mathscr{E}, \nabla_\hbar]\}$

$\mathscr{E}$ holomorphic vector bundle in the Higgs pair $[\mathscr{E}, \varphi]$

$\nabla_\hbar$ holomorphic $\hbar$-connection $\hbar\partial_x + A(x)$

$A(x)_{\hbar=0}$

quantisation of the curves $(\hbar^2\partial_x^2 - q_\hbar(x))\chi(x) = 0$

Emerging geometric picture

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quantisation of the curves $(\hbar^2\partial_x^2 - q_\hbar(x))\chi(x) = 0$

- Cover $\mathscr{Z}$ with local patches $\mathscr{U}_\iota$ and Darboux coordinates $(\,\mathbf{x}_\iota(\hbar)\,, \check{\mathbf{x}}^\iota(\hbar)\,)$
  - from exact WKB analysis define an $\hbar$-family of deformations of $\mathscr{M}_H(Y)$ $\Omega = \sum_{r=1}^{d} dx_\iota^r \wedge d\check{x}_r^\iota$
  - asymptotic behaviour $x_r \simeq a_r/\hbar + \mathscr{O}(\hbar^0)\,,\ \check{x}^r \simeq \check{a}^r/\hbar + \mathscr{O}(\hbar^0)$ homological coordinates $(\,a_r\,, \check{a}^r\,)$ on $\mathscr{B}$

$a_r = \int_{\alpha_r} y dx\ ,\ \check{a}_r = \int_{\beta_r} y dx \qquad y^2 + \mathrm{tr}(\varphi^2) = 0$

Emerging geometric picture

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- FG-FG change of coordinates as $\hbar$ crosses a ray $\ell \in \mathbb{C}^\times$

  for $\mathrm{X}^\iota_\gamma = e^{2\pi i \langle \gamma, \mathbf{x}_\iota \rangle}$, where $\langle \gamma, \mathbf{x}_\iota \rangle = \mathrm{p}^\iota_r x^r_\iota - \mathrm{q}^r_\iota \check{x}^\iota_r$ and $\gamma \in$ charge lattice

  $$\mathrm{X}^J_{\gamma'} = \mathrm{X}^\iota_{\gamma'}(1 - \mathrm{X}_\gamma)^{\langle \gamma', \gamma \rangle \Omega(\gamma)}$$

  determined by BPS invariants $\Omega(\gamma)$ satisfying the Kontsevich-Soibelman WCF

- Cluster transformations relate FG-coord. $\longrightarrow$ their collection $\equiv$ input to RH problems

  [Bridgeland '16,'17]

- include FN coordinates accumulation rays get associated FN-coord.

Emerging geometric picture

- The moduli space of quantum curves $\mathscr{Z} = \mathscr{M}_H(Y) \times \mathbb{C}^\times$ torus fibration $\mathscr{M}_H(C) \longrightarrow \mathscr{B}$

  ~ space of $\{[\hbar\,, \mathscr{E}, \nabla_\hbar]\}$

- Cover $\mathscr{Z}$ with local patches $\mathscr{U}_\imath$ and Darboux coordinates $(\,\mathbf{x}_\imath(\hbar)\,, \check{\mathbf{x}}^\imath(\hbar)\,)$
  - from exact WKB analysis   define an $\hbar$-family of deformations of $\mathscr{M}_H(Y)$
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- Normalised isomonodromic tau-functions $\mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}^\imath)$ associated to the quantum curve
  - are sections of a holomorphic line bundle $\mathscr{L}$ on $\mathscr{Z}$ ~ hyperholomorphic bundle on $\mathscr{M}_H$
  - defined by transition functions $F_{\imath\jmath}(\mathbf{x}_\imath, \mathbf{x}_\jmath)$ on overlaps $\mathscr{U}_\imath \cap \mathscr{U}_\jmath$

    difference generating functions for changes of coordinates $\mathbf{x}_\imath(\mathbf{x}_\jmath, \check{\mathbf{x}}_\jmath)$

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$$\hookrightarrow \quad \mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{2\pi \mathrm{i}(\mathbf{n}, \check{\mathbf{x}}_\imath)}\, Z_{\mathrm{top}}(\mathbf{x}_\imath + \mathbf{n}) \quad ♣ \qquad\qquad \mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = F_{\imath\jmath}(\mathbf{x}_\imath, \mathbf{x}_\jmath)\mathscr{T}_\jmath(\mathbf{x}_\jmath, \check{\mathbf{x}}_\jmath)$$

Vision

Use the quantum curve $\hat{\Sigma}$ as a key in the description of Z_{top}

… as analytic objects, ultimately want to describe Z_{top} as sections of a holomorphic line bundle over the moduli space of quantum curves $\mathscr{Z}$

Points to address

A) How to quantise Σ ? … describe as a D-module; allow $\hbar$-corrections

- $\mathscr{T}$-function describes isomonodromic deformations of the D-module

B) Find "good" coordinates $\mathbf{x}$ and corresponding normalisations for tau-functions

$$\mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}(\mathbf{n},\check{\mathbf{x}}_\imath)} Z_{\mathrm{top}}(\mathbf{x}_\imath + \mathbf{n}) \quad ♣$$

Comparison with Z_{top} derived using the topological vertex SU(2) SYM, pure and Nf=4

Quantum curves and integrability

- Seiberg-Witten curve $\Sigma = \{(x,y), y^2 + q(x) = 0\} \subset T^*C$

  quantised by substituting $y \to -i\hbar\partial_x$ so $[y,x] = -i\hbar$

- Describe the quantum curve $\hat{\Sigma}$

  - as a D-module $\left(\hbar^2\partial_x^2 - q_\hbar(x)\right)\chi(x) = 0$

    the quadratic differential receives $\hbar$-corrections $q_\hbar(x) = q(x) + \mathcal{O}(\hbar)$

  - equivalently, pairs $(\mathcal{E}, \nabla_\hbar)$ with $\hbar$-connection $\nabla_\hbar$ and flat section $\nabla_\hbar\Psi(x) = 0$

    locally $\nabla_\hbar = \hbar\partial_x - A(x)$ with $A(x) = \begin{pmatrix} 0 & q_\hbar(x) \\ 1 & 0 \end{pmatrix} \in sl_2(\mathbb{C})$ $\qquad \Psi = \begin{pmatrix} \chi_-' & \chi_+' \\ \chi_- & \chi_+ \end{pmatrix}$

    more generally any $A(x) = \begin{pmatrix} \varphi_0 & \varphi_+ \\ \varphi_- & -\varphi_0 \end{pmatrix}$ can be brought to oper form by $\nabla_{\mathrm{Op}} = h^{-1}\cdot\nabla_\hbar\cdot h$

$$q_\hbar(x) = \varphi_0^2 + \varphi_+\varphi_- + \mathcal{O}(\hbar)$$

$q(x)$

Quantum curves and integrability

Oper with apparent singularities

$$q_\hbar(x) = \sum_{r=1}^{n} \frac{H_r}{x - z_r} + \frac{\delta_r}{(x-z_r)^2} - \hbar \sum_{i=1}^{d} \left(\frac{v_i}{x-u_i} - \frac{3}{4}\frac{\hbar}{(x-u_i)^2} \right)$$

regular singularities

apparent

Constraints

$q_0^{(i)} = \hbar^2 v_i^2, \; i = 1, \ldots, n-3$

$q_\hbar(x) = \sum_{n=-2} (x-u_i)^n q_n^{(i)}$

$q_\hbar(x) dx^2$ regular at $x \to \infty$

Constraints determine $H_r(u, v)$

[Okamoto '86]

$$\frac{\partial u_k}{\partial z_r} = \frac{\partial H_r}{\partial v_k}, \quad \frac{\partial v_k}{\partial z_r} = -\frac{\partial H_r}{\partial u_k}$$

ensure that the monodromy data μ for $\nabla_\hbar$ is unchanged

- H_r are the Hamiltonians generating isomonodromic deformations
- generated by the isomonodromic tau-function [Jimbo, Miwa, Ueno '81]

$$\boxed{H_r(\mu; \hbar) = \partial_{z_r} \log \mathcal{T}(\mu; \mathbf{z})}$$

SU(2) N f=4 [Gavrylenko, Lisovyy '16]

pure SU(2) [Its, Lisovyy, Tykhyy '15]

The space of monodromy data $\mathcal{M}_{\mathrm{ch}} = \{\rho : \pi_1(C) \to SL_2(\mathbb{C})\} / \sim$

algebraic structure expressed using trace functions

FG / FN coordinates for μ $\longleftrightarrow$ triangulations / pants decompositions of C

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for monodromy data μ

$$\mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}(\mathbf{n},\check{\mathbf{x}}_\imath)} Z_{\mathrm{top}}(\mathbf{x}_\imath + \mathbf{n}) \quad ♣$$

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Coordinates for monodromy data from exact WKB

Exact WKB assigns distinguished coordinates to different regions of $\mathscr{B}$

Solutions to $(\hbar^2\partial_x^2 - q_\hbar(x))\chi(x) = 0$

$$\chi_\pm^{(b)}(x) = \frac{1}{\sqrt{S_{odd}(x)}}\exp\left[\pm\int^x dx'\ S_{odd}(x')\right]$$

$$S_{odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$$

formal series $S^{(\pm)}(x) = \sum_{k=-1}^{\infty} \hbar^k S_k^{(\pm)}(x)$

$S^{(\pm)}(x)$ solutions to the associated Ricatti equation $q_\hbar = \hbar^2(S^2 + S')$, where $S_k^{(\pm)}$ satisfy recursion relations

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Borel summability of S — away from Stokes lines on C $\quad \mathrm{Im}\left(e^{-i\arg(\hbar)} \int_a^x dx' \sqrt{q(x')}\right) = 0$

[Koike, Schäfke] [Nikolaev '20]

zero of q(x)

$V_\beta = \int_\beta dx\; S_{odd}(x)$ — Voros symbols e^{V_β} — Borel summable*, define coordinates on $\mathscr{X}$

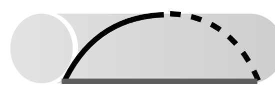
(* depends on the Stokes graph)

Coordinates for monodromy data from exact WKB

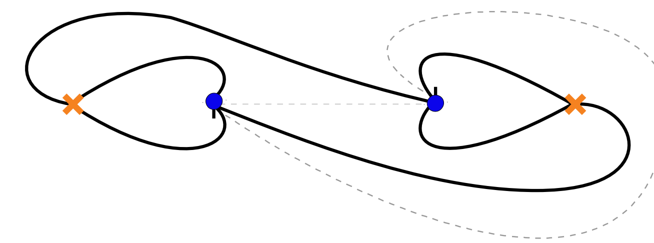
Regions in $\mathscr{B}$ are distinguished by types of Stokes graphs

[Gaiotto, Moore Neitzke '09]
[Hollands, Neitzke '13, Hollands, Kidwai '17] …

- on the cylinder



triangulation



Coordinates for monodromy data from exact WKB

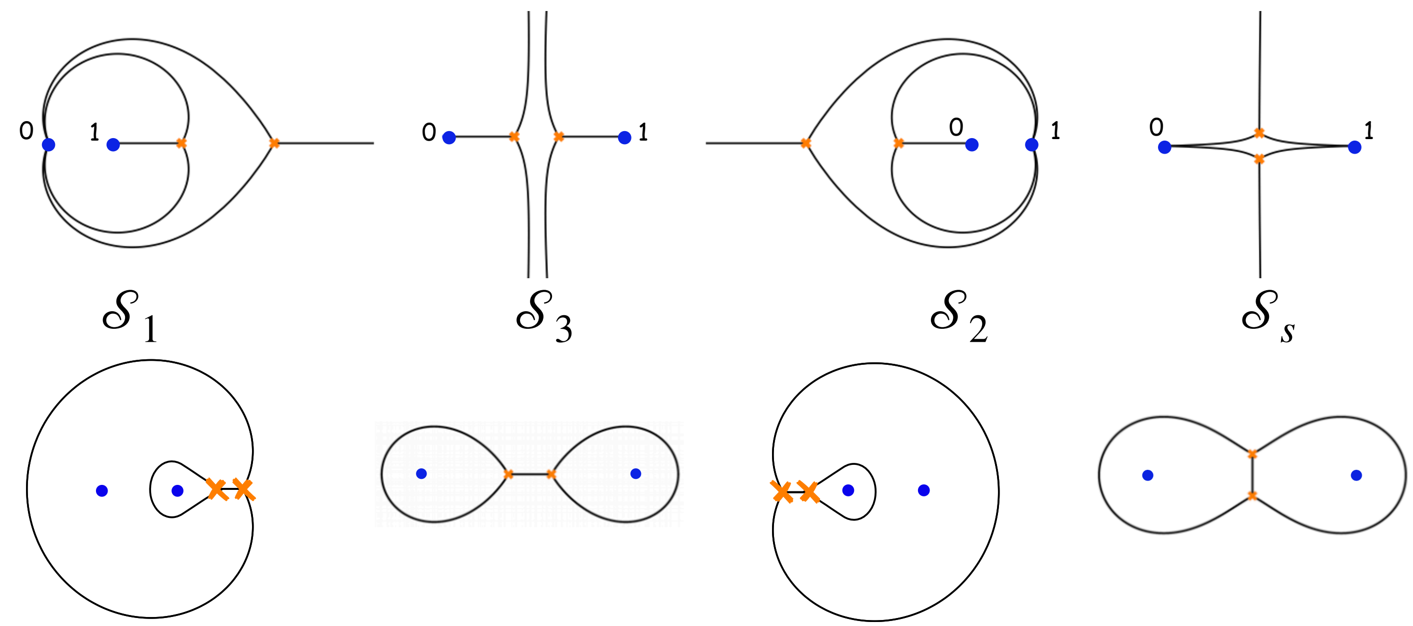
Regions in $\mathscr{B}$ are distinguished by types of Stokes graphs

- on the three-punctured sphere

FG Stokes graphs

[Aoki, Tanda '13]

FN Stokes graphs



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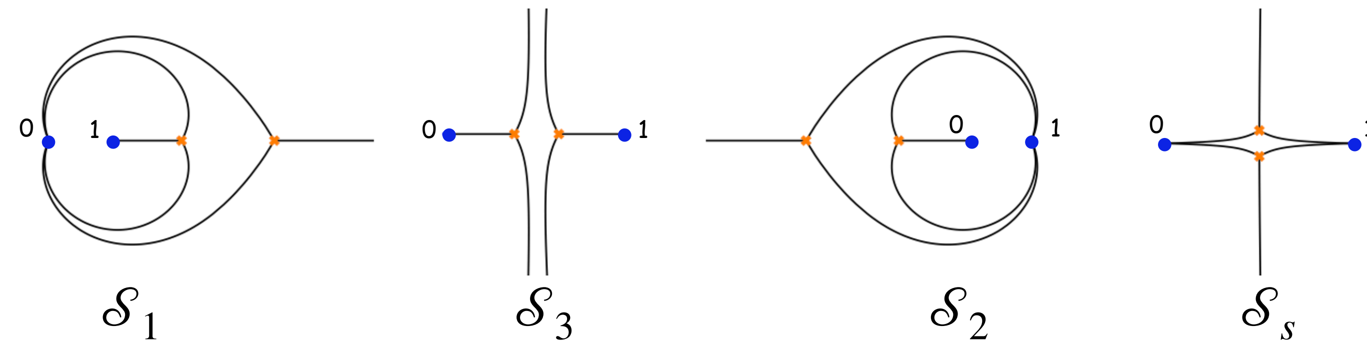
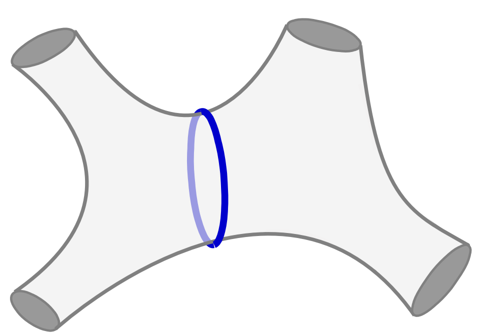


- on $C_{0,4}$



Borel sums of Voros symbols e^{V_β} ~ FG coord. associated to dual WKB triangulation

[Delabaere, Dillinger, Pham '93; Iwaki, Nakanishi '14; Allegretti '19]

Transformations of e^{V_β} associated to flip of WKB triangulation ⇔ change of topology of FG-graph

take the form of cluster mutations of FG coord. ⇒ examples of Stokes phenomena

[Bridgeland '16]

Coordinates on $\mathscr{Z}$ ~ V_β $\qquad$ $X^{J}_{\gamma'} = X^{\iota}_{\gamma'}(1 - X_\gamma)^{\langle \gamma', \gamma \rangle \Omega(\gamma)}$

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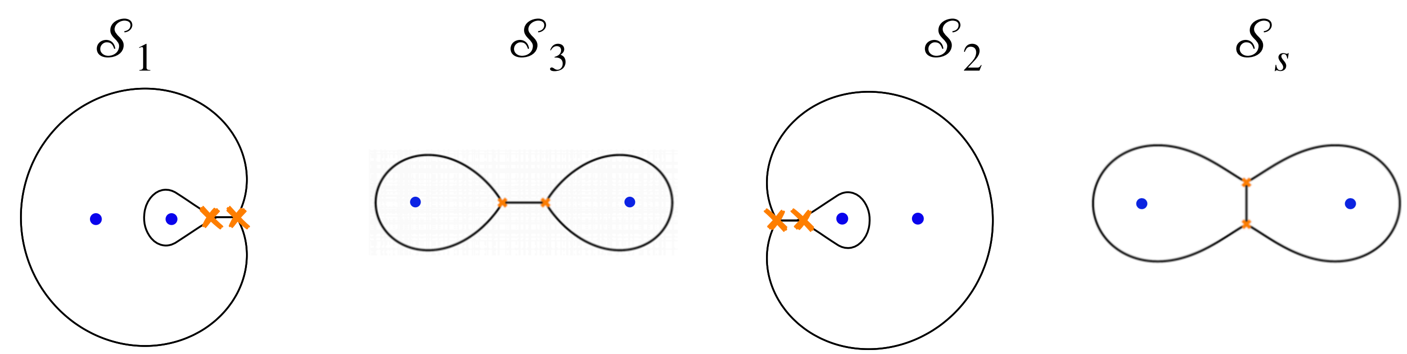

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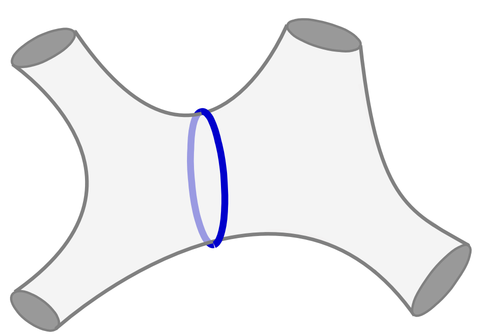
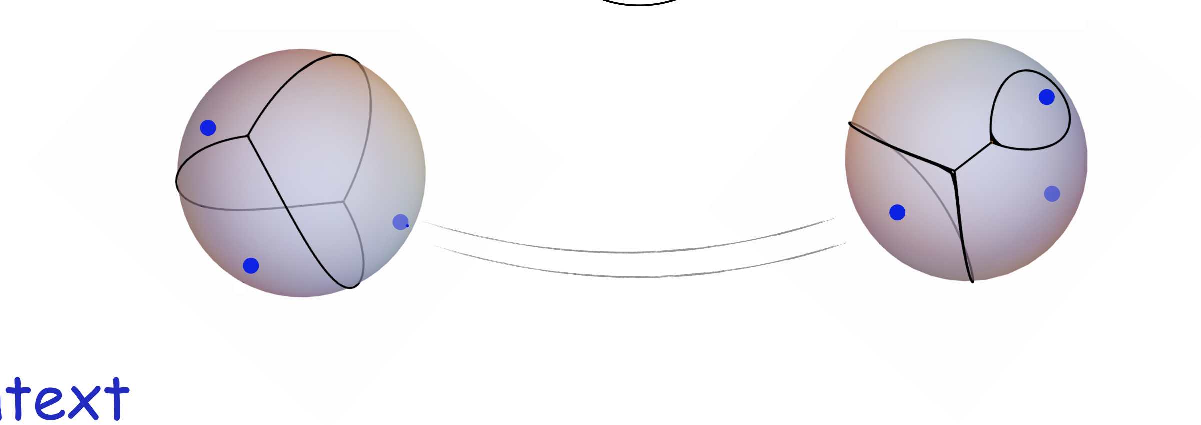
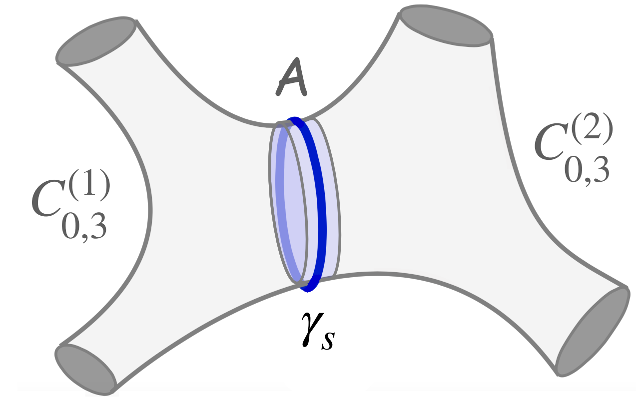




FN coord. relevant in topological strings context

FN Stokes graphs decompose C into annuli & punctured discs

FN-type coordinates on $C_{0,4}$



$\rho : \pi_1(\gamma) \to SL_2\mathbb{C}$

Monodromy data FN-type coordinates (σ, η) on $C_{0,4}$

- quantum curve $\left(\hbar^2\partial_x^2 - q_\hbar(x)\right)\chi(x) = 0 \quad \Leftrightarrow \quad \left(\hbar\partial_x - A(x)\right)\, \Psi(x) = 0$

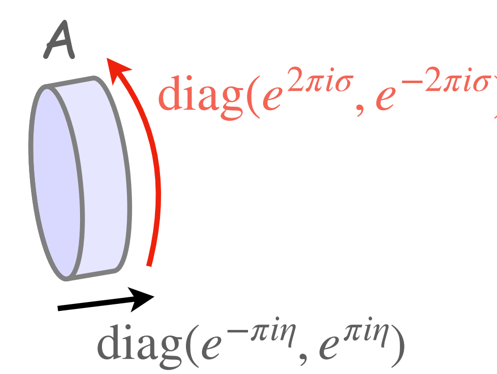
$$\Psi = \begin{pmatrix} \chi_-' & \chi_+' \\ \chi_- & \chi_+ \end{pmatrix} \quad A = \begin{pmatrix} 0 & q_\hbar(x) \\ 1 & 0 \end{pmatrix}$$

From the gluing construction of $C_{0,4}$

$\Psi(x) \sim$ solutions $\Phi^{(i)}(x_i)$ on $C^{(i)}_{0,3}$ with diagonal monodromy $\rho(\gamma_s)$ ~ eigenvalues σ

across the connecting cylinder $\Phi^{(2)}(x) = \Phi^{(1)}(x)\,\mathrm{diag}(e^{-\pi i\eta}, e^{\pi i\eta})$

η is fixed by fixing $\Phi^{(i)}(x_i)$



Freedom in definition of η $\longrightarrow$ normalisation factors n (complex numbers)

can fix a reference η_0

other η are related to this $e^{2\pi i\eta_0} = e^{2\pi i\eta}\,\mathrm{n}^{(1)}\,\mathrm{n}^{(2)}$

Coordinates for monodromy data from exact WKB

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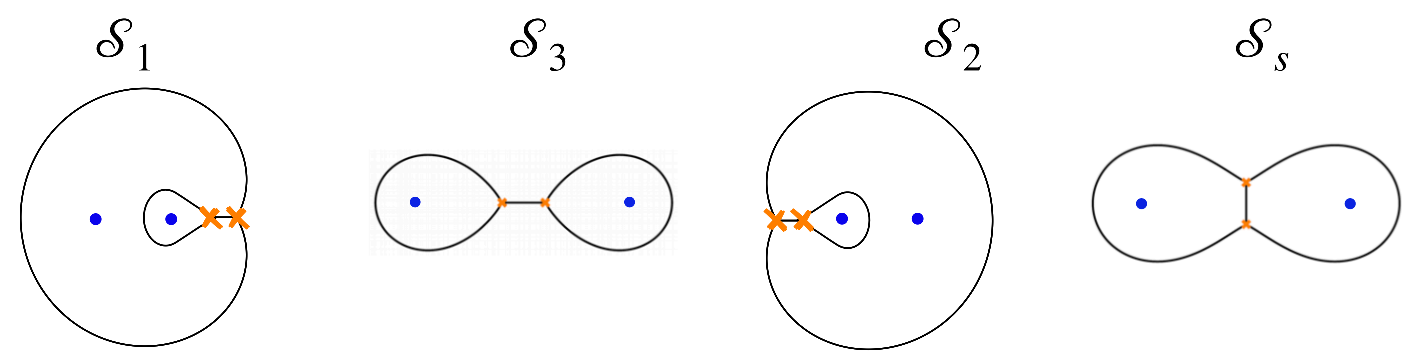
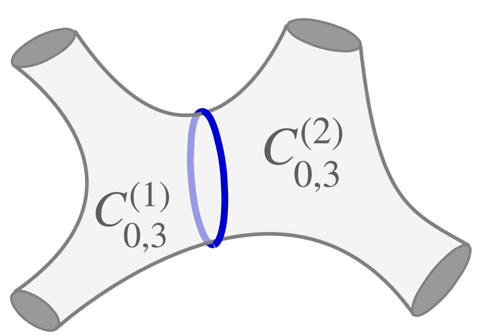
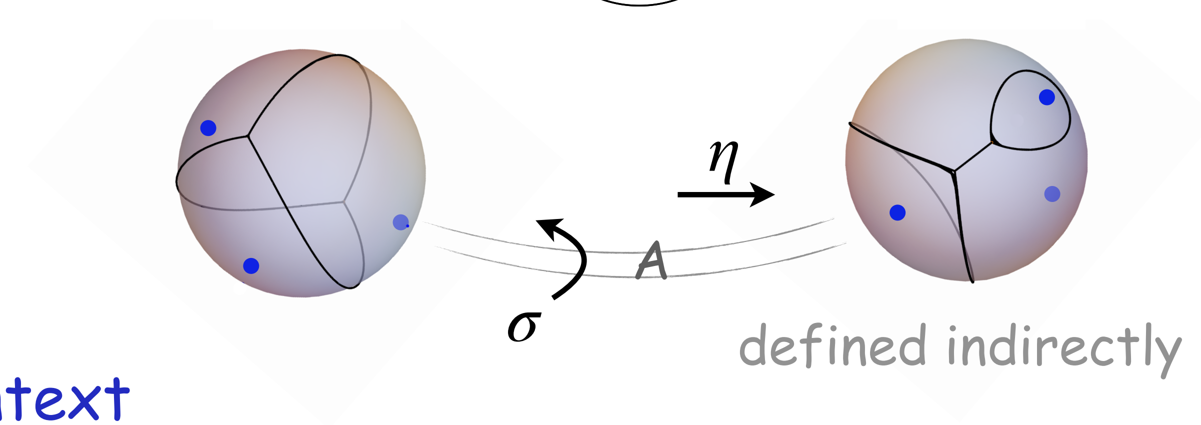


- on $C_{0,4}$





FN coord. (η, σ) relevant in topological strings context

- FN graph for special $(q, \hbar_\star)$; a small perturbation $\hbar \neq \hbar_\star$ makes graph saddle free
- Exact WKB analysis determines solutions $\Phi^{(i)}(x_i)$ on $C_{0,3}^{(i)}$
- η from relative normalisation of $\pm$ solutions; σ defined by their holonomy around A

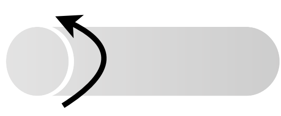
FN-type coordinates

FN & FG coordinates are closely related

- **FG-FN transition**

  - $C_{0,2}$

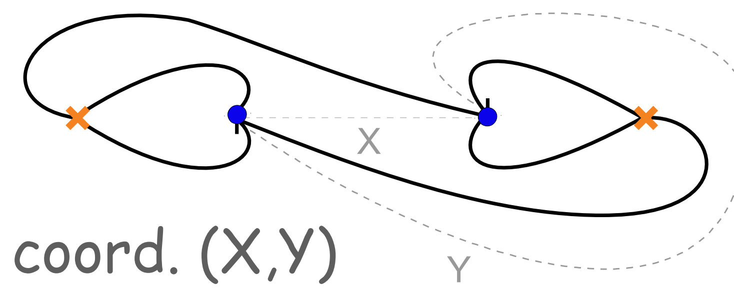
$\mathrm{diag}(U, U^{-1})$



$\mathrm{diag}(V, V^{-1})$

FN coord. (U,V)

$U = e^{2\pi i\sigma}\ , \quad V = \mathrm{i}e^{2\pi i\eta}$

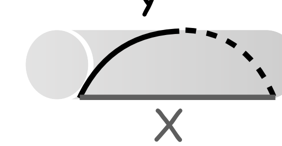


FG coord. (X,Y)

diagonal monodromies

$$X = \left(\frac{V+V^{-1}}{U-U^{-1}}\right)^2 , \quad Y = \left(\frac{U-U^{-1}}{UV+U^{-1}V^{-1}}\right)^2$$

triangulation



FN-coord. (U,V) are limits of FG-coord. (X,Y)

flip + Dehn relabelling $\quad X \to Y^{-1} \quad Y \to X(1+Y^{-1})^{-2}$

$\Rightarrow \quad U \to U \quad V \to UV$

Infinite sequence of flips $\quad U = \lim_{n\to\infty} \frac{1}{\sqrt{X(n)Y(n)}} \qquad V = (U^2-1) \lim_{n\to\infty} U^{-n} \sqrt[4]{\frac{X(n)}{Y(n)}}$

GMN “juggle”

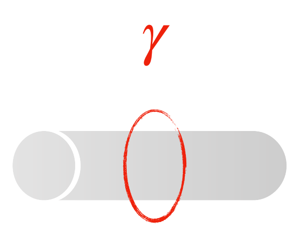
FN-type coordinates

FN & FG coordinates are closely related

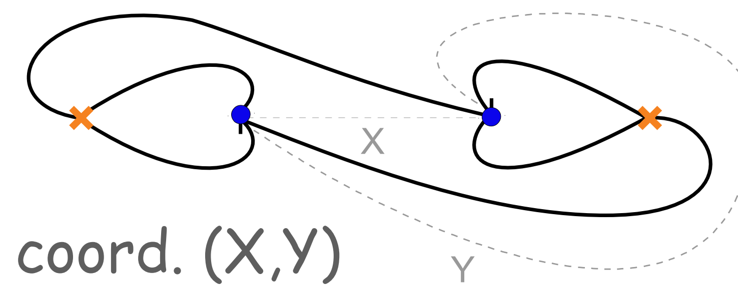
- FG-FN transition

this generalises

- ▹ $C_{0,2}$



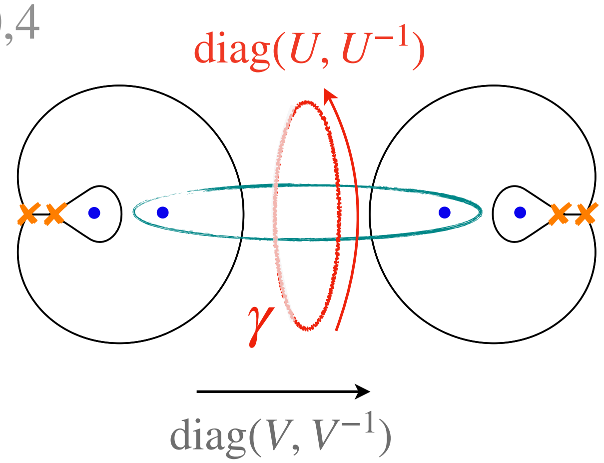
FN coord. (U,V)

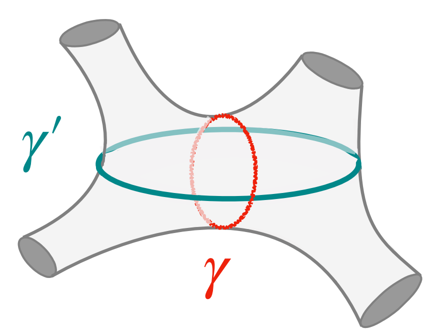


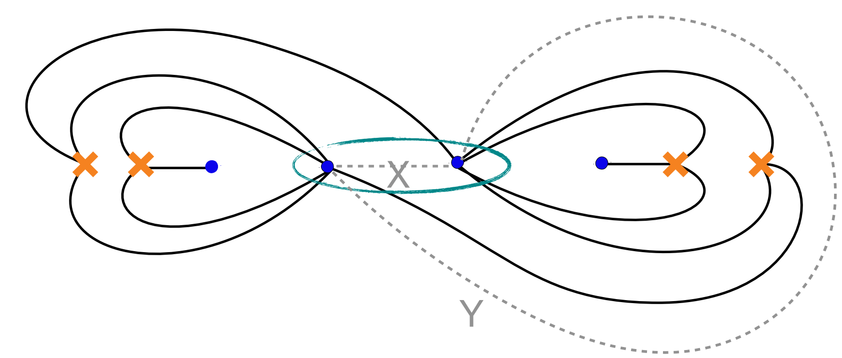
FG coord. (X,Y)

$$X = \left(\frac{V + V^{-1}}{U - U^{-1}}\right)^2, \quad Y = \left(\frac{U - U^{-1}}{UV + U^{-1}V^{-1}}\right)^2$$

- ▹ $C_{0,4}$







- also FN-FN transitions

Vision

Use the quantum curve $\hat{\Sigma}$ as a key in the description of Z_{top}

… as analytic objects, ultimately want to describe Z_{top} as sections of a holomorphic line bundle over the moduli space of quantum curves $\mathscr{Z}$

Points to address

A) How to quantise Σ ? … describe as a D-module; allow $\hbar$-corrections

$\mathscr{T}$-function describes isomonodromic deformations of the D-module

B) Find "good" coordinates $\mathbf{x}$ and corresponding normalisations for tau-functions

for monodromy data μ

$$\boxed{\mathscr{T}_\iota(\mathbf{x}_\iota, \check{\mathbf{x}}_\iota) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}(\mathbf{n},\check{\mathbf{x}}_\iota)} Z_{\mathrm{top}}(\mathbf{x}_\iota + \mathbf{n}) \quad ♣}$$

Comparison with Z_{top} derived using the topological vertex SU(2) SYM, pure and Nf=4

Examples

$$\mathcal{T}_\iota(\mathbf{x}_\iota, \check{\mathbf{x}}_\iota) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i(\mathbf{n},\check{\mathbf{x}}_\iota)} Z_{\mathrm{top}}(\mathbf{x}_\iota + \mathbf{n}) \quad ♣$$

- **Painlevé III tau-function** $q(z) = \frac{\Lambda^2}{z} + \frac{2u}{z^2} + \frac{\Lambda^2}{z^3}$; base curve $C = C_{0,2}$

irregular singularities at $z \to 0,\infty$

$$\mathcal{T}(\sigma,\eta;t) \underset{\substack{r\to 0\\ t\sim r^4}}{=} \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta} \frac{t^{\sigma^2}}{G(1+2\sigma)G(1-2\sigma)} \mathcal{F}(\sigma+n,t) \;\longleftrightarrow\; \mathcal{T}(\nu,\rho;r) \underset{r\to\infty}{=} F(\sigma,\nu) \sum_{n\in\mathbb{Z}} e^{4\pi i n\rho}\, \mathcal{G}(\nu+in,r)$$

$\Lambda^2 = \frac{r^2}{64}$

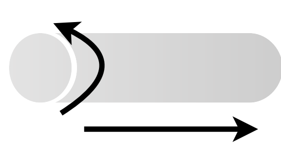
~ Z_{top}

[Its, Lisovyy, Tykhyy '15]

difference generating function ~ transition function for $\mathcal{L}$

(U,V) ~ FN-coord.

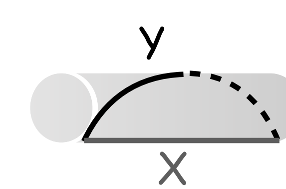
$U = e^{2\pi i\sigma}$, $V = ie^{2\pi i\eta}$



$$X = \left(\frac{V+V^{-1}}{U-U^{-1}}\right)^2, \quad Y = \left(\frac{U-U^{-1}}{UV+U^{-1}V^{-1}}\right)^2$$

(X,Y) ~ FG-coord.

$X = -e^{2\pi\nu}$, $Y = -e^{8\pi i\rho - 2\pi\nu}$



Examples

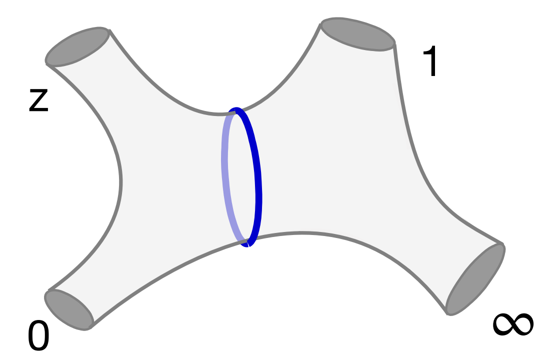
$$\mathcal{T}_\iota(\mathbf{x}_\iota, \check{\mathbf{x}}_\iota) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}(\mathbf{n},\check{\mathbf{x}}_\iota)} Z_{\mathrm{top}}(\mathbf{x}_\iota + \mathbf{n}) \;\clubsuit$$

- **Painlevé VI tau-function** $\quad q(x) = \sum_{r=1}^{4} \frac{E_r}{x - z_r} + \frac{\delta_r}{(x-z_r)^2}$ and $C = C_{0,4}$

$$\mathcal{T}_{(\eta)}(\sigma,\eta) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\eta} \mathcal{G}_{(\eta)}(\sigma + n)$$

[Gamayun, Iorgov, Lisovyy '12]
[Iorgov, Lisovyy, Teschner '15]

- the precise form is sensitive to the choice of coordinates
- change of coordinates $\longleftrightarrow$ change of normalisation $\mathcal{T}_\iota(\mathbf{x}_\iota, \check{\mathbf{x}}_\iota) = F_{\iota\jmath}(\mathbf{x}_\iota, \mathbf{x}_\jmath)\mathcal{T}_\jmath(\mathbf{x}_\jmath, \check{\mathbf{x}}_\jmath)$



Examples

$$\mathscr{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}(\mathbf{n},\check{\mathbf{x}}_\imath)}\, Z_{\mathrm{top}}(\mathbf{x}_\imath + \mathbf{n}) \ \clubsuit$$

- **Painlevé VI tau-function** $q(x) = \sum_{r=1}^{4} \frac{E_r}{x-z_r} + \frac{\delta_r}{(x-z_r)^2}$ and $C = C_{0,4}$

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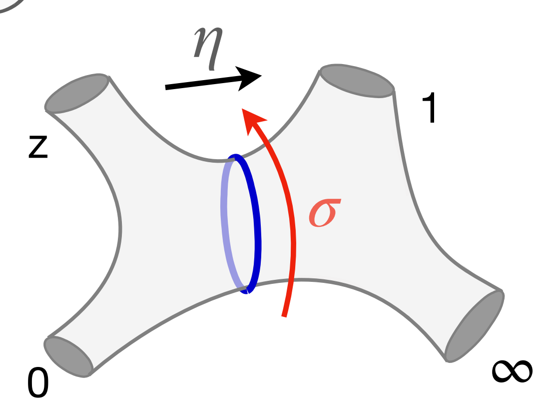


... a closer look at the structure of the series expansion

- z-corrections controlled by isomonodromy
- Exact WKB controls the normalisation in limit $z \to 0$

$$\mathscr{T}_{(\eta)}(\sigma,\eta;z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\eta}\, \mathscr{N}^{(\eta)}(\,\sigma + n\,)\, \mathscr{F}(\,\sigma + n\,;z)$$

power series in z

$\mathscr{T}_{(\eta)} = \mathscr{N}^{(\eta)}\mathscr{T}_{(\eta_0)}$, $e^{2\pi i\eta_0} = e^{2\pi i\eta}\,\mathrm{n}^{(1)}\mathrm{n}^{(2)}$ where $\mathrm{n}^{(1)}\mathrm{n}^{(2)} = \dfrac{\mathscr{N}^{(\eta)}(\sigma)}{\mathscr{N}^{(\eta)}(\sigma-1)}$

Voros symbols define $\mathrm{n}^{(i)} = e^{2V^{(p_ib_i)}}$

[Iwaki, Nakanishi '14, Aoki, Takahashi, Tanda '17]

$$e^{\pm V(p_ib_i)} = \psi_\pm^{(b_i)}(x)\,/\,\psi_\pm^{(p_i)}(x)$$

Normalisation factors of tau-functions from Exact WKB

Voros symbols define a relative normalisation of exact WKB solutions

[Iwaki, Nakanishi '14, Aoki, Takahashi, Tanda '17]

definition of Voros symbols can be generalised to paths starting and ending at poles of q

p is a double pole of q, b is a branch point in the same chart with coord. x

$$V^{(pb)} = V^{(pb)}_{>0} + V^{(pb)}_{\leq 0}$$

$$V^{(pb)}_{\leq 0} = \frac{1}{2} \lim_{x \to x_p} \left(\int_{\gamma_x} dx' \, S^{\mathrm{odd}}_{\leq 0}(x') + (2\sigma - 1)\log(x) \right), \quad V^{(pb)}_{>0} = \frac{1}{2} \int_{\gamma_{x_p}} dx' S^{\mathrm{odd}}_{>0}(x')$$

$$\sigma = \frac{1}{2} + \operatorname*{Res}_{x=0}(S_{\mathrm{odd}})$$

closed contour starting at x_p and encircling b

$$\nabla_\hbar \Psi(x) = 0 \, , \quad \Psi = \begin{pmatrix} \chi'_- & \chi'_+ \\ \chi_- & \chi_+ \end{pmatrix}$$

$\psi^{(b_i)}_\pm$

- one-to-one correspondence flat sections Ψ on $C \leftrightarrow \Psi_i$ on $C^{(i)}_{0,3}$ in the pants decomposition
- reference flat section $\Psi^{(0)}_i$ defined by $\psi^{(p_i)}_\pm$ ~ distinguished by relative normalisation $\mathrm{n}^{(i)} = 1$

Relative normalisation $e^{\pm V(p_i b_i)} = \psi^{(b_i)}_\pm(x) \, / \, \psi^{(p_i)}_\pm(x)$ captured by the Voros symbol

Ratios of exact WKB solutions $\mathrm{n}^{(i)} = e^{2V^{(p_i b_i)}} \rightsquigarrow$ relate different coordinates $\mathbf{x}_\imath \longleftrightarrow \mathbf{x}_\jmath$

and suitably normalised tau-functions $\mathcal{T}_\imath \longleftrightarrow \mathcal{T}_\jmath$

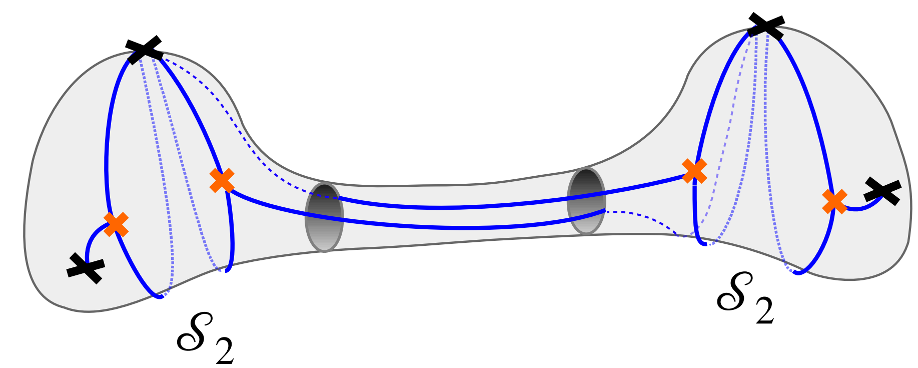
Theta series of tau-functions from Exact WKB

To determine theta-series expansions of tau-functions $\mathcal{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}^\imath)$ from Exact WKB

$C_{0,4}$ example

depends on the type $\mathcal{S}_*$ of Stokes graph on each $C_{0,3}$

(s-type) pants decomposition fixes the σ coordinate but not η



- $\mathcal{S}_2$ -type Stokes graphs on each $C^{(i)}_{0,3}$ $\rightsquigarrow$ $V^{(p_i b_i)}$

[Aoki, Takahashi, Tanda '17; Iwaki, Kohei, Takei '18]

$$\mathsf{n}(\vartheta_1, \vartheta_2, \vartheta_3) := e^{-2V^{(p_i b_i)}}$$

$$e^{2\pi i \eta_\imath}\, \mathsf{n}_1\, \mathsf{n}_2 = e^{2\pi i \eta_0}$$

$$\mathcal{T}_{(\imath)}(\sigma, \eta_\imath) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta_\imath}\, Z_{\mathrm{top},\imath}(a+n)$$

$$\mathsf{n}_1 = \mathsf{n}(\sigma, \theta_2, \theta_1)$$
$$\mathsf{n}_2 = \mathsf{n}(\sigma, \theta_3, \theta_4)$$

$$\mathcal{T}_{(\imath)} = \mathcal{N}^{(\imath)}(\sigma)\,\mathcal{T}$$

$$\mathsf{n}^{(1)}\mathsf{n}^{(2)} = \frac{\mathcal{N}^{(\imath)}(\sigma)}{\mathcal{N}^{(\imath)}(\sigma-1)}$$

$$\sigma = ia/\hbar$$

$$\mathcal{N}^{(\imath)}(\sigma) = \mathcal{N}_2(\sigma, \theta_2, \theta_1)\mathcal{N}_2(\sigma, \theta_3, \theta_4)$$

$$\mathcal{N}_2(\vartheta_1, \vartheta_2, \vartheta_3) = \frac{\prod_{\epsilon,\epsilon'=\pm} G(1+\vartheta_2+\epsilon\vartheta_1+\epsilon'\vartheta_3)}{(2\pi)^{\vartheta_1} G(1) \prod_{r=1}^{3} G(1+2\vartheta_r)}$$

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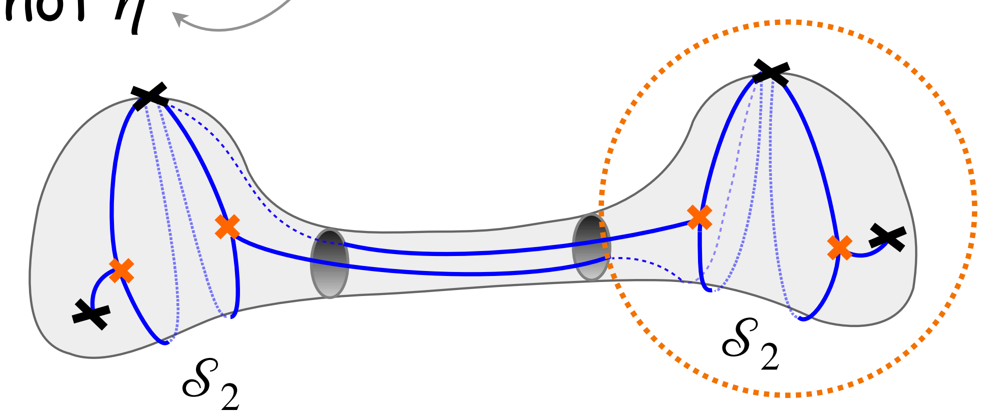

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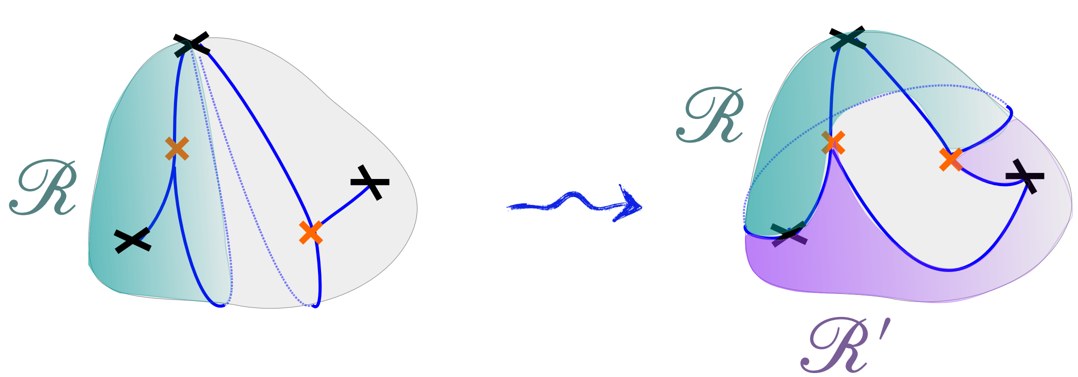
$$\mathscr{T}_{(\imath)}(\sigma, \eta_\imath) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta_\imath}\, Z_{\mathrm{top},\imath}(\,a + n\,)$$

- change of Stokes graph $\mathcal{S}_2 \to \mathcal{S}_s$



- Stokes graph topology and Voros symbol jump

define $n_2^{(s)} := e^{-V^{(0b_+)} - V^{(0b_-)}}$ $\Rightarrow$ $\eta_\jmath, \mathscr{N}^{(\jmath)}$

Theta series of tau-functions from Exact WKB

Systematic match: tau-functions associated to FN-graphs reproduce Z_{top} from vertex

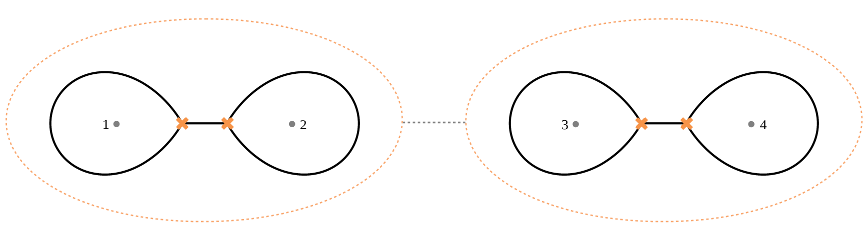
[IC, Pomoni, Teschner '18]
[IC, Longhi, Teschner '20]

- normalised tau-functions $\mathcal{T}_*(\sigma, \eta_*)$ admitting expansion

$$\mathcal{T}_\iota(\mathbf{x}_\iota, \check{\mathbf{x}}_\iota) = \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{2\pi i (\mathbf{n}, \check{\mathbf{x}}_\iota)} Z_{\text{top}}(\mathbf{x}_\iota + \mathbf{n}) \quad \clubsuit$$

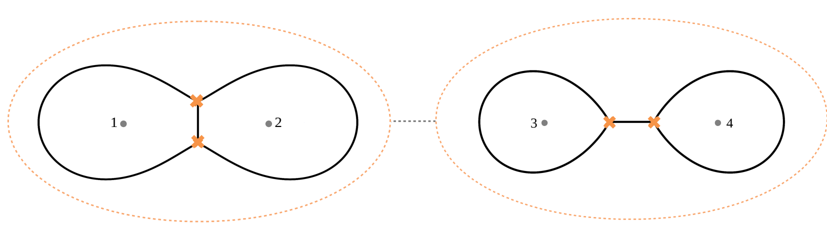
Generalised theta-series

- topological string partition functions in chambers of $\mathcal{M}_{K\ddot{a}hler}$

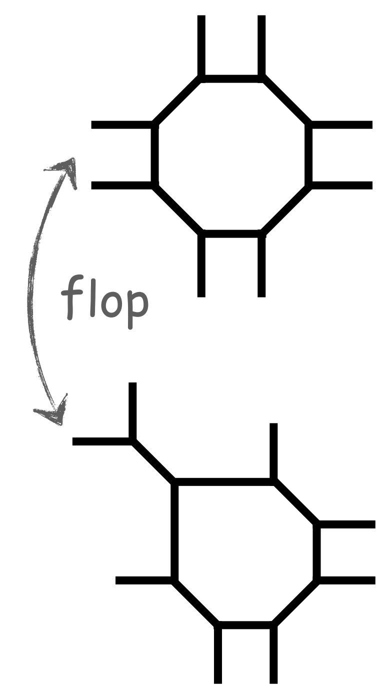


$$\mathcal{T}_{33}(\sigma, \eta_{33}) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta_{33}} Z_{33}(a + n)$$

$$e^{2\pi i \eta_{33}} = 2 \sin \pi(\sigma - (\theta_1 + \theta_2))\, e^{2\pi i \eta_{s3}} \quad \rightsquigarrow \quad \frac{\mathcal{N}_s(\sigma + n, \theta_2, \theta_1)}{\mathcal{N}_s(\sigma, \theta_2, \theta_1)} = \frac{\mathcal{N}_3(\sigma + n, \theta_2, \theta_1)}{\mathcal{N}_3(\sigma, \theta_2, \theta_1)} \left(2 \sin \pi(\sigma - (\theta_1 + \theta_2))\right)^n$$



$$\mathcal{T}_{s3}(\sigma, \eta_{s3}) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta_{s3}} Z_{s3}(a + n)$$



$$\mathcal{N}_i(\vartheta_1, \vartheta_2, \vartheta_3) = \frac{\prod_{\epsilon, \epsilon' = \pm} G(1 + \vartheta_i + \epsilon \vartheta_{i+1} + \epsilon' \vartheta_{i+2})}{(2\pi)^{\vartheta_1} G(1) \prod_{r=1}^{3} G(1 + 2\vartheta_r)}$$

and

$$\mathcal{N}_s(\vartheta_1, \vartheta_2, \vartheta_3) = G(1 + \sum_{i=1}^{3} \vartheta_i) \frac{\prod_{j=1}^{3} G(1 + \sum_{i=1}^{3} \vartheta_i - 2\vartheta_j)}{(2\pi)^{\vartheta_1} G(1) \prod_{r=1}^{3} G(1 + 2\vartheta_r)}$$

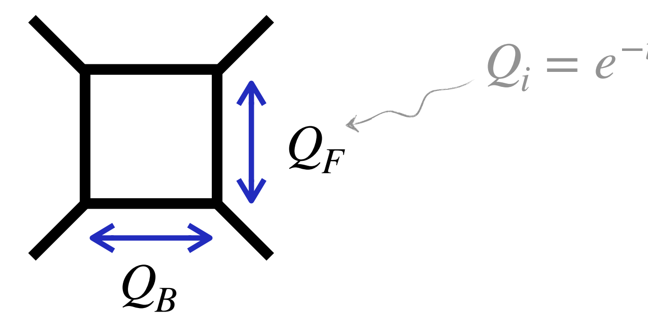
Z_{top} from the topological vertex

Type IIA string theory on the CY3 mirror dual to Y_Σ, the toric CY3 X

X characterised by a toric diagram:

Kähler moduli $\mathbf{t}(\mathbf{m})$ are periods of ydx around cycles of Σ

pure SU(2) SYM



Can compute $Z_{top}(\mathbf{t};\hbar)$ with the topological vertex

$C_{0,2}$

[Aganagic, Klemm, Marino, Vafa '03]

$$Z_{top}^{\text{box}} = \mathcal{M}(Q_F)^2 \sum_{Y_1,Y_2} \prod_{i=1}^{2} (Q_B/Q_F)^{|Y_i|} \frac{1}{\mathcal{N}_{Y_iY_i}(1)\mathcal{N}_{Y_1Y_2}(Q_F)\mathcal{N}_{Y_2Y_1}(Q_F^{-1})}$$

$$\mathcal{M}(Q) = \prod_{i,j=0}^{\infty} (1 - Qq^{i+j+1})^{-1}, \quad |q| < 1$$

$$\mathcal{N}_{RP}(Q) = \prod_{(i,j)\in R} \left(1 - Qq^{R_i+P_j^t-i-j+1}\right) \prod_{(i,j)\in P} \left(1 - Qq^{-P_i-R_j^t+i+j-1}\right)$$

Z_{top} from the topological vertex

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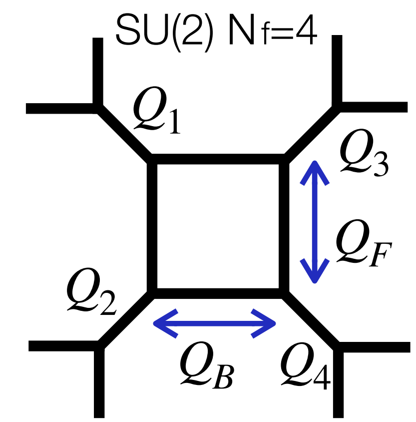
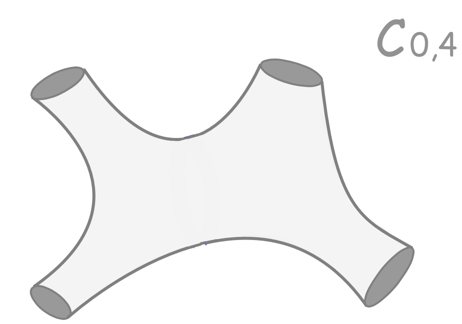


Can compute Z_{top} with the topological vertex

[Aganagic, Klemm, Marino, Vafa '03]



$$Z_{top} = Z_{in} Z_{out} Z_{inst}$$

$$Z_{in} = \frac{\mathscr{M}(Q_F)\mathscr{M}(Q_1Q_2Q_F)}{\prod_{i=1}^{2}\mathscr{M}(Q_i)\mathscr{M}(Q_iQ_F)} \qquad Z_{out} = \frac{\mathscr{M}(Q_F)\mathscr{M}(Q_3Q_4Q_F)}{\prod_{i=3}^{4}\mathscr{M}(Q_i)\mathscr{M}(Q_iQ_F)}$$

$$\mathscr{M}(Q) = \prod_{i,j=0}^{\infty}(1 - Qq^{i+j+1})^{-1}, \quad |q| < 1$$

$$Z_{inst} = \sum_{Y_1,Y_2}(Q_1Q_4Q_B)^{|Y_2|}(Q_2Q_3Q_B)^{|Y_1|}q^{\frac{\kappa_{Y_2}}{2}-\frac{\kappa_{Y_1}}{2}}\prod_{i=1}^{2}s_{Y_i}(q^\rho)s_{Y_i^t}(q^\rho) \times \frac{\prod_{i=2,3}\mathscr{N}_{Y_1\varnothing}(Q_i^{-1})\mathscr{N}_{\varnothing Y_2}(Q_iQ_F)\prod_{i=1,4}\mathscr{N}_{\varnothing Y_2}(Q_i^{-1})\mathscr{N}_{Y_1\varnothing}(Q_iQ_F)}{\mathscr{N}_{Y_1Y_2}(Q_F)\mathscr{N}_{Y_1Y_2}(Q_F)}$$

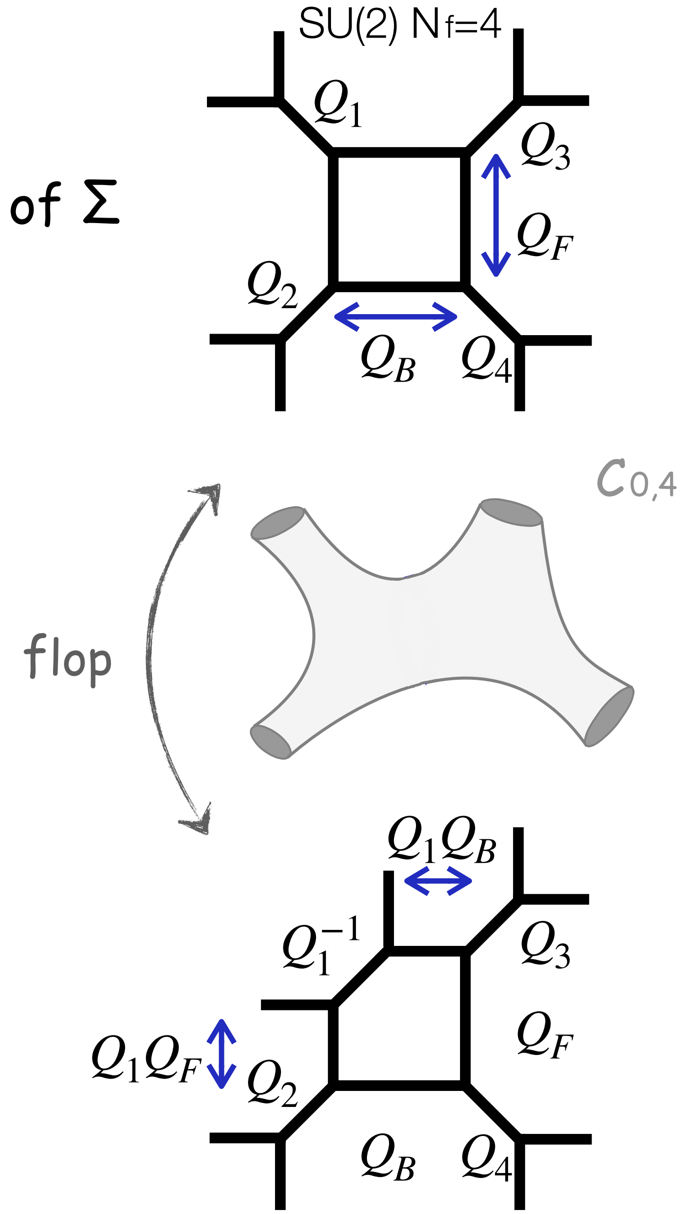
Nekrasov instanton partition function

Z_{top} from the topological vertex

Type IIA string theory on the CY_3 mirror dual to Y_Σ, the toric CY_3 X

X characterised by a toric diagram:

Kähler moduli $\mathbf{t}(\mathbf{m})$ are periods of ydx around cycles of Σ



Can compute Z_{top} with the topological vertex

[Aganagic, Klemm, Marino, Vafa '03]

$$Z'_{top} = Z'_{in} Z_{out} Z_{inst}$$

$$Z'_{in} = \frac{\mathcal{M}(Q_1)}{\mathcal{M}(Q_1^{-1})} Z_{in}$$

$Q_i = e^{-t_i}$

$Z_{top}(\mathbf{t})$ is not continuous on the parameters $\mathbf{t}$

... in the geometric characterisation of Z_{top} expect this to jump across walls in $\mathscr{B}$

Intuition from string dualities

Predictions from a sequence of string dualities relate Z top in different pictures

- geometric picture — B-model topological string from type IIB string theory on CY3 Y

[Maulik, Nekrasov, Okounkov, Pandharipande '06]

- D-brane picture — type IIA string theory on $\mathbb{R}^3 \times S^1 \times$ X with D6-brane on $S^1 \times$ X

$Z_{top}(\mathbf{t};\hbar) \sim Z_{\mathrm{DT}}(\mathbf{t};\hbar)$ generating function of Donaldson Thomas invariants counting bound states of D-branes

D0-D2-D6 system

$$Z'_{\mathrm{DT}}(\xi,\mathbf{t};\hbar) = \sum_{\mathbf{n}\in H^2(Y,\mathbb{Z})} e^{\mathbf{n}.\xi} Z_{top}(\mathbf{t}+\hbar\mathbf{n};\hbar)$$ D0-D2-D4-D6 system

- fermionic picture — free fermion partition function on a quantisation of the SW curve

$$Z'_{\mathrm{DT}}(\xi,\mathbf{t};\hbar) \sim Z_{\mathrm{ff}}(\xi,\mathbf{t};\hbar)$$

[Dijkgraaf, Hollands, Sulkowski, Vafa '08]

Intuition from string dualities

Predictions from a sequence of string dualities relate Z top in different pictures

[Dijkgraaf, Hollands, Sulkowski, Vafa '08]

$$Z_{\text{ff}}(\xi, \mathbf{t}; \hbar) = \sum_{\mathbf{n} \in H^2(X,\mathbb{Z})} e^{n\cdot\xi}\, Z_{\text{top}}(\mathbf{t} + \hbar\mathbf{n}; \hbar)$$

Identification Z ff ~ tau-function of the associated integrable system

$$Z_{\text{ff}}(\mu; \mathbf{z}) \sim \mathcal{T}(\mu; \mathbf{z})$$ [Gamayun, Iorgov, Lisovyy '12][Iorgov, Lisovyy, Teschner '15]

isomonodromic deformations of the quantum SW curve $\hat{\Sigma}$

quantisation replaces the y parameter of Σ with $y \to -i\hbar\partial_x$

Expansion relating the tau-function and Z top

$$\mathcal{T}(\mu(\mathbf{x}, \check{\mathbf{x}})) \sim \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{2\pi\mathrm{i}(\mathbf{n},\check{\mathbf{x}})}\, Z_{\text{top}}(\mathbf{x} + \mathbf{n}) \quad ♣$$

[IC, Pomoni, Teschner '18]

Emerging geometric picture

- The moduli space of quantum curves $\mathscr{Z} = \mathscr{M}_H(Y) \times \mathbb{C}^\times$ torus fibration $\mathscr{M}_H(C) \longrightarrow \mathscr{B}$

  ~ space of $\{[\hbar\,,\mathscr{E},\nabla_\hbar]\}$

  $\mathscr{E}$ holomorphic vector bundle in the Higgs pair $[\mathscr{E},\varphi]$

  $\nabla_\hbar$ holomorphic $\hbar$-connection $\hbar\partial_x + A(x)$

- Cover $\mathscr{Z}$ with local patches $\mathscr{U}_\iota$ and Darboux coordinates $(\,\mathbf{x}_\iota(\hbar)\,,\check{\mathbf{x}}^\iota(\hbar)\,)$
  - from exact WKB analysis define an $\hbar$-family of deformations of $\mathscr{M}_H(Y)$
  - asymptotic behaviour $x_r \simeq a_r/\hbar + \mathscr{O}(\hbar^0)\,,\ \check{x}^r \simeq \check{a}^r/\hbar + \mathscr{O}(\hbar^0)$ homological $(\,a_r\,,\check{a}^r\,)$ coordinates on $\mathscr{B}$

- FG-FG change of coordinates as $\hbar$ crosses a ray $\ell \in \mathbb{C}^\times$ $\mathsf{X}^J_{\gamma'} = \mathsf{X}^\iota_{\gamma'}(1-\mathsf{X}_\gamma)^{\langle\gamma',\gamma\rangle\Omega(\gamma)}$

  determined by BPS invariants $\Omega(\gamma)$ satisfying the Kontsevich-Soibelman WCF

- Cluster transformations relate FG-coord. $\to$ their collection $\equiv$ input to RH problems

  [Bridgeland '16,'17]

- include FN coordinates accumulation rays get associated FN-coord.

Emerging geometric picture

- The moduli space of quantum curves $\mathcal{Z} = \mathcal{M}_H(Y) \times \mathbb{C}^\times$ torus fibration $\mathcal{M}_H(C) \longrightarrow \mathcal{B}$

  ~ space of $\{[\hbar, \mathcal{E}, \nabla_\hbar]\}$

  $\mathcal{E}$ holomorphic vector bundle in the Higgs pair $[\mathcal{E}, \varphi]$

  $\nabla_\hbar$ holomorphic $\hbar$-connection $\hbar\partial_x + A(x)$

- Cover $\mathcal{Z}$ with local patches $\mathcal{U}_\imath$ and Darboux coordinates $(\mathbf{x}_\imath(\hbar), \check{\mathbf{x}}^\imath(\hbar))$
  - from exact WKB analysis — define an $\hbar$-family of deformations of $\mathcal{M}_H(Y)$
  - asymptotic behaviour $x_r \simeq a_r/\hbar + \mathcal{O}(\hbar^0)$, $\check{x}^r \simeq \check{a}^r/\hbar + \mathcal{O}(\hbar^0)$ — homological $(a_r, \check{a}^r)$ coordinates on $\mathcal{B}$

- Normalised isomonodromic tau-functions $\mathcal{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}^\imath)$ associated to the quantum curve
  - are sections of a holomorphic line bundle $\mathcal{L}$ on $\mathcal{Z}$
  - defined by transition functions $F_{\imath\jmath}(\mathbf{x}_\imath, \mathbf{x}_\jmath)$ on overlaps $\mathcal{U}_\imath \cap \mathcal{U}_\jmath$

  given by difference generating functions for changes of coordinates $\mathbf{x}_\imath(\mathbf{x}_\jmath, \check{\mathbf{x}}_\jmath)$

[IC, Pomoni, Teschner '18]
[IC, Longhi, Teschner '20]

$$\hookrightarrow \mathcal{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i(\mathbf{n}, \check{\mathbf{x}}_\imath)} Z_{\text{top}}(\mathbf{x}_\imath + \mathbf{n}) \quad ♣ \qquad \mathcal{T}_\imath(\mathbf{x}_\imath, \check{\mathbf{x}}_\imath) = F_{\imath\jmath}(\mathbf{x}_\imath, \mathbf{x}_\jmath)\mathcal{T}_\jmath(\mathbf{x}_\jmath, \check{\mathbf{x}}_\jmath)$$

Outlook

- Uplift to 5d gauge theories - q-deformed functions, exponential networks

  [Bonelli, Grassi, Tanzini '17, Bershtein, Gavrylenko, Marshakov '17, Bonelli, Del Monte, Tanzini '20] [Banerjee, Longhi, Romo '18]

- Relation geometry of hypermultiplet moduli spaces

  [Alexandrov, Persson, Pioline '10]
  [Neitzke '11]

  the transition functions of $\mathscr{L}$ are equivalent to those of the hyperholomorphic line bundle

- 2d-4d wall crossing, isomonodromic tau function & free fermions

  [Gaiotto, Moore, Neitzke '11] [Iorgov, Lisovyy, Teschner '14]

- Relation to topological recursion

  [Chekhov, Eynard, Orantin '06, Eynard, Orantin '07, Eynard, Garcia-Failde '19, Iwaki '19]

- Comparisons to other nonperturbative definitions of Z_{top}

  [Grassi, Hatsuda, Marino '14, Grassi, Gu, Marino '19]

Thank you