Geometric description of topological string partition functions from quantum curves and DT invariants

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Donaldson-Thomas invariants and Resurgence
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Topological string partition functions

Topological string partition functions are objects of interest

- **Physics**
  - instanton partition functions
  - index counting BPS states

- **Mathematics**
  - enumerative invariants

(Gromov-Witten, Donaldson-Thomas, Gopakumar-Vafa)

\[
\log Z_{top} = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g
\]

Approaches towards computation

holomorphic anomaly, topological vertex, topological recursion, spectral theory

[Bershadsky, Cecotti, Ooguri, Vafa ’94] [Aganagic, Klemm, Marino, Vafa ’05] [Chekhov, Eynard, Orantin ’06] [Grassi, Hatsuda, Marino ’14]
Topological string partition functions

Aim: geometric characterisation of the topological string partition function $Z_{\text{top}}$

... through its relation to quantum curves and DT invariants

Topological strings come in two variants related by mirror symmetry

- **A model**
  - type IIA string theory on a Calabi-Yau threefold $X$
    - Kähler moduli $t$
      - $\log Z_{\text{top}}(t, \lambda) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)$

- **B model**
  - type IIB string theory on a Calabi-Yau threefold $Y$
    - Complex moduli $m$
      - mirror map $t(m)$

Are $Z_{\text{top}}(t)$ locally defined functions on $\mathcal{M}_{\text{Kähler}}(X) \leftrightarrow \mathcal{M}_{\text{Cplx}}(Y)$?
Geometric setup

Type IIB string theory on a local CY $Y_{\Sigma}$

$Y_{\Sigma} : vw - P(x, y) = 0$ \quad where \quad $P(x, y) = y^2 + q(x) \ , \quad x \in C$

$q(x) = \sum_{r=1}^{n} \frac{E_r}{x - z_r} + \frac{\delta_r}{(x - z_r)^2} + \ldots$

Riemann surface $C_{0,n}$

$q(x)dx^2$ quadratic differential on $C$

irregular singularities $\leftrightarrow$ poles of order >2

$SW$ curve $\Sigma = \{(x, y), P(x, y) = 0\} \subset T^*C$

$\text{Moduli space } \mathcal{M}_{\text{cplx}}(Y) \text{ of pairs } (C, q)$

complex structure moduli $m = \{E_r, \delta_r, z_r\}$

Geometric engineering of 4d, $\mathcal{N} = 2$ theories of class $S$

[Katz, Klemm, Vafa '96]  [Gaiotto '09]

Examples

- $C_{0,2}$ \quad pure SU(2) super Yang-Mills theory
- $C_{0,4}$ \quad SU(2) 4-flavour super Yang-Mills
Geometric setup

Type IIB string theory on a local CY $Y_{\Sigma}$

$Y_{\Sigma} : vw - P(x, y) = 0$ where $P(x, y) = y^2 + q(x)$, $x \in C$

$q(x) = \sum_{r=1}^{n} \frac{E_r}{x - z_r} + \frac{\delta_r}{(x - z_r)^2} + \ldots$

$SW$ curve $\Sigma = \{(x, y), P(x, y) = 0\} \subset T^*C$

Geometric engineering of 4d, $\mathcal{N} = 2$ theories of class $S$

Special geometry $\mathcal{B} := \mathcal{M}_{cplx}(Y)$ homological coordinates $a_r = \int_{\alpha_r} ydx, \; \bar{a}^r = \int_{\beta_r} ydx = \frac{\partial \mathcal{F}}{\partial a_r}$

Hitchin moduli space $\mathcal{M}_H(Y)$ of Higgs pairs $(\mathcal{E}, \phi)$ torus fibration $\mathcal{M}_H(Y) \longrightarrow \mathcal{B}$ where $[\mathcal{E}, \phi] \mapsto \text{tr}(\phi^2) \sim q(x)$
Topological string partition functions

Guiding idea

close relation between $Z_{top}$ and a quantisation of the SW curve

Broad questions

How are the variables of $Z_{top}$ related to the parameters of the quantum curve?

Exact WKB - how does this approach work to define these parameters?
Topological string partition functions come in two variants

- **A model**
  - type IIA string theory on a Calabi-Yau threefold $X$
  - Kähler moduli $t$

- **B model**
  - type IIB string theory on a Calabi-Yau threefold $Y$
  - Complex moduli $m$

The topological string partition function $Z_{top}(t)$ is given by:

$$
\log Z_{top}(t) = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g(t)
$$

$Z_{top}(t)$ are locally defined functions on $\mathcal{M}_{Kähler}(X) \leftrightarrow \mathcal{M}_{Cplx}(Y)$

Aim: geometric characterisation of the topological string partition function $Z_{top}$ ... through its relation to quantum curves and DT invariants

Physics predicts that higher-genus corrections are encoded in quantum SW curves

[Aganagic, Dijkgraaf, Klemm, Marino, Vafa ’03], [Dijkgraaf, Hollands, Sulkowski, Vafa ‘08] …
Emerging geometric picture

- The moduli space of quantum curves $\mathcal{E}$

- Cover $\mathcal{E}$ with local patches $\mathcal{U}_i$ and Darboux coordinates $(x_i, \dot{x}_i)$

- Normalised isomonodromic tau-functions $\mathcal{T}_i(x_i, \dot{x}_i)$

\[
\mathcal{T}_i(x_i, \dot{x}_i) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i(n, \dot{x}_i)} Z_{\text{top}}(x_i + n) \quad \mathcal{T}_i(x_i, \dot{x}_i) = F_{ij}(x_i, x_j) \mathcal{T}_j(x_j, \dot{x}_j)
\]
Emerging geometric picture

- The moduli space of quantum curves $\mathcal{Z} = \mathcal{M}_H(Y) \times \mathbb{C}^\times$

  space of $\{[\hbar, \mathcal{E}, \nabla_\hbar]\}$

$\mathcal{E}$ holomorphic vector bundle in the Higgs pair $[\mathcal{E}(\phi)]$

$\nabla_\hbar$ holomorphic $\hbar$-connection $\hbar \partial_x + A(x)$

quantisation of the curves $(\hbar^2 \partial_x^2 - q_\hbar(x))\chi(x) = 0$

torus fibration $\mathcal{M}_H(C) \to \mathcal{B}$
Emerging geometric picture

- The moduli space of quantum curves $\mathcal{E} = M_H(Y) \times \mathbb{C}^\times$ isomorphic to space of $\{[\hbar, \mathcal{E}, \nabla_\hbar]\}$

- Torus fibration $\mathcal{M}_H(C) \to \mathcal{B}$

- From exact WKB analysis, define an $\hbar$-family of deformations of $\mathcal{M}_H(Y)$

- Asymptotic behaviour $x_r \simeq a_r/\hbar + O(\hbar^0), \check{x}^r \simeq \check{a}^r/\hbar + O(\hbar^0)$

- Homological coordinates $(a_r, \check{a}^r)$ on $\mathcal{B}$

\[ a_r = \int_{\alpha_r} y dx, \check{a}^r = \int_{\beta_r} y dx \]

- Quantisation of the curves $\mathcal{M}_H$ (WKB analysis)

- Cover $\mathcal{E}$ with local patches $\mathcal{U}_i$ and Darboux coordinates $(x_i(\hbar), \check{x}^i(\hbar))$

- $\hbar^2 \partial^2_x - q_h(x)\check{\chi}(x) = 0$
Emerging geometric picture

- The moduli space of quantum curves $\mathcal{E} = \mathcal{M}_H(Y) \times \mathbb{C}^\times$ \hspace{1cm} torus fibration $\mathcal{M}_H(C) \to \mathcal{B}$
  \hspace{2cm} $\sim$ space of $[[\hbar, \mathcal{C}, \nabla_\hbar]]$

- Cover $\mathcal{E}$ with local patches $\mathcal{U}_i$ and Darboux coordinates $(x_i(\hbar), \dot{x}_i(\hbar))$
  \hspace{2cm} from exact WKB analysis define an $\hbar$-family of deformations of $\mathcal{M}_H(Y)$
  \hspace{2cm} asymptotic behaviour $x_r \simeq a_r/\hbar + \mathcal{O}(\hbar^0)$, $\dot{x}_r \simeq \dot{a}_r/\hbar + \mathcal{O}(\hbar^0)$ homological coordinates $(a_r, \dot{a}_r)$ on $\mathcal{B}$

- FG-FG change of coordinates as $\hbar$ crosses a ray $\ell \in \mathbb{C}^\times$
  \hspace{2cm} for $X^i_\gamma = e^{2\pi i \langle \gamma, x_i \rangle}$, where $\langle \gamma, x_i \rangle = p_i^{\gamma} x_i^\gamma - q_i^\gamma \dot{x}_i^\gamma$ and $\gamma \in$ charge lattice
  \hspace{2cm} $X^i_\gamma = X^i_\gamma (1 - X_i^\gamma \langle \gamma', \gamma \rangle \Omega(\gamma))$

- Cluster transformations relate FG-coord. → their collection $\equiv$ input to RH problems

- include FN coordinates accumulation rays get associated FN-coord.

[Bridgeland '16,'17]
Emerging geometric picture

○ The moduli space of quantum curves \( \mathcal{E} = \mathcal{M}_H(Y) \times \mathbb{C}^\times \)
  \( \sim \) space of \( \{ [\hbar, \xi, \nabla\hbar] \} \)

○ Cover \( \mathcal{E} \) with local patches \( \mathcal{U}_i \) and Darboux coordinates \( (x_i(\hbar), \dot{x}_i(\hbar)) \)
  ○ from exact WKB analysis define an \( \hbar \)-family of deformations of \( \mathcal{M}_H(Y) \)
  ○ asymptotic behaviour \( x_r \simeq a_r/\hbar + o(\hbar^0), \dot{x}_r \simeq \dot{a}_r/\hbar + o(\hbar^0) \)
    homological coordinates \( (a_r, \dot{a}_r) \) on \( \mathcal{B} \)

○ Normalised isomonodromic tau-functions \( \mathcal{T}_i(x_i, \dot{x}_i) \)
  ○ are sections of a holomorphic line bundle \( \mathcal{L} \) on \( \mathcal{E} \)
  \( \sim \) hyperholomorphic bundle on \( \mathcal{M}_H \)
  ○ defined by transition functions \( F_{ij}(x_i, x_j) \) on overlaps \( \mathcal{U}_i \cap \mathcal{U}_j \)
  difference generating functions for changes of coordinates \( x_i(x_j, \dot{x}_j) \)

\[ \mathcal{T}_i(x_i, \dot{x}_i) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i(n, \dot{n})} Z_{\text{top}}(x_i + n) \quad \mathcal{T}_i(x_i, \dot{x}_i) = F_{ij}(x_i, x_j) \mathcal{T}_j(x_j, \dot{x}_j) \]
Use the quantum curve $\hat{\Sigma}$ as a key in the description of $Z_{\text{top}}$

... as analytic objects, ultimately want to describe $Z_{\text{top}}$ as sections of a holomorphic line bundle over the moduli space of quantum curves $\mathcal{E}$

Points to address

A) How to quantise $\Sigma$? ... describe as a D-module; allow $\hbar$-corrections

  - $\mathcal{T}$-function describes isomonodromic deformations of the D-module

B) Find “good” coordinates $x$ and corresponding normalisations for tau-functions

$$\mathcal{T}_i(x_i, \dot{x}_i) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i (n, \dot{x}_i)} Z_{\text{top}}(x_i + n)$$

Comparison with $Z_{\text{top}}$ derived using the topological vertex $SU(2)$ SYM, pure and $N_f=4$
Quantum curves and integrability

- Seiberg-Witten curve $\Sigma = \{(x, y), y^2 + q(x) = 0\} \subset T^*C$
  
  quantised by substituting $y \rightarrow -i\hbar \partial_x$ so $[y, x] = -i\hbar$

- Describe the quantum curve $\hat{\Sigma}$
  
  as a D-module $\left(\hbar^2 \partial_x^2 - q_\hbar(x)\right) \chi(x) = 0$
  
  the quadratic differential receives $\hbar$-corrections $q_\hbar(x) = q(x) + \mathcal{O}(\hbar)$

- equivalently, pairs $(\mathcal{E}, \nabla_\hbar)$ with $\hbar$-connection $\nabla_\hbar$ and flat section $\nabla_\hbar \Psi(x) = 0$

  locally $\nabla_\hbar = \hbar \partial_x - A(x)$ with $A(x) = \begin{pmatrix} 0 & q_\hbar(x) \\ 1 & 0 \end{pmatrix} \in sl_2(\mathbb{C})$

  $\Psi = \begin{pmatrix} \chi_- & \chi_+ \\ \chi_- & \chi_+ \end{pmatrix}$

  more generally any $A(x) = \begin{pmatrix} \varphi_0 & \varphi_+ \\ \varphi_- & -\varphi_0 \end{pmatrix}$ can be brought to oper form by $\nabla_{\text{Op}} = \hbar^{-1} \cdot \nabla_\hbar \cdot \hbar$

  
  $q_\hbar(x) = \varphi_0^2 + \varphi_+ \varphi_- + \mathcal{O}(\hbar)$
Quantum curves and integrability

Oper with apparent singularities

\[ q_\hbar(x) = \sum_{r=1}^{n} \frac{H_r}{x-z_r} + \frac{\delta_r}{(x-z_r)^2} - \hbar \sum_{i=1}^{d} \left( \frac{v_i}{x-u_i} - \frac{3}{4} \frac{\hbar}{(x-u_i)^2} \right) \]

Constraints determine \( H_r(u, v) \)

\[ \frac{\partial u_k}{\partial z_r} = \frac{\partial H_r}{\partial v_k}, \quad \frac{\partial v_k}{\partial z_r} = - \frac{\partial H_r}{\partial u_k} \]

[Okamoto '86] ensure that the monodromy data \( \mu \) for \( \nabla \hbar \) is unchanged

- \( H_r \) are the Hamiltonians generating isomonodromic deformations
- \( H_r \) generated by the isomonodromic tau-function

\[ H_r(\mu; \hbar) = \partial_{z_r} \log \mathcal{T}(\mu; \mathcal{Z}) \]

The space of monodromy data \( \mathcal{M}_{\text{ch}} = \{ \rho : \pi_1(C) \to SL_2(\mathbb{C}) \} / \sim \)

algebraic structure expressed using trace functions

FG / FN coordinates for \( \mu \) \( \longleftrightarrow \) triangulations / pants decompositions of \( C \)
Use the quantum curve $\hat{\Sigma}$ as a key in the description of $Z_{\text{top}}$

... as analytic objects, ultimately want to describe $Z_{\text{top}}$ as sections of a holomorphic line bundle over the moduli space of quantum curves $\mathcal{X}$

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$\mathcal{T}$-function describes isomonodromic deformations of the D-module

B) Find “good” coordinates $\mathbf{x}$ and corresponding normalisations for tau-functions

$$\mathcal{T}_t(\mathbf{x}_t, \tilde{\mathbf{x}}_t) = \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{2\pi i (\mathbf{n}, \tilde{\mathbf{x}}_t)} Z_{\text{top}}(\mathbf{x}_t + \mathbf{n})$$

Comparison with $Z_{\text{top}}$ derived using the topological vertex $\text{SU}(2)\text{ SYM, pure and } N_f=4$
Coordinates for monodromy data from exact WKB

Exact WKB assigns distinguished coordinates to different regions of $\mathcal{B}$

**Solutions to** $(\hbar^2 \partial_x^2 - q_\hbar(x)) \chi(x) = 0$

\[
\chi^{(b)}_\pm(x) = \frac{1}{\sqrt{S_{\text{odd}}(x)}} \exp \left[ \pm \int^x dx' \ S_{\text{odd}}(x') \right]
\]

\[
S_{\text{odd}} = \frac{1}{2} (S^{(+)} - S^{(-)})
\]

formal series $S^{(\pm)}(x) = \sum_{k=-1}^{\infty} \hbar^k S_k^{(\pm)}(x)$

$S^{(\pm)}(x)$ solutions to the associated Ricatti equation $q_\hbar = \hbar^2 (S^2 + S')$, where $S_k^{(\pm)}$ satisfy recursion relations
Coordinates for monodromy data from exact WKB

Exact WKB assigns distinguished coordinates to different regions of $ℬ$

Solutions to $(\hbar^2 \partial_x^2 - q_\hbar(x)) \chi(x) = 0$

$$\chi_{\pm}^{(b)}(x) = \frac{1}{\sqrt{S_{odd}(x)}} \exp \left[ \pm \int^x dx' \ S_{odd}(x') \right]$$

$$S_{odd} = \frac{1}{2} (S^+ - S^-)$$

formal series $S^{(\pm)}(x) = \sum_{k=-1}^\infty \hbar^k S_{k}^{(\pm)}(x)$

$S^{(\pm)}(x)$ solutions to the associated Ricatti equation $q_\hbar = \hbar^2 (S^2 + S')$, where $S_{k}^{(\pm)}$ satisfy recursion relations

Borel summability of $S$ — away from Stokes lines on $C$

$$\text{Im} \left( e^{-i \text{arg}(\hbar)} \int_a^x dx' \sqrt{q(x')} \right) = 0$$

zero of $q(x)$

$V_\beta = \int_{\beta} dx \ S_{odd}(x) —$ Voros symbols $e^{V_\beta} —$ Borel summable*, define coordinates on $ℬ$

(* depends on the Stokes graph)
Regions in $\mathcal{B}$ are distinguished by types of Stokes graphs

- on the cylinder

[Hollands, Neitzke '13, Hollands, Kidwai '17]
Coordinates for monodromy data from exact WKB

Regions in $\mathcal{B}$ are distinguished by types of Stokes graphs

- on the three-punctured sphere

**FG Stokes graphs**

[Aoki, Tanda '13]

**FN Stokes graphs**
Coordinates for monodromy data from exact WKB

Regions in $\mathcal{B}$ are distinguished by types of Stokes graphs

- on the three-punctured sphere
  [[FG Stokes graphs]]

- on $C_{0,4}$

Borel sums of Voros symbols $e^{V_\beta} \sim$ FG coord. associated to dual WKB triangulation

Transformations of $e^{V_\beta}$ associated to flip of WKB triangulation $\Leftrightarrow$ change of topology of FG-graph take the form of cluster mutations of FG coord. $\Rightarrow$ examples of Stokes phenomena

Coordinates on $\mathcal{Z} \sim V_\beta$

$X_{\gamma'}^I = X_{\gamma'}^I (1 - X_{\gamma})^{(\gamma', \gamma)} \Omega(\gamma)$

[Delabaere, Dillinger, Pham '93; Iwaki, Nakanishi '14; Allegretti '19]

[Bridgeland '16]
Regions in $\mathcal{B}$ are distinguished by types of Stokes graphs

- on the three-punctured sphere
- on $C_{0,4}$

FN coord. relevant in topological strings context

FN Stokes graphs decompose $C$ into annuli & punctured discs
**FN-type coordinates on** $C_{0,4}$

**Monodromy data FN-type coordinates** $(\sigma, \eta)$ on $C_{0,4}$

- quantum curve $\left( \hbar^2 \partial_x^2 - q_\hbar(x) \right) \chi(x) = 0 \iff \left( \hbar \partial_x - A(x) \right) \Psi(x) = 0$

  $$\Psi = \begin{pmatrix} \chi' & \chi' \\ \chi & \chi' \end{pmatrix} \quad A = \begin{pmatrix} 0 & q_\hbar(x) \\ 1 & 0 \end{pmatrix}$$

**From the gluing construction of** $C_{0,4}$

$\Psi(x) \sim$ solutions $\Phi^{(i)}(x_i)$ on $C_{0,3}^{(i)}$ with diagonal monodromy $\rho(\gamma_s) \sim$ eigenvalues $\sigma$

across the connecting cylinder $\Phi^{(2)}(x) = \Phi^{(1)}(x) \text{diag}(e^{-\pi i \eta}, e^{\pi i \eta})$

$\eta$ is fixed by fixing $\Phi^{(i)}(x_i)$

**Freedom in definition of** $\eta$ $\rightarrow$ normalisation factors $n$ (complex numbers)

- can fix a reference $\eta_0$
- other $\eta$ are related to this $e^{-2\pi i \eta_0} = e^{2\pi i n^{(1)} n^{(2)}}$
Coordinates for monodromy data from exact WKB

Regions in $\mathcal{B}$ are distinguished by types of Stokes graphs

- on the three-punctured sphere

FN Stokes graphs

- on $C_{0,4}$

FN coord. $(\eta, \sigma)$ relevant in topological strings context

- FN graph for special $(q, \hbar_*)$; a small perturbation $\hbar \neq \hbar_*$ makes graph saddle free
- Exact WKB analysis determines solutions $\Phi^{(i)}(x_i)$ on $C_{0,3}^{(i)}$
- $\eta$ from relative normalisation of $\pm$ solutions; $\sigma$ defined by their holonomy around $A$
FN-type coordinates

FN & FG coordinates are closely related

- FG-FN transition
  - $C_{0,2}$
  - FN coord. $(U, V)$
    - $U = e^{2\pi i \sigma}$, $V = ie^{2\pi i \eta}$
  - diagonal monodromies
    - $X = \left( \frac{V + V^{-1}}{U - U^{-1}} \right)^2$
    - $Y = \left( \frac{U - U^{-1}}{UV + U^{-1}V^{-1}} \right)^2$

FG coord. $(X, Y)$

FN-coord. $(U, V)$ are limits of FG-coord. $(X, Y)$

- flip + Dehn relabelling
  - $X \to Y^{-1}$
  - $Y \to X(1 + Y^{-1})^{-2}$
  - $U \to U$
  - $V \to UV$

Infinite sequence of flips

$U = \lim_{n \to \infty} \frac{1}{\sqrt{X(n)Y(n)}}$

$V = (U^2 - 1) \lim_{n \to \infty} U^{-n} \sqrt[4]{\frac{X(n)}{Y(n)}}$

GMN "juggle"
FN-type coordinates

FN & FG coordinates are closely related

- FG-FN transition

\[ C_{0,2} \]

\[ \gamma \]

FN coord. \((U, V)\)

\[ X = \left( \frac{V + V^{-1}}{U - U^{-1}} \right)^2, \quad Y = \left( \frac{U - U^{-1}}{UV + U^{-1}V^{-1}} \right)^2 \]

- also FN-FN transitions
Use the quantum curve $\hat{\Sigma}$ as a key in the description of $Z_{\text{top}}$

... as analytic objects, ultimately want to describe $Z_{\text{top}}$ as sections of a holomorphic line bundle over the moduli space of quantum curves $\mathcal{E}$

Points to address

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$\mathcal{T}$-function describes isomonodromic deformations of the D-module

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$\mathcal{T}_t(x_t, \dot{x}_t) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i (n, \dot{x}_t)} Z_{\text{top}}(x_t + n)$

Comparison with $Z_{\text{top}}$ derived using the topological vertex $\text{SU(2) SYM, pure and N}_f=4$
Examples

\[ \mathcal{T}_I(x_I, \dot{x}_I) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i (n, \dot{x}_I)} Z_{\text{top}}(x_I + n) \]

- **Painlevé III tau-function**

\[ q(z) = \frac{\Lambda^2}{z} + \frac{2u}{z^2} + \frac{\Lambda^2}{z^3}; \text{ base curve } C = C_{0,2} \]

irregular singularities at \( z \to 0, \infty \)

\[ \mathcal{T}(\sigma, \eta; t) = \sum_{n \in \mathbb{Z}} e^{4\pi i n \eta} \frac{t^{\sigma^2}}{G(1 + 2\sigma)G(1 - 2\sigma)} \mathcal{F}(\sigma + n, t) \]

\[ \mathcal{T}(\nu, \rho; r) = F(\sigma, \nu) \sum_{n \in \mathbb{Z}} e^{4\pi i n \rho} G(\nu + in, r) \]

[Its, Lisovyy, Tykhyy '15]

\[ \mathcal{T}_I(\tilde{x}_I, \tilde{\tilde{x}}_I) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i (n, \tilde{x}_I)} Z_{\text{top}}(x_I + n) \]

\[ (U, V) \sim \text{FN-coord.} \]

\[ U = e^{2\pi i \sigma}, \quad V = i e^{2\pi i \eta} \]

\[ \text{diag}(U, U^{-1} \rightarrow \text{diag}(V, V^{-1}) \]

\[ X = \left( \frac{V + V^{-1}}{U - U^{-1}} \right)^2, \quad Y = \left( \frac{U - U^{-1}}{UV + U^{-1}V^{-1}} \right)^2 \]

\[ (X, Y) \sim \text{FG-coord.} \]

\[ X = -e^{2\pi \nu}, \quad Y = -e^{8\pi i \rho - 2\pi \nu} \]

\[ \text{difference generating function} \sim \text{transition function for } \mathcal{L} \]
Examples

\[ \mathcal{T}_i(x_i, \hat{x}_i) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i (n, \hat{x}_i)} Z_{\text{top}}(x_i + n) \]

○ Painlevé VI tau-function

\[ q(x) = \sum_{r=1}^{4} \frac{E_r}{x - z_r} + \frac{\delta_r}{(x - z_r)^2} \text{ and } C = C_{0,4} \]

\[ \mathcal{T}_{(\eta)}(\sigma, \eta) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta} G_{(\eta)}(\sigma + n) \]  

[Gamayun, Iorgov, Lisovyy '12]  
[Iorgov, Lisovyy, Teschner '15]

○ the precise form is sensitive to the choice of coordinates

○ change of coordinates \( \leftrightarrow \) change of normalisation  

\[ \mathcal{T}_i(x_i, \hat{x}_i) = F_j(x_i, x_j) \mathcal{T}_j(x_j, \hat{x}_j) \]
Examples

\[ \mathcal{T}_i(x_i, \dot{x}_i) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i(n, \dot{x}_i)} Z_{\text{top}}(x_i + n) \]

- Painlevé VI tau-function
  \[ q(x) = \sum_{r=1}^{4} \frac{E_r}{x - z_r} + \frac{\delta_r}{(x - z_r)^2} \text{ and } C = C_{0,4} \]

\[ \mathcal{T}(\eta) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta} \mathcal{G}(\eta)(\sigma + n) \]

- the precise form is sensitive to the choice of coordinates
- change of coordinates \(\leftrightarrow\) change of normalisation \(\mathcal{T}_i(x_i, \dot{x}_i) = F_{ij}(x_i, x_j)\mathcal{T}_j(x_j, \dot{x}_j)\)

... a closer look at the structure of the series expansion

- \(z\)-corrections controlled by isomonodromy
- Exact WKB controls the normalisation in limit \(z \to 0\)

\[ \mathcal{T}(\eta)(\sigma, \eta; z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta} \mathcal{N}(\eta)(\sigma + n) \mathcal{T}(\sigma + n; z) \]

\[ \mathcal{T}(\eta) = \mathcal{N}(\eta)\mathcal{T}_{\eta_0}, e^{2\pi i \eta_0} = e^{2\pi i n(1)} n(2) \quad \text{where} \quad n(1)n(2) = \frac{\mathcal{N}(\eta)(\sigma)}{\mathcal{N}(\eta)(\sigma - 1)} \]

Voros symbols define \(n(i) = e^{2V(p_i b_i)}\)

[Gamayun, Iorgov, Lisovyy ’12]
[Iorgov, Lisovyy, Teschner ’15]

[Iwaki, Nakanishi ’14, Aoki, Takahashi, Tanda ’17]
Normalisation factors of tau-functions from Exact WKB

**Voros symbols define a relative normalisation of exact WKB solutions**

- The definition of Voros symbols can be generalised to paths starting and ending at poles of \( q \).
- \( p \) is a double pole of \( q \), \( b \) is a branch point in the same chart with coord. \( x \).

\[
V(pb) = V_{>0}^{(pb)} + V_{\leq0}^{(pb)}
\]

\[
\nabla_h \Psi(x) = 0, \quad \Psi = \begin{pmatrix} \chi^- & \chi^+ \\ \chi^- & \chi^+ \end{pmatrix}
\]

- One-to-one correspondence flat sections \( \Psi \) on \( C \leftrightarrow \Psi_i \) on \( C^{(i)}_{0,3} \) in the pants decomposition.
- Reference flat section \( \Psi_i^{(0)} \) defined by \( \psi_{\pm}^{(p_i)} \sim \) distinguished by relative normalisation \( n^{(i)} = 1 \).

Relative normalisation \( e^{\pm V(p_i b_i)} = \psi_{\pm}^{(b_i)}(x) / \psi_{\pm}^{(p_i)}(x) \) captured by the Voros symbol.

Ratios of exact WKB solutions \( n^{(i)} = e^{2V(p_i b_i)} \) relate different coordinates \( x_i \leftrightarrow x_j \) and suitably normalised tau-functions \( \mathcal{F}_i \leftrightarrow \mathcal{F}_j \).

[Iwaki, Nakanishi '14, Aoki, Takahashi, Tanda '17]
Theta series of tau-functions from Exact WKB

To determine theta-series expansions of tau-functions $\mathcal{T}_i(x_i, \dot{x}_i)$ from Exact WKB

$C_{0,4}$ example

(s-type) pants decomposition fixes the $\sigma$ coordinate but not $\eta$

- $\mathcal{S}_2$ -type Stokes graphs on each $C_{0,3}^{(i)} \sim V^{(p_i b_i)}$

$e^{2\pi i n i} n_1 n_2 = e^{2\pi i n_0}$

$\mathcal{T}_{(i)}(\sigma, \eta_i) = \sum_{n \in \mathbb{Z}} e^{2\pi i n i} Z_{\text{top},i} (a + n)$

$n_1 = n(\sigma, \theta_2, \theta_1)$
$n_2 = n(\sigma, \theta_3, \theta_4)$

$\mathcal{N}^{(i)}(\sigma) = \mathcal{N}_2(\sigma, \theta_2, \theta_1) \mathcal{N}_2(\sigma, \theta_3, \theta_4)$

$\mathcal{N}_2(\vartheta_1, \vartheta_2, \vartheta_3) = \frac{1}{(2\pi)^{\vartheta_1}} G(1 + \vartheta_2 + \epsilon \vartheta_1 + \epsilon' \vartheta_3) \prod_{r=1}^{3} G(1 + 2 \vartheta_r)$

[Refs: Aoki, Takahashi, Tanda '17; Iwaki, Kohei, Takei '18]
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$\mathcal{T}_{(i)}(\sigma, \eta_i) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta_i} Z_{\text{top},i}(a + n)$

- change of Stokes graph $S_2 \rightarrow S_s$

- Stokes graph topology and Voros symbol jump

$\eta_j, \mathcal{N}(j)$

$\mathcal{R} \rightarrow \mathcal{R}'$

[Aoki, Takahashi, Tanda ‘17; Iwaki, Kohei, Takei ‘18]
Systematic match: tau-functions associated to FN-graphs reproduce $Z_{\text{top}}$ from vertex

- Normalised tau-functions $\mathcal{T}_*(\sigma, \eta_*)$ admitting expansion
  
  $$\mathcal{T}_*(x, \dot{x}) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i(n, \dot{x})} Z_{\text{top}}(x + n)$$

- Generalised theta-series

- Topological string partition functions in chambers of $\mathcal{M}_{\text{K"ahler}}$

\[
\mathcal{T}_{33}(\sigma, \eta_{33}) = \sum_{n \in \mathbb{Z}} e^{2\pi in\eta_{33}} Z_{33}(a + n)
\]

\[
\mathcal{T}_{s3}(\sigma, \eta_{s3}) = \sum_{n \in \mathbb{Z}} e^{2\pi in\eta_{s3}} Z_{s3}(a + n)
\]

\[
\mathcal{N}_i(\theta_1, \theta_2, \theta_3) = \prod_{\epsilon, \epsilon' = \pm} G(1 + \theta_i + \epsilon \theta_{i+1} + \epsilon' \theta_{i+2}) \quad \text{and} \quad \mathcal{N}_s(\theta_1, \theta_2, \theta_3) = G(1 + \sum_{i=1}^3 \theta_i) \prod_{j=1}^3 G(1 + \sum_{i=1}^3 \theta_i - 2\theta_j) \prod_{r=1}^3 G(1 + 2\theta_r)
\]
Kähler moduli \( t(m) \) are periods of \( ydx \) around cycles of \( \Sigma \)

\[ Z_{\text{top}} \text{ from the topological vertex} \]

Type IIA string theory on the CY\(_3\) mirror dual to \( Y_\Sigma \), the toric CY\(_3\) \( X \)

\( X \) characterised by a toric diagram:

Kähler moduli \( t(m) \) are periods of \( ydx \) around cycles of \( \Sigma \)

Can compute \( Z_{\text{top}}(t; \hbar) \) with the topological vertex

\[ Z_{\text{box}}^{\text{top}} = \mathcal{M}(Q_F)^2 \sum_{Y_1, Y_2} \prod_{i=1}^2 (Q_B/Q_F)^{|Y_i|} \frac{1}{\mathcal{N}_{Y_1 Y_i}(1) \mathcal{N}_{Y_1 Y_2}(Q_F) \mathcal{N}_{Y_2 Y_1}(Q_F^{-1})} \]

\[ \mathcal{M}(Q) = \prod_{i,j=0}^{\infty} (1 - Q q^{i+j+1})^{-1}, \quad |q| < 1 \]

\[ \mathcal{N}_{RP}(Q) = \prod_{(i,j) \in R} \left( 1 - Q q^{R_i + P_j - i - j + 1} \right) \prod_{(i,j) \in P} \left( 1 - Q q^{-P_i - R_j + i + j - 1} \right) \]
**Z**\textsubscript{top} from the topological vertex

Type IIA string theory on the CY\textsubscript{3} mirror dual to \( Y_{\Sigma} \), the toric CY\textsubscript{3} \( X \)

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Kähler moduli \( t(m) \) are periods of \( ydx \) around cycles of \( \Sigma \)

Can compute \( Z_{\text{top}} \) with the topological vertex

\[ Z_{\text{top}} = Z_{\text{in}} Z_{\text{out}} Z_{\text{inst}} \]

\[ Z_{\text{in}} = \frac{\mathcal{M}(Q_{F}) \mathcal{M}(Q_{1}Q_{2}Q_{F})}{\prod_{i=1}^{2} \mathcal{M}(Q_{i}) \mathcal{M}(Q_{i}Q_{F})} \quad Z_{\text{out}} = \frac{\mathcal{M}(Q_{F}) \mathcal{M}(Q_{3}Q_{4}Q_{F})}{\prod_{i=3}^{4} \mathcal{M}(Q_{i}) \mathcal{M}(Q_{i}Q_{F})} \]

\[ Z_{\text{inst}} = \sum_{Y_{1},Y_{2}} (Q_{1}Q_{4}Q_{B})^{y_{2}}(Q_{2}Q_{3}Q_{B})^{y_{1}}q^{\frac{y_{2}q_{1}}{2} - \frac{y_{1}q_{2}}{2}} \prod_{i=1}^{2} s_{Y_{i}}(q^{\rho}) s_{Y_{i}}(q^{\nu}) \times \frac{\prod_{i=2,3} \mathcal{N}_{Y_{1}\phi}(Q_{i}^{-1}) \mathcal{N}_{\phi Y_{2}}(Q_{i}Q_{F}) \prod_{i=1,4} \mathcal{N}_{\phi Y_{2}}(Q_{i}^{-1}) \mathcal{N}_{Y_{1}\phi}(Q_{i}Q_{F})}{\prod_{i=1,4} \mathcal{N}_{Y_{1}Y_{2}}(Q_{F}) \mathcal{N}_{Y_{1}Y_{2}}(Q_{F})} \]

Nekrasov instanton partition function

\[ \mathcal{M}(Q) = \prod_{i,j=0}^{\infty} (1 - Q q^{i+j+1})^{-1}, \quad |q| < 1 \]
Kähler moduli $t(m)$ are periods of $ydx$ around cycles of $\Sigma$

$X$ characterised by a toric diagram:

Can compute $Z_{\text{top}}$ with the topological vertex

$$Z'_{\text{top}} = Z'_{\text{in}} Z_{\text{out}} Z_{\text{inst}}$$

$$Z'_{\text{in}} = \frac{\mathcal{M}(Q_1)}{\mathcal{M}(Q_1^{-1})} Z_{\text{in}}$$

$Z_{\text{top}}(t)$ is not continuous on the parameters $t$

... in the geometric characterisation of $Z_{\text{top}}$ expect this to jump across walls in $\mathcal{B}$
Intuition from string dualities

Predictions from a sequence of string dualities relate $Z_{\text{top}}$ in different pictures

- geometric picture — B-model topological string from type IIB string theory on CY3 $Y$
  
[Maulik, Nekrasov, Okounkov, Pandharipande ’06]

- D-brane picture — type IIA string theory on $\mathbb{R}^3 \times S^1 \times X$ with D6-brane on $S^1 \times X$

  \[ Z_{\text{top}}(t; \hbar) \sim Z_{\text{DT}}(t; \hbar) \] generating function of Donaldson Thomas invariants counting bound states of D-branes

  $D0$-$D2$-$D6$ system

- fermionic picture — free fermion partition function on a quantisation of the SW curve

  \[ Z'_{\text{DT}}(\xi, t; \hbar) \sim Z_{\text{ff}}(\xi, t; \hbar) \]

[Dijkgraaf, Hollands, Sulkowski, Vafa ’08]
Intuition from string dualities

Predictions from a sequence of string dualities relate $Z_{\text{top}}$ in different pictures

$$Z_{\text{ff}}(\xi, \mathbf{t}; \hbar) = \sum_{n \in H^2(X, \mathbb{Z})} e^{n \cdot \xi} Z_{\text{top}}(\mathbf{t} + \hbar \mathbf{n}; \hbar)$$

Identification $Z_{\text{ff}} \sim$ tau-function of the associated integrable system

$$Z_{\text{ff}}(\mu; z) \sim \mathcal{T}(\mu; z)$$

isomonodromic deformations of the quantum SW curve $\hat{\Sigma}$
quantisation replaces the $y$ parameter of $\Sigma$ with $y \rightarrow -i\hbar \partial_x$

Expansion relating the tau-function and $Z_{\text{top}}$

$$\mathcal{T}(\mu(\mathbf{x}, \mathbf{x}^\prime)) \sim \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{2\pi i (\mathbf{n}, \mathbf{x}^\prime)} Z_{\text{top}}(\mathbf{x} + \mathbf{n})$$
Emerging geometric picture

The moduli space of quantum curves $\mathcal{E} = \mathcal{M}_H(Y) \times \mathbb{C}^\times$ torus fibration $\mathcal{M}_H(C) \rightarrow \mathcal{B}$

~ space of $\{[\hbar, \mathcal{E}, \nabla_{\hbar}]\}$

$\mathcal{E}$ holomorphic vector bundle in the Higgs pair $[\mathcal{E}, \varphi]$ $\nabla_{\hbar}$ holomorphic $\hbar$-connection $\hbar \partial_x + A(x)$

Cover $\mathcal{E}$ with local patches $\mathcal{U}_i$ and Darboux coordinates $(x_i(\hbar), \tilde{x}^i(\hbar))$

- from exact WKB analysis define an $\hbar$-family of deformations of $\mathcal{M}_H(Y)$
- asymptotic behaviour $x_r \simeq a_r / \hbar + O(\hbar^0)$, $\tilde{x}^r \simeq \tilde{a}^r / \hbar + O(\hbar^0)$ homological $(a_r, \tilde{a}^r)$ coordinates on $\mathcal{B}$

FG-FG change of coordinates as $\hbar$ crosses a ray $\ell \in \mathbb{C}^\times$ $X^l_{\gamma'} = X^l_{\gamma'} (1 - X_r)_{\gamma', \gamma} \Omega(\gamma)$ determined by BPS invariants $\Omega(\gamma)$ satisfying the Kontsevich-Soibelman WCF

Cluster transformations relate FG-coord. $\rightarrow$ their collection $\equiv$ input to RH problems

include FN coordinates accumulation rays get associated FN-coord.

[Bridgeland ʼ16,ʼ17]
Emerging geometric picture

- The moduli space of quantum curves \( \mathcal{E} = \mathcal{M}_H(Y) \times \mathbb{C}^\times \) torus fibration \( \mathcal{M}_H(C) \to \mathcal{B} \)

  \( \sim \) space of \( \{ [\hbar, \mathcal{E}, \nabla_\hbar] \} \)

  \( \mathcal{E} \) holomorphic vector bundle in the Higgs pair \([\mathcal{E}, \varphi]\)  

  \( \nabla_\hbar \) holomorphic \( \hbar \)-connection \( \hbar \partial_x + A(x) \)

- Cover \( \mathcal{E} \) with local patches \( \mathcal{U}_i \) and Darboux coordinates \( (x_i(\hbar), \tilde{x}_i(\hbar)) \)

  o from exact WKB analysis define an \( \hbar \)-family of deformations of \( \mathcal{M}_H(Y) \)

  o asymptotic behaviour \( x_r \approx a_r / \hbar + \mathcal{O}(\hbar^0) \), \( \tilde{x}_r \approx \tilde{a}_r / \hbar + \mathcal{O}(\hbar^0) \) homological \( (a_r, \tilde{a}_r) \) coordinates on \( \mathcal{B} \)

- Normalised isomonodromic tau-functions \( \mathcal{T}_i(x_i, \tilde{x}_i) \) associated to the quantum curve

  o are sections of a holomorphic line bundle \( \mathcal{L} \) on \( \mathcal{E} \)

  o defined by transition functions \( F_{ij}(x_i, x_j) \) on overlaps \( \mathcal{U}_i \cap \mathcal{U}_j \)

  given by difference generating functions for changes of coordinates \( x_i(x_j, \tilde{x}_j) \)

\[ \mathcal{T}_i(x_i, \tilde{x}_i) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i (n, \tilde{x}_i)} Z_{\text{top}}(x_i + n) \quad \mathcal{T}_i(x_i, \tilde{x}_i) = F_{ij}(x_i, x_j) \mathcal{T}_j(x_j, \tilde{x}_j) \]
Outlook

- Uplift to 5d gauge theories - q-deformed functions, exponential networks
  [Bonelli, Grassi, Tanzini '17, Bershtein, Gavrylenko, Marshakov '17, Bonelli, Del Monte, Tanzini '20]
  [Banerjee, Longhi, Romo '18]

- Relation geometry of hypermultiplet moduli spaces
  the transition functions of $\mathcal{L}$ are equivalent to those of the hyperholomorphic line bundle
  [Alexandrov, Persson, Pioline '10]
  [Neitzke '11]

- 2d-4d wall crossing, isomonodromic tau function & free fermions
  [Gaiotto, Moore, Neitzke '11]
  [Iorgov, Lisovyy, Teschner '14]

- Relation to topological recursion
  [Chekhov, Eynard, Orantin '06, Eynard, Orantin '07, Eynard, Garcia-Failde '19, Iwaki '19]

- Comparisons to other nonperturbative definitions of $Z_{\text{top}}$
  [Grassi, Hatsuda, Marino '14, Grassi, Gu, Marino '19]

Thank you