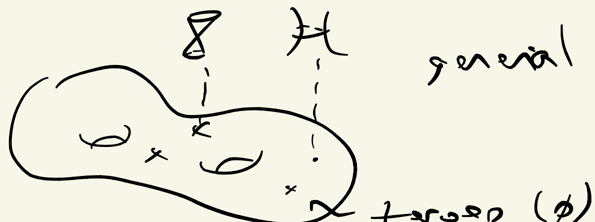
Quadratic Differentials as stability conditions

(joint work with Tom Bridgeland, circa late Cretaceous)

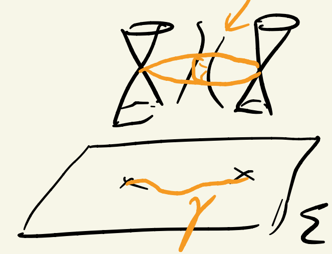
Fix $g \geq 2$, a R. Surface Σ of genus g, & a quadratic differential $\phi \in H^0(K_\Sigma^{\otimes 2})$ with distinct zeroes.

$$\rightsquigarrow \quad Y_\phi = \{ (a,b,c) \in K_\Sigma^{\oplus 3} \mid ab + c^2 = \phi \}$$

$$\begin{array}{c} Y_\phi \\ \downarrow \\ \Sigma \end{array}$$



general fibre T^*S^2 $\quad S^2 \subseteq T^*S^2$



Path γ between zeroes of ϕ

$\rightarrow$ 3-sphere $L_\gamma \subseteq Y_\phi$, Lagrangian.

Consider $\mathcal{F}(Y_\phi)$ the Fukaya category of Y_ϕ generated by the L_γ.

$\exists$ 2:1 cover $\hat{\Sigma} \xrightarrow[2:1]{} \Sigma$? Zeroes (ϕ) : $g(\hat{\Sigma}) = 4g-3$.

$H_3(Y_\phi) = H_1(\hat{\Sigma})^-$ antiinvariant part for involution

rank $6g-6$.

$K^0(\mathcal{F}(Y_\phi)) \longrightarrow H_3(Y_\phi, \mathbb{Z})$.

<u>Conjecture</u>: $\mathrm{Stab}(\mathcal{F}(Y_\phi)) / \mathrm{Auteq}(\mathcal{F}) = \mathrm{Quad}(g)$

q. differentials with simple zeroes

$\underset{\text{open}}{\subset}$ $V \longrightarrow M_g$

$V_S := H^0(K_S^{\otimes 2})$.

<u>Note</u>

There are also 3-folds fibred over Σ with fibre the smoothing of $\mathbb{C}^2/\mathbb{Z}/m$ (A_{m-1}-Milnor fibre).

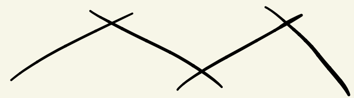
If $\phi_2 \dots \phi_m$; $\phi_i \in H^0(K_\Sigma^{\otimes i})$ Suppose $K_\Sigma = L_1 \otimes L_2$

$$Y_\Phi = \{ (a,b,c) \in L_1^{\otimes m} \oplus (L_1 L_2) \oplus (L_2)^{\otimes m} \mid ac = b^m + \phi_2 b^{m-2} + \dots + \phi_m \}$$

$$\downarrow$$

$$\Sigma$$

general fibre



Plumbing of copies of T^*S^2.

$\exists$ open subset $U \subseteq \widehat{\bigoplus_{i=2}^{m}} H^0(K_\Sigma^{\otimes i})$ s.t. Y_Φ is smooth.

Guess : $U \underset{\text{high codimension.}}{\hookrightarrow} \mathrm{Stab}(\mathcal{F}(Y_\Phi)) / \mathrm{Aut}(\mathcal{F}(Y_\Phi))$

<u>Now</u> $H_3(Y_\Phi)$ & rank $K^0(\mathcal{F}(Y_\Phi)) \gg \dim_{\mathbb{C}} U$.

<u>Variation</u>: Meromorphic quadratic differentials.

Take (S,D), S Riemann surface, $\emptyset \neq D \subseteq S$ divisor.

$\phi \in H^0(K_S(D)^{\otimes 2})$. Suppose ϕ has distinct zeroes & double poles at the points of D.

$K_S(D) = L_1 \otimes L_2$

Consider $Y_\phi \to S$: $\left\{ (a,b,c) \in L_1^2 \oplus L_1 L_2 \oplus L_2^2(-D) \;\middle|\; ac\delta = b^2 + \phi \right\}$ δ cuts out D.

Now $Y_\phi \to S$ (still q-proj. CY total space) has general fibre T^*S^2,  at zeroes of ϕ, fibre $\mathbb{C}^2 \amalg \mathbb{C}^2$ at $p \in D$.

$\exists$ class $\eta \in H^2(Y_\phi; \mathbb{Z}/2)$ P. Dual to the sum of <u>one</u> component of the reducible fibre at p, summed over $p \in D$.

Theorem: (Bridgeland, S.) $\left(\begin{matrix} \text{If } g(S) = 0, & \# D \geq 3 \\ g(S) > 0, & \# D \geq 2 \end{matrix} \right)$

$\exists$ a distinguished subcategory $\mathcal{C} \subseteq \mathcal{F}(Y_\phi; \mathbb{Z})$ s.t.

$\mathcal{C} \underset{?}{=}$ Subcat. gen'd by Lagrangian spheres)

$$\mathrm{Stab}_{\dagger}(\mathcal{C}) \Big/ \mathrm{Aut}_{\dagger}(\mathcal{C}) = \mathrm{Quad}_{\heartsuit}(S)$$

$\mathrm{Quad}_{\heartsuit} \underset{\text{open}}{\supseteq}$ q.d. with distinct zeroes at points other than D.

BUT we have a divisor at infinity where a zero lands

on a point of D & ϕ has a simple pole at that point.

(& $\mathbb{Z}/2$-orbifold structure along that divisor).

Remark: One can allow higher order poles in ϕ; corresponding

$Y_\phi \to S$ has empty fibres at such points of $D_{\geq 2}$.

Now we again consider $\hat{S} \xrightarrow[2:1]{q} S \supseteq \text{zeroes}(\phi)$ ($\&$ odd order poles)

$$H_1(\hat{S}_0)^-$$ ← anti-invariant part

$\hat{S} \setminus$ preimages of poles of order ≥ 2

$$\|$$

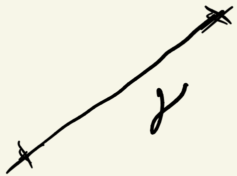
$$H_3(T_\phi) = K^0(\mathcal{C}) \xrightarrow[Z_\sigma]{} \mathbb{C} \quad \text{for } \sigma \in \text{Stab}.$$

Then $q^*\phi = \psi \otimes \psi$ for a meromorphic differential.

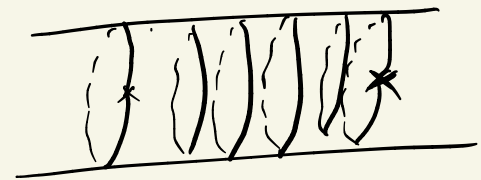
$$Z_\sigma(\gamma) := \int_\gamma \psi \qquad \gamma \in \hat{H}(S) := H_1(\hat{S}_0)^-.$$

Theorem (continued):

For a q. differential ϕ, the stable objects of the associated stability condition are the saddle trajectories for ϕ.



$DT(\gamma) = 1$



$DT(\gamma) = -2$

$\gamma \leftrightarrow L_\gamma \cong S^3 \subseteq Y_\phi$

$\gamma \leftrightarrow$ Lagrangian $S^1 \times S^2 \subseteq Y_\phi$

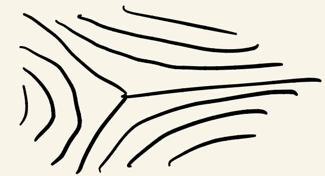
Geometry of a (mero^=) quadratic differential:

If $\phi \in H^0(K_\Sigma^{\otimes 2})$ locally away from zeroes & poles

$\exists$ a co-ordinate z s.t. $\phi = dz^{\otimes 2}$. (Defined up to $z' = \pm z + \text{constant}$)

→ well-defined Euclidean metric, & the horizontal foliation (lines $\mathrm{Im}(z)$ = constant in our "natural" co-ordinate).

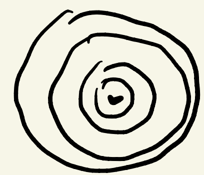
At simple zero of ϕ:



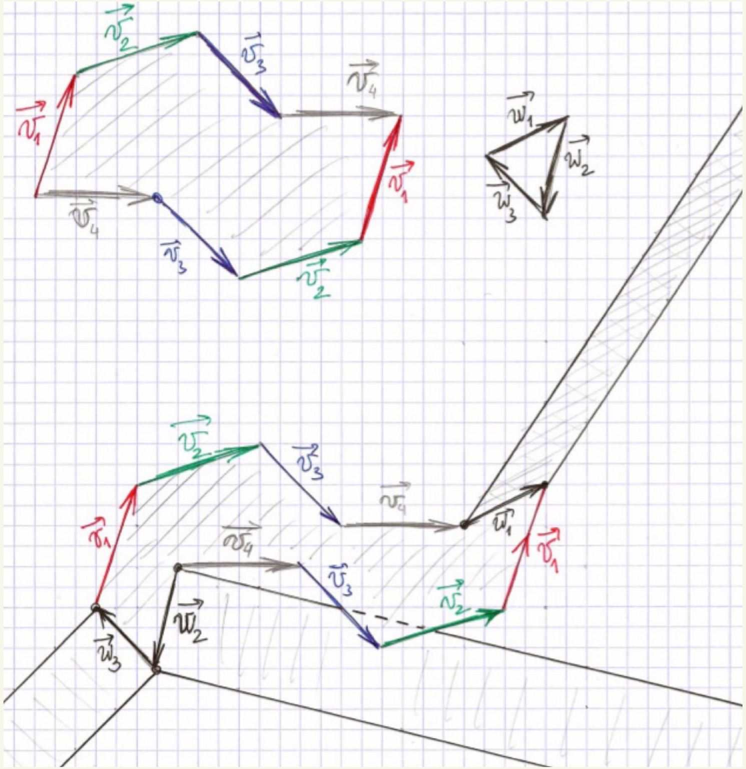
At a double pole of ϕ, $\exists$ a residue $\phi \sim c.\,\dfrac{dz^{\otimes 2}}{z^2}$ $c \in \mathbb{C}$.

$\mathrm{Im}(c) \neq 0$

$c \in \mathbb{R}_{<0}$



Locally metric is like a $\frac{1}{2}$-infinite cylinder.



Gaiotto – Moore – Neitzke :

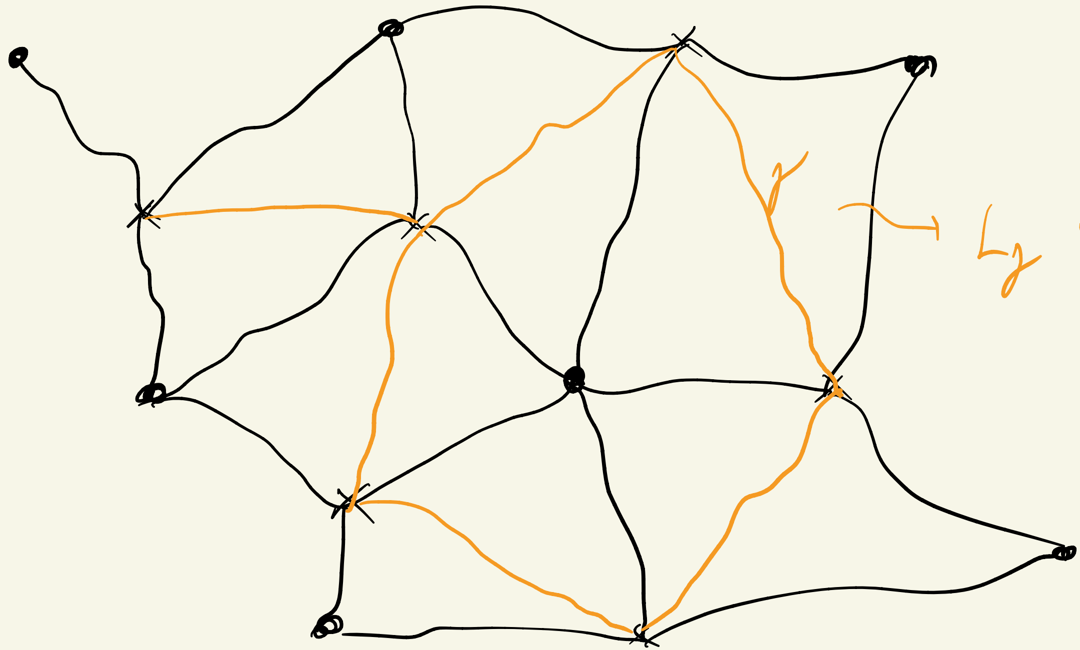
whereas for hol^{c} q. diff^{l},

the set of

$$\{\vartheta \in S^1 \mid \exists \text{ a } \underline{\text{straight}} \text{ trajectory for } e^{-i\vartheta}\phi \text{ between } 2 \text{ zeroes of } \phi\} \text{ is } \underline{\text{dense}}$$

<u>But</u> for ϑ mero^{c} with a pole of order at least 2, this set is closed.

$\phi \in H^0(K_S(D)^{\otimes 2})$ Suppose ϕ is generic.



$\bullet$ = pts of D

$\times$ = zeros of ϕ.

$\longrightarrow L_\gamma \subseteq Y_\phi$.

S is cut into horizontal strips $\rightsquigarrow$ finite distinguished collection of Lagrangian 3-spheres L_γ $\rightsquigarrow$ $\bigoplus_{i,j} HF^*(L_{\gamma_i}, L_{\gamma_j})$ f. dim^1 A_∞-algebra

Using that ϕ is mero$^{\equiv}$, this algebra (over $\mathbb{C}$) is governed by a quiver with potential.

Vertices $\longleftrightarrow$ Lagr. spheres

Edges $\longleftrightarrow$ intersections between spheres

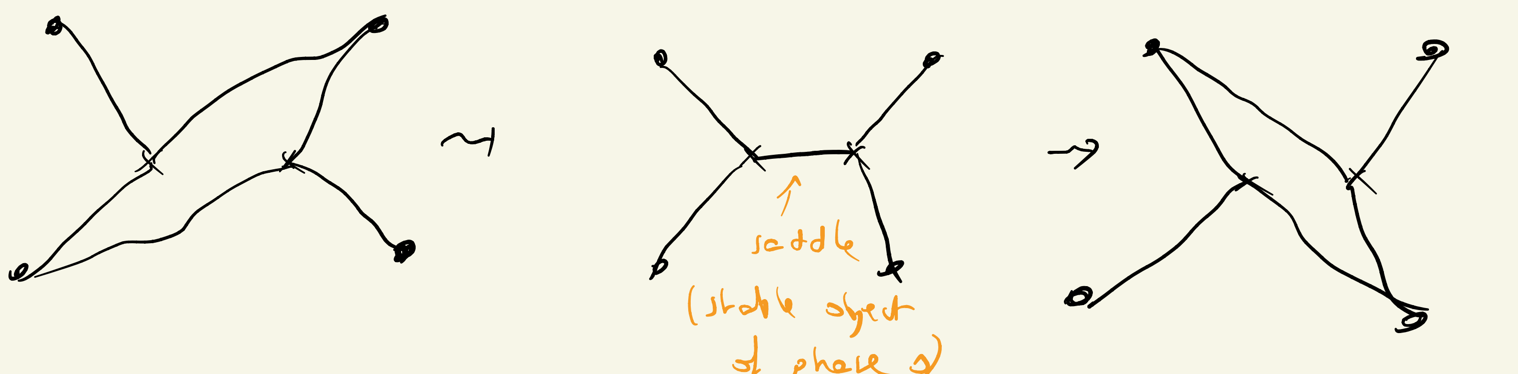
Potential $\longleftrightarrow$ A_∞-operations / holo$^{\equiv}$ polygons.

$$\mathrm{Quad} = \bigcup_{\text{ideal } \Delta\text{ns}} (\text{chambers})$$

locus distinct zeroes & double poles

periods wrt differential of ms

preferred arcs $\gamma \to \mathbb{H}^{\#\text{vert}}$



Special Example: If $g=0$, consider differentials with

a single higher order pole at ∞: pole of order $n+5$:

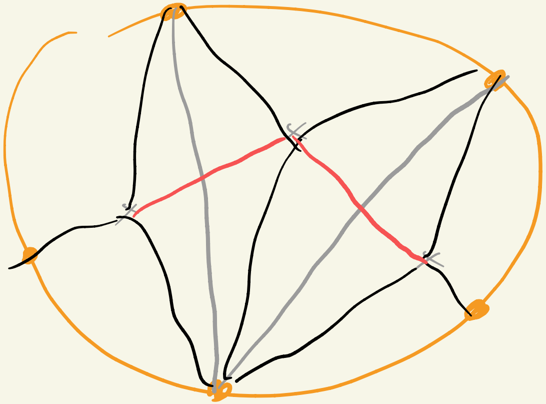
$$\mathrm{Quad}\ (= \mathrm{Quad}_\heartsuit) = \left\{ p_{n+1}(z)\, dz^{\otimes 2} \right\} = \left\{ \begin{array}{l} \text{deg } n+1 \text{ polys} \\ \text{with distinct} \\ \text{zeroes} \end{array} \right\} \Big/ \mathrm{Aut}(\mathbb{P}^1, \infty)$$

$$= \left\{ \prod_{i=1}^{n+1} (z-a_i)\, dz^{\otimes 2} \;\middle|\; a_i \neq a_j,\ \textstyle\sum a_i = 0 \right\} \Big/ \mathbb{Z}/(n+5).$$

Concretely: "A_2-case"

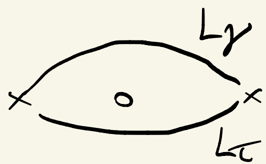
$$\Big\{ \underbrace{(z^2 + az + b)\, dz^{\otimes 2}}_{q(z)\, dz^{\otimes 2}} \;\Big|\; 4a^3 + 27b^2 \neq 0 \Big\}$$

Periods (central charges) $\displaystyle\int_{\gamma_i} \sqrt{q(z)}\, dz$ $\quad \gamma_i \in H_1(E)$, $E: q(z) = w^2$.



Final remarks

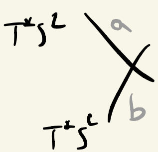
(i) $\mathbb{P}^1$ poles orders 2, 4 : 2 zeroes.



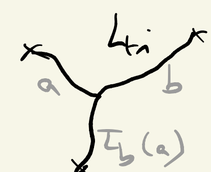
2 matching spheres L_γ, L_τ.

$HF(L_\gamma, L_\tau; \eta) = 0$ BUT in untwisted Fukaya category, $HF(L_\gamma, L_\tau) = \mathbb{C}^2$.

(ii) $Y_\phi \to \Sigma$ has A_2-Milnor fibre as general fibre,



"Tripod" matching spheres



$\rightsquigarrow$  $\exists$ classes $\gamma \in K^0$ s.t. $DT(n\gamma) \approx e^n$.

$$\begin{tikzcd} C \arrow[r, phantom, "\subseteq"] & (\mathbb{C}^*)^2 \\ & Y \arrow[u] \end{tikzcd}$$

$uv = f$ $\quad$ f defines C.

$u, v \in \mathbb{C}$