Calibrated submanifolds $\subset$ Ambient space with special holonomy.

BPS counting:

1) depends only on isotopy class of a more soft structure

2) controlled pointwise by an open domain in $\mathbb{C} \times$ mfd

3) using 1) = Convex cones in $T \times \mathbb{R}$ charge lattice
   Containing $\text{Supp}(\text{BPS})$
1. GW invariants: 
\[(X, \omega) \text{ compact}\]
\[\omega \text{ symplectic metric}\]

Def.: almost complex structure \(J \in \text{End} T_x\)
\[J^2 = -1\]
is compatible with \(\omega\):
\[\omega(J\cdot\cdot\cdot , J\cdot\cdot\cdot ) > 0\]
\[\forall u \in T_x - 0\]

\[\Rightarrow\] Gromov compactness:
\[\{ J\text{-holom. curves} \}\]
of given \(\int_{C} c_{1}(X, \omega)\)
is compact \((\Leftrightarrow \text{bounded area})\)
Charge lattice: $\Gamma = \text{Pic}(X, \mathbb{Z})$

GW invariant: \# of curves of class $\beta \in \Gamma$

generic = $g$

intersecting homology classes

$2$ classes

$2$-cycle of effective cycle for $\Gamma$

same GW inv.$\omega \rightarrow 5 - \omega' - 5' - \omega$

depends only on \{ all symplectic structures on $X$ \}

Corr. corr. of $\omega \in \{ \delta \in H_2(X, \mathbb{Z}) \mid \delta \omega' \geq 0 \}$

w/ con. $\omega$
Open domain:

\[ \forall \omega \in L^2(\mathbb{R}^n)^* \text{ s.t. for} \]

\[ U \omega \subseteq \mathcal{G}_n^1(\mathbb{C}^{2n}) \]

\[ \text{open} \]

\[ = \left\{ V \subseteq \mathbb{C}^n \mid \dim V = n, \quad V \cap \overline{V} = 0 \quad \omega(V, \overline{V}) > 0 \right\} \]

\[ \forall v \in V_0 \]

\[ U \omega \text{ is contractible} \]
Advantage of general Kähler metric vs CY.

Easy: "SYT" collapse as $\theta$ small to $B = \text{Base}$.

$\Rightarrow$ $3$-holomorphic curves

$\approx$ in the limit

$\approx$ $s'$ geodesics in $(S')^*$

Piece-wise linear graphs

Tropical counting
2. Special Lagrangians

(and DT invariants in dim $X = 3$)

$(X^2, w) \rightarrow C^\infty$ symplectic manifold $c_1(T_X) = 0$

$\Omega^{n,0}$ holomorphic form

Kähler–Einstein metric $Ricci = 0$

(Calabi–Yau)

M. Douglas: Strong structure

$\rightarrow$ special Lagrangian $L \subset X^{2n}$ with $\omega|_L = 0$

slope $\theta = \arg (L^{2n})$ (Im $e^{-\theta} \Omega^{2n}$) |$L$ = 0
Weaker structure: \( \Omega \sim (x, w) \)

\[ \Omega \in \Gamma (X, \Pi^h_x) \otimes \mathcal{F} \quad ds^2 = 0 \]

\( \Omega \) prime is pointwise domain

\( \Omega \) prime is open

\[ \tilde{U} \subset \left( \mathbb{R}^n \right)^n \times \mathbb{R}^r \]

\( \tilde{U} \) is open

\( \tilde{U} \) prime is invariant

F. Haider:

\[ \tilde{U} \sim S^1 \] homotopy
Convex cone \( \Gamma = H_\theta^\infty (x, \mathbb{Z}) \cap \mathbb{Z}^\infty (e^{i\theta}(1,2)) \)

\[ C_\theta \subset H_\theta^\infty (x, \mathbb{Z}) \cap \mathbb{Z}^\infty \]

\[ Z(y) = \int \mathcal{M}^o \]

\[ \omega_{\ell}^\infty = 0 \]

\[ (\Omega_\ell = \Re e^{i\theta} \omega_{\ell}^\infty) \|_L > 0 \]

\[ (\Omega_\ell = \Im e^{-i\theta} \omega_{\ell}^\infty) \|_L = 0 \]

Seek:

\[ \Pi_x & \in \Gamma(x, S_n^\infty) \; \mathrm{d} \Pi_x = \nu \]

\[ S_{\Omega_\ell} \|_L > 0 \] automatically \( \Rightarrow \) Support property
Conjecture: weak n.f. from gives \( F(x, w) \) non-arch. field (formal power series)

Bridgeland's Stability structure on \( F(x, w) \)

SS objects of slope \( \Theta \):

May be singular \( \text{spec} \) \( \text{Special} \) Lagrangian \( LC \times \ldots \)

+ object of \( \text{local} \ F(U, x, w) \)

+ solution of Maurer-Cartan equation for perturbation by discs \( D^1 \) (J-holom) \( \overline{D^1 CL} \)
Special case $n = 3$.

Limiting $D T$-invariants "bounding cochain" (solution of Maurer-Cartan).

$L^2 \subset X^6 + \text{local system } G \ni \pi_1(L^2) \to GL(N,\mathbb{C})$

$L^1 \to \infty$ real Higgs field

$= \text{multivalued harmonic } 1.$ form

$L^3 \left[ (L, P) = \text{lim} (L_s, \text{root}=1) \text{ multiple cover} \right]$
Categorical version should work in $A$ dim

Kähler collapse (mod $SU(2)$)

\[
\begin{align*}
X^{2m} \\
Y^{2m} \quad \text{dim } Y < \text{dim } X
\end{align*}
\]

From: $2m-1$ dim CY

\[\text{also Kähler}\]

Iterated stability

\[\text{limiting spec. logs.}\]

\[\text{XZ sls \& folds}\]

\[\text{ZPL graph}\]

\[\text{see recent preprint by Haideh, Katzarkov \& Simpson}\]
3. DT invariants for complex symplectic manifolds

**Strong structure**

\[ L^2 \subset X^{4n} \]

\[ \text{hyper-Kähler manifold} \]

\[ J, \text{holom.} \]

\[ w^j, w^k, \text{lagrangian}. \]

**Weaker structure**

\[ X^{4n} \times \mathbb{C}, \quad w_1 + i w_2 = w_3 \]

two symplectic forms

\[ \text{open domain in} \]

\[ (\mathbb{C}^*)^{12n} \]
$L \subset (X, \omega^{2\nu})$ (could be singular)

holomorphic Lagrangian

$\dim_{\mathbb{C}} L = n$, \quad $\dim_{\mathbb{C}} X = 2n$

Fix generic $\Theta \in \mathbb{R}/2\pi \mathbb{Z}$

Consider Fukaya category $\mathcal{F}(X, \frac{1}{\hbar} \omega^{2\nu}) = \mathcal{F}(X, \Theta)$.

B. Reil Herbst $\Im(\frac{1}{\hbar} \omega^{2\nu})$

$\hbar \in \mathbb{C} - 0$

$0 < |\hbar| \leq 1$

(or better, normal)

along rays

\Rightarrow fully faithful embedding

$\mathcal{F}_{\text{local}}(L) \subset \mathcal{F}(X, \frac{1}{\hbar} \omega^{2\nu})$

Novikov field
Reason. A $J_\theta$-pseudoholomorphic disc $D^2 \subset X$ with $D^2 \subset L$.

Here $J_\theta$: almost complex structure compatible
with $\text{Re}(e^{-i\theta} w^{2\theta}) = \text{Arg} w^{2\theta}$

if $\theta \neq \text{Arg} \left( \int_C w^{2\theta} e^{C_{\theta}} \right)$ for all $\theta \in H_2(X, L; \mathbb{Z})$

charge lattice $T$

Pointwise property. For curve $C$

$$\int_C e^{-i\theta} w^{2\theta} \in \mathbb{R}_{>0}$$
Slopes say, move a bit to the left/right:

\[ \theta \rightarrow \text{Two different fully faithful functors} \]

\[ \mathcal{F}(L) \hookrightarrow \mathcal{F}(X, \frac{1}{h} w') \]

overlap

\[ \rightarrow \text{Automorphism} \]

of \( \mathcal{F}(L)/\mathcal{O}(\mathfrak{g}) \)

several parameters

take domain in \( \text{Hom}(\Gamma, G_m) \)
Conjecture...

Assume \([e^{2i\theta}] = 0\)

\((X\text{ is not compact}\)
\(\Rightarrow X = \frac{1}{1+i} Y\)

\(\mathbb{Z}(\Gamma) \subseteq \mathbb{Z} + \sqrt{1+i}\mathbb{Z}\)

\(\Rightarrow\) composition of \(A\) and \(F\) from \(0^\circ\) to \(90^\circ\) is algebraic

draw this in a log-compactification

(see MK-Y. Sooibelman "Analytic WC"
4. Holomorphic Morse-Novikov Theory

("twisted masses," by G. Moore)

\[ X \text{ real } \rightarrow \text{ manifold (compact, to make life easy)} \]

\[ \alpha \in \Omega^1(X) \otimes \mathbb{C} \quad dd = 0 \]

Only one axiom:

\[ \forall x \in X \text{ either } d|_{T_x}x = 0 \]

or \[ \text{Re} \alpha \wedge \text{Im} \alpha |_{T_x}x \neq 0 \]

\[ \{ u \in (\mathbb{R}^N)^* \big| v \wedge \nabla \neq 0 \} \text{ Local open \mathbb{C}-domain} \]
Such $\lambda$ : zeroes of $\lambda$ is always of even codimension.

"Morse zero" : $\lambda = dQ$

$Q \in \text{Sym}^2 (\mathbb{R}^n)^* \times \mathbb{R}$

$\lambda + i \alpha$

$\det (\lambda_i Q + \alpha_2 Q) \neq 0$

$\forall (\alpha_i, \alpha_2, \neq 1, 0) < 12$

$\rightarrow \eta \in \mathbb{Z}$ both $\alpha_i, \alpha_2$

have signature $(\eta, \eta)$
If $X$ is $C^r$ manifold, $\alpha$ is holom. closed 1-form.

$\gamma_i = \text{graph}(\alpha) = L_i$

$\omega|_L$ is symplectic but $L = L_1 \cup L_2$ can be obstructed for some $\theta$.
Theorem: If \( \Gamma = H_i(\mathbb{X}, \text{zeroes}(d), \mathbb{Z}) \rightarrow \mathcal{D} \) takes values in \( \mathbb{Z} + i \cdot \mathbb{Z} \) (generalizes square tiled surfaces), then

\[ \Rightarrow \text{Rational isomorphisms (in the sense of alg geom) between } H^i(\text{zeroes}, \text{vanishing cycles}) \subseteq H^i(\mathbb{X}, \mathcal{D}) \]

\[ \Rightarrow \text{Gel matrix } \text{Mat}(M \times M) - \text{valued rational function in } N \text{ variables} \]

\[ M = r_k H^i(\text{zeroes}, \text{vanishing cycles}), \quad N = \text{rk } \Gamma = r_k H_i(\mathbb{X}, \text{zeroes}) \]
Simplest example: holomorphic torus with 2 torus square-tiled.

\[
\begin{vmatrix}
1 + \frac{(a^3 f - a^2 f)(b^2 g - b^3 g) + (a - a^3 f)(b^3 g - b)}{(1-a^3 f)(1-b^3 g)} & 1 + \frac{(a-a^3 f)b^2 + (a^3 f - a^2 f)b}{(1-a^3 f)(1-b^3 g)} - \frac{(a^2 f - a f)b}{(1-a^3 f)(1-b)} \\
\frac{a^2(b^3 g - b) + a(b^2 g - b^3 g)}{(1-a^3 f)(1-b^3 g)} - \frac{a(bg - b^2 g)}{(1-a)(1-b^3 g)} & 1 + \frac{a^2 b^2 + a b}{(1-a^3 f)(1-b^3 g)} - \frac{a^3 f b}{(1-a^3 f)(1-b)} - \frac{a b^3 g}{(1-a)(1-b^3 g)}
\end{vmatrix}
\]

\( \in \text{SL}(2, \mathbb{Q}(a, b, f, g)) \)

\( T = \prod_{ \theta \in \mathbb{R} } T_{\theta} \)

\( T_{\theta} = T_{\theta} (a^{k_1} b^{k_2}, f, g) \)

\( k_1/k_2 = \tan \theta \)

Counts saddle connections with slope \( \theta \).