12/10/2020

INTRODUCTION TO BRIDGELAND STABILITY

AIM WORKSHOP: STABILITY IN MIRROR SYMMETRY

OUTLINE:

1) Motivation
2) Triangulated categories
3) Bridgeland stability conditions
4) Space of stability conditions

"DUY Theorem" (Donaldson '85, Uhlenbeck - Yau '86):

X ... compact Kähler, $E \to X$ ... holomorphic vector bundle

E admits HYM metric $\Leftrightarrow$ E is slope polystable

Hermitian metric h on E with associated connection D with curvature $F = F_D$ such that:

$$F \wedge \omega^{n-1} = \lambda \omega^n$$

(λ constant)

$$\mu(E) := \frac{1}{rk(E)} \int_X c_1(E) \wedge \omega^{n-1}$$

E is stable: $\forall\ F \in Coh(E)$,

$$0 \neq F \subsetneq E \Rightarrow \mu(F) < \mu(E)$$

polystable = direct sum of stable bundles of the same slope

geometric PDE problem

algebraic criterion

Analogy, loosely based on Mirror Symmetry:

complex manifold (X, J)	symplectic manifold (M, ω)
Kähler $(1,1)$-form ω	compatible cx. structure J and holomorphic vol. form Ω
holomorphic bundle E with metric h	Lagrangian submanifold $L \subset M$
changing h	Hamiltonian isotopy of L
HYM condition	special Lagrangian condition
$\mathrm{Coh}(X)$	$\mathrm{Fuk}(M)$
slope stability	??

not abelian category!

Triangulated category : 1) morphism $f: X \to Y$ has cone

(vague idea) 2) suspension $X \mapsto \mathrm{Cone}(X \to *)$

is invertible (c.f. stable homotopy theory)

cochain complexes

- suspension = shift

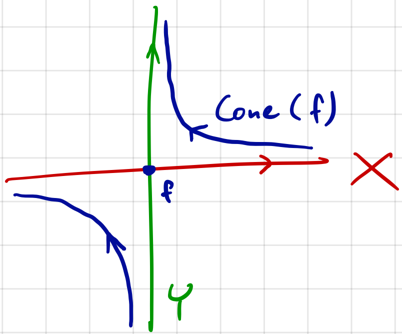
$$(X^\bullet, d)[1] := (X^{\bullet+1}, -d)$$

- $\mathrm{Cone}(f: X \to Y) :=$

$$= \left(X[1] \oplus Y \,,\, \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix} \right)$$

Fukaya category

- shift choice of $\mathrm{Arg}(\Omega|_L)$ by π
- cone often corresponds to surgery on $X \cup Y$



Triangulated Categories

A triangulated category is given by:

1) additive category $\mathcal{C}$ (zero object, $\oplus$)
2) auto-equivalence $[1] : \mathcal{C} \to \mathcal{C}$, shift
3) distinguished sequences, the exact triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

satisfying conditions which axiomatize properties of sequences $X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to X[1]$ of chain complexes.

BRIDGELAND STABILITY

$\mathcal{C}$ ... triangulated category

Grothendieck group $K_0(\mathcal{C}) :=$ abelian group generated by $Ob(\mathcal{C})$

with relations $X + Z = Y$ if $X \to Y \to Z \xrightarrow{+1}$ exact

fix $\Gamma \cong \mathbb{Z}^r$, homomorphism $cl : K_0(\mathcal{C}) \longrightarrow \Gamma$

A stability condition on $\mathcal{C}$ is given by

1) additive map $Z : \Gamma \longrightarrow \mathbb{C}$, central charge

2) full additive subcategories $\mathcal{C}_\phi \subset \mathcal{C}$, $\phi \in \mathbb{R}$

of semistable objects of phase ϕ

such that:

1) $\mathcal{C}_\phi[1] = \mathcal{C}_{\phi+\pi}$

2) $\phi_1 < \phi_2$, $A_i \in \mathcal{C}_{\phi_i}$ $\Rightarrow$ $\mathrm{Hom}(A_2, A_1) = 0$

3) $\forall E \in \mathcal{C}$ $\exists$ tower of exact triangles

$$
\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n \cong E \arrow[dl] \\
& A_1 \arrow[ul, "+1"] & & A_2 \arrow[ul, "+1"] & & & & A_n \arrow[ul, "+1"] &
\end{tikzcd}
$$

with $0 \neq A_i \in \mathcal{C}_{\phi_i}$, $\phi_1 > \phi_2 > \dots > \phi_n$

Harder - Narasimhan filtration

4) $0 \neq A \in \mathcal{C}_\phi \Rightarrow Z(A) := Z(cl(A)) \in \mathbb{R}_{>0}\, e^{i\phi}$

5) $\exists$ norm $\|.\|$ on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, $C > 0$:

$$\|cl(A)\| \leq C\,|Z(A)|, \quad A \in \mathcal{C}_\phi$$

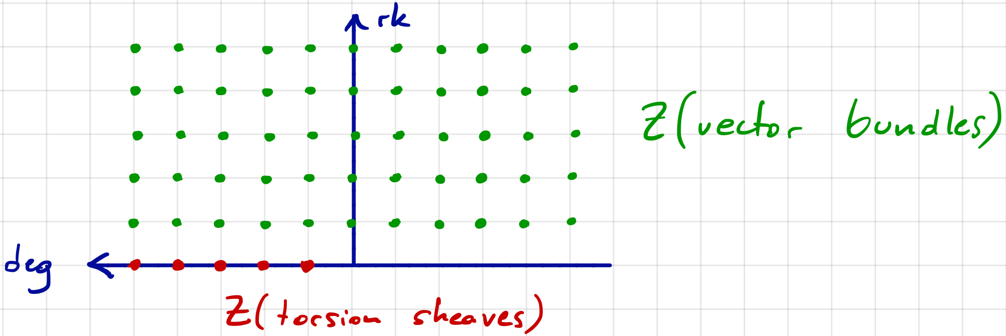
support property

(That's it!)

Example: $\mathcal{C} = D^b(\mathrm{Coh}(\text{smooth curve } X/\mathbb{C}))$

$$\Gamma = K_{num}(\mathcal{C}) = \mathbb{Z}^2 \xleftarrow[cl = (rk, deg)]{} K_0(\mathcal{C})$$

$$Z(E) := -\deg E + i\, rk(E)$$



semistable objects:

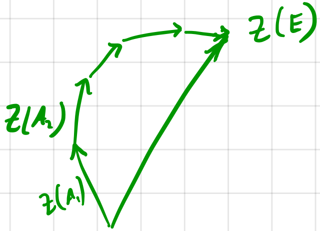
$\mathcal{C}_\pi$ = torsion sheaves on X

$\phi \in (0,\pi)$: $\mathcal{C}_\phi$ = slope-semistable vector bundles on X of slope $\mu = \tan(\phi - \frac{\pi}{2})$

Topology on $\mathrm{Stab}(\mathcal{C})$

Given the HN filtration of an object E:

$$\begin{cases} \begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[dl, "+1"'] \arrow[rr] & & E_2 \arrow[dl, "+1"'] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n \cong E \arrow[dl] \\ & A_1 \arrow[ul] & & A_2 \arrow[ul] & & & & A_i \arrow[ul, "+1"] & \end{tikzcd} \\ \text{with} \quad 0 \neq A_i \in \mathcal{C}_{\phi_i}, \quad \phi_1 > \phi_2 > \dots > \phi_n \end{cases}$$



define $m(E) = \sum' |Z(A_i)|$, $\phi_-(E) = \phi_n$, $\phi_+(E) = \phi_1$

Set of stability conditions: $\mathrm{Stab}(\mathcal{C}) = \mathrm{Stab}(\mathcal{C}, \Gamma, cl)$ has natural topology by requiring that

- Z converges in $\mathrm{Hom}(\Gamma, \mathbb{C}) \cong \mathbb{C}^r$
- $m(E)$, $\phi_-(E)$, $\phi_+(E)$ converge uniformly in E

(topology is metrizable)

Stab($\mathcal{C}$) AS COMPLEX MANIFOLD

THEOREM (Bridgeland): The map

$$\begin{aligned} \mathrm{Stab}(\mathcal{C}) &\longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C}) \cong \mathbb{C}^r \\ ((\mathcal{C}_\phi)_{\phi\in\mathbb{R}}, Z) &\longmapsto Z \end{aligned}$$

is a local homeomorphism. (Intuitively: If Z is changed by small amount, one can compute new sets $\mathcal{C}_\phi, \phi \in \mathbb{R}$ in a canonical way.)

COROLLARY: Stab($\mathcal{C}$) has the structure of a complex manifold of dimension $r = \mathrm{rk}\,\Gamma$.

REFORMULATION WITH t-STRUCTURES

Fix stability condition on $\mathcal{C}$.

$\rightsquigarrow$ full subcategory

$\mathcal{A} := \mathcal{C}_{(0,\pi]} := \{ E \in \mathcal{C} \mid 0 < \phi_-(E) \leq \phi_+(E) \leq \pi \} \subset \mathcal{C}$

is abelian! Also $K_0(\mathcal{C}) = K_0(\mathcal{A})$.

Conversely, if $\mathcal{A} \subset \mathcal{C}$ is heart of bounded t-structure (e.g. $\mathcal{A} \subset D^b(\mathcal{A})$), $Z : \Gamma \to \mathbb{C}$ with

1) $0 \neq E \in \mathcal{A} \Rightarrow Z(E) \in \{ \operatorname{Im} z > 0 \} \cup (-\infty, 0)$

2) any $E \in \mathcal{A}$ has HN filtration with slope-semistable factors

then $(\mathcal{A}, Z)$ determines stability condition on $\mathcal{C}$.