Line defects, UV-IR map, and exact WKB

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Outlook

Based on review of work by Gaiotto-Moore-Neitzke, as well as some recent developments.
4d N=2 theories of class S

Class S theories $\mathcal{T}[\mathfrak{g}, C]$ are 4d N=2 supersymmetric theories originating from twisted compactification of a 6d (2, 0) theory of type $\mathfrak{g}$ ($\mathfrak{g} \in \{A, D, E\}$) on a Riemann surface $C$ with appropriate decorations (punctures or twisted lines), in the zero area limit of $C$.

[Gaiotto],[Gaiotto-Moore-Neitzke]
Class $S$ theories $\mathcal{T}[\mathfrak{g}, C]$ are 4d $N=2$ supersymmetric theories originating from twisted compactification of a 6d $(2, 0)$ theory of type $\mathfrak{g}$ ($\mathfrak{g} \in \{A, D, E\}$) on a Riemann surface $C$ with appropriate decorations (punctures or twisted lines), in the zero area limit of $C$.

[1],[2],[3],[4]

$\mathcal{T}[\mathfrak{g}, C]$ has a subspace of vacua $\mathcal{B}$ called the Coulomb branch, where the low energy effective theory is $U(1)^r$ gauge theory.

\[
\mathcal{B} \subset \bigoplus_{k=1}^{r} H^0 \left( C, K_C^{\otimes d_k} \left( \sum_i p_{d_k}^{(i)} z_i \right) \right)
\]

[1],[2],[3],[4] ... 

A point in $\mathcal{B}$ corresponds to a branched covering $\tilde{C} \to C$, where $\tilde{C} \subset T^* C$ is the Seiberg-Witten curve.
Relation to Hitchin systems

Further compactifying $\mathcal{T}[\mathfrak{g}, C]$ on $S^1_R$, the low energy effective theory is a 3d $N = 4$ sigma model with target space $M_H(G, C)$. 
Relation to Hitchin systems

Further compactifying $\mathcal{T}[\mathfrak{g}, C]$ on $S^1_R$, the low energy effective theory is a 3d $N = 4$ sigma model with target space $M_H(G, C)$. Starting from 6d and changing the order of compactification on $C \times S^1_R$ [Gaiotto-Moore-Neitzke], $M_H(G, C)$ is identified with the moduli space of solutions to Hitchin's equations:

\[
F_A + R^2 [\Phi, \bar{\Phi}] = 0,
\]
\[
\bar{\partial}_A \Phi = 0, \quad \partial_A \bar{\Phi} = 0.
\]

$\partial + A$ is a $G$-connection in a top. trivial $G$-bundle $V \to C$, $\Phi \in \Omega^{1,0}(\text{End} V)$ is the Higgs field.

$(A, \Phi)$ have specified boundary conditions at punctures of $C$. (regular: simple pole, irregular: higher order pole)

Today we take $\mathfrak{g} = \mathfrak{gl}(N)$ or $\mathfrak{sl}(N)$, $G = U(N)$ or $SU(N)$.
Relation to Hitchin systems

$M_H(G, C)$ is hyperkähler, has a $\mathbb{CP}^1$-worth of complex structures $J_\zeta$. Different $J_\zeta$ expose different features of $M_H(G, C)$:

[Hitchin], [Simpson], [Biquard-Boalch], [Gaiotto-Moore-Neitzke]...

- $\zeta \in \mathbb{C}^\times$: Hitchin’s equations indicate $\partial + A$ is flat, with

$$\mathcal{A} := \frac{R}{\zeta} \Phi + A + R\zeta \bar{\Phi}$$

$(M_H, J_\zeta)$ diff. to a moduli space of flat $G_\mathbb{C}$-connections on $C$. 
Relation to Hitchin systems

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- $ζ ∈ \mathbb{C}^\times$: Hitchin’s equations indicate $\partial + A$ is flat, with

$$A := \frac{R}{ζ} Φ + A + Rζ \bar{Φ}$$

$(M_H, J_ζ)$ diff. to a moduli space of flat $G_\mathbb{C}$-connections on $C$.

- $ζ = 0$: $(M_H, J_0)$ diff. to moduli space of Higgs bundles $M_{Higgs}$, which is a complex integrable system: $M_{Higgs} → B$ with generic fiber being compact tori.

The Seiberg-Witten curve $\tilde{C}$ identified with spectral curve:

$$\tilde{C} = \{(z ∈ C, λ ∈ T^*_z C) : \text{Det} (Φ(z) − λ) = 0\} ⊂ T^* C$$
Line defects in class S theories

\[ \mathcal{T}[g, C] \] admits families of line defects \( \mathbb{L}(\zeta) \) extending along \( \mathbb{R}^t \)-direction, where \( \zeta \in \mathbb{C}^\times \) parametrizes preserved supercharges. [Kapustin], [Kapustin-Saulina], [Drukker-Morrison-Okuda], [Drukker-Gaiotto-Gomis], [Drukker-Gomis-Okuda-Teschner] [Gaiotto-Moore-Neitzke], [C´ordova-Neitzke], [Aharony-Seiberg-Tachikawa], [Moore-Royston-van den Bleeken]...

\( \mathbb{L}(\zeta, p, \mathcal{R}) \) depends on path \( p \) on \( C \), carrying representation \( \mathcal{R} \) of \( g \).
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[Gaiotto-Moore-Neitzke], [Cárdova-Neitzke], [Aharony-Seiberg-Tachikawa], [Moore-Royston-van den Bleeken]…

\( \mathbb{L}(\zeta, p, \mathcal{R}) \) depends on path \( p \) on \( C \), carrying representation \( \mathcal{R} \) of \( g \).

- \( g = A_1 \), \( C \) only has regular punctures:  
  \( p \) is a non-self-intersecting closed curve on \( C \).

- \( C \) contains irregular punctures:  
  \( p \) corresponds to integral laminations [Fock-Goncharov], [Gaiotto-Moore-Neitzke], collection of paths either closed or open with ends on marked points corres. to Stokes directions at irregular punctures.

- In general \( p \) could contain junctions, where paths carrying different \( \mathcal{R}_i \) meet, associated with certain \( g \)-invariant tensor.  
  [Sikora], [Le], [Xie], [Saulina], [Coman-Gabella-Teschner], [Tachikawa-Watanabe], [Gabella]…
Upon circle compactification, the vacuum expectation values of $\mathbb{L}(\zeta)$ wrapping $S^1_R$ are $J_\zeta$-holomorphic functions on $M_{\text{flat}}(G_{\mathbb{C}}, C)$. 

[Gaiotto-Moore-Neitzke]
Line defects in class S theories

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\[ \text{[Gaiotto-Moore-Neitzke]} \]

- \( g = A_1, C \) has only regular punctures:

\[
\langle \mathbb{L}(\zeta, p, R) \rangle = \text{Tr}_R \text{Hol}_p \left( \frac{R\Phi}{\zeta} + A + R\zeta\bar{\Phi} \right)
\]

\[ = \text{Tr}_R \text{Hol}_p A(\zeta). \]

The dependence on \( p \) is only through its homotopy class.

- In general, compute parallel transport of \( A(\zeta) \) along paths, contract together via \( g \)-invariant tensors.
The UV-IR map for line defects

A useful way to study $\mathbb{L}(\zeta)$ in class S theories, is deforming to a point $u$ on the Coulomb branch $\mathcal{B}$ and follow the defect into IR.

\[ \mathbb{L}(\zeta) \]
The UV-IR map for line defects

A useful way to study $\mathbb{L}(\zeta)$ in class S theories, is deforming to a point $u$ on the Coulomb branch $\mathcal{B}$ and follow the defect into IR.

The IR limit of $\mathbb{L}(\zeta)$ is a superposition of supersymmetric line defects in the abelian theory, with integer coefficients in this superposition given by framed BPS index $\overline{\Omega}(\mathbb{L}(\zeta), \gamma, u)$.

[Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Cirafici-Del Zotto],[Coman-Gabella-Teschner],
[Moore-Royston-van den Bleeken],[Ito-Okuda-Taki],[Galakhov-Longhi-Moore], . . .

The UV-IR map for line defects:

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_{\gamma} \overline{\Omega}(\mathbb{L}(\zeta), \gamma, u) X_{\gamma}(\zeta)$$

$X_{\gamma}(\zeta)$ represent IR Wilson-’t Hooft lines with charge $\gamma$. 
The IR line defects

Recall that, a point $u$ on the Coulomb branch $B$ corresponds to a branched covering $\tilde{C} \rightarrow C$ (spectral curve/Seiberg-Witten curve): Geometrically the IR line defect $X_\gamma(\zeta)$ correspond to loops $\tilde{p} \subset \tilde{C}$ in class $\gamma \in H_1(\tilde{C}, \mathbb{Z})$. 
The IR line defects

Upon circle compactification, the VEV $X_\gamma(\zeta)$ of $X_\gamma(\zeta)$ wrapping $S^1_R$ are local Darboux coordinates on $M_{\text{flat}}(G_\mathbb{C}, C)$: Fock-Goncharov, complexified Fenchel-Nielsen, or more general spectral coordinates.

[Fock-Goncharov],[Fenchel-Nielsen],[Gaiotto-Moore-Neitzke],[Nekrasov-Rosly-Shatashvili],

[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev],[Jeong-Nekrasov],[Coman-Longhi-Teschner] ...

c.f. Bridgeland’s talk and Coman’s talk
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[Hollands-Neitzke], [Hollands-Kidwai], [Allegretti], [Nikolaev], [Jeong-Nekrasov], [Coman-Longhi-Teschner] ... c.f. Bridgeland’s talk and Coman’s talk

- The holomorphic Poisson brackets are given by

$$\{X_\gamma, X_{\gamma'}\} = \langle \gamma, \gamma' \rangle X_\gamma X_{\gamma'}$$

- $X_\gamma$ has nice asymptotic behavior as $\zeta \to 0$ and $\zeta \to \infty$; has discontinuities across “BPS rays” controlled by Kontsevich-Soibelman symplectomorphisms. $X_\gamma$ are solutions to a certain Riemann-Hilbert problem.

[Gaiotto-Moore-Neitzke], [Gaiotto], [Bridgeland], [Barbieri], [Bridgeland-Barbieri-Stoppa]...
The UV-IR map as the trace map

- $\langle L(\zeta) \rangle$ are $J_\zeta$-holomorphic trace functions on $M_{\text{flat}}(G_\mathbb{C}, C)$.
- $\mathcal{X}_\gamma(\zeta) := \langle X_\gamma(\zeta) \rangle$ are Darboux-coordinates on $M_{\text{flat}}(G_\mathbb{C}, C)$.
The UV-IR map as the trace map

- $\langle \mathbb{I}(\zeta) \rangle$ are $J_\zeta$-holomorphic trace functions on $M_{\text{flat}}(G_C, C)$.
- $\mathcal{X}_\gamma(\zeta) := \langle \mathcal{X}_\gamma(\zeta) \rangle$ are Darboux-coordinates on $M_{\text{flat}}(G_C, C)$.
- The UV-IR map then implies the trace map:

$$\text{Tr}_R \text{Hol}_p \mathcal{A}(\zeta) = \sum_{\gamma} \overline{\Omega}(\mathbb{I}(\zeta), \gamma) \mathcal{X}_\gamma(\zeta)$$

$\mathcal{A}(\zeta)$: flat $G_C$-connection on $C$, $p$: path on $C$.
- Alternatively we can understand $\mathcal{X}_\gamma(\zeta)$ as:

$$\mathcal{X}_\gamma(\zeta) = \text{Hol}_\gamma \nabla^{ab}(\zeta), \gamma \in H_1(\tilde{C}, \mathbb{Z}).$$

The UV-IR map is interpreted as a nonabelianization map.

[Gaiotto-Moore-Neitzke],[Hollands-Neitzke]
Line defects OPE

The algebra structure on space of $J_\zeta$-holomorphic functions corresponds to line defects operator products (OPE):

$$\langle \mathbb{L}_1(\zeta)\mathbb{L}_2(\zeta) \rangle = \langle \mathbb{L}_1(\zeta) \rangle \langle \mathbb{L}_2(\zeta) \rangle$$

This algebra structure admits a quantization via skein algebras.

[Reshetikhin-Turaev], [Turaev], [Witten], [Alday-Gaiotto-Gukov-Tachikawa-Verlinde], [Gaiotto-Moore-Neitzke], [Drukker-Gomis-Okuda-Teschner], [Tachikawa-Watanabe], [Coman-Gabella-Teschner], [Gabella]...
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Turning on $\Omega$-deformation on a $\mathbb{R}^2$-plane, susy line defects sit along a fixed axis in $\mathbb{R}^3 \rightsquigarrow$ non-commutative associative OPE $\ast$

[Nekrasov-Shatashvili], [Gaiotto-Moore-Neitzke], [Ito-Okuda-Taki], [Yagi], [Oh-Yagi],...
Line defects OPE

- In the UV, non-commutative line defects OPE is complicated.
- In the IR, the OPE is given by quantum torus algebra:
  \[ X_{\gamma_1}(\zeta) \ast X_{\gamma_2}(\zeta) = (-q)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}(\zeta). \]

  \( \langle \cdot, \cdot \rangle \) is the Dirac-Schwinger-Zwanziger pairing, identified with intersection pairing in \( H_1(\tilde{C}, \mathbb{Z}) \).

  In particular,
  \[ X_{\gamma_1}(\zeta) \ast X_{\gamma_2}(\zeta) = (-q)^{2\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_2}(\zeta) \ast X_{\gamma_1}(\zeta) \]

  quantizes the Poisson algebra of coordinate functions \( X_{\gamma} \).
- This hints existence of a quantum trace map, embedding the UV skein algebra into a quantum torus algebra.
The UV skein algebra

For example, taking $\mathfrak{g} = \mathfrak{gl}(2)$, the space of line defects equipped with OPE is described by the $\mathfrak{gl}(2)$ HOMFLY skein algebra of $M = \mathbb{C} \times \mathbb{R}_h$, defined as the space of formal $\mathbb{Z}[q^{\pm 1}]$-linear combinations of framed oriented links (up to isotopy) in $M$, modulo the following relations: (N=2 in the figure)

\[(I)\quad - \quad = (q - q^{-1})\]

\[(II)\quad = q^N\]

\[(III)\quad = \frac{q^N - q^{-N}}{q - q^{-1}}\]

The algebra structure is defined by “stacking” links along the $\mathbb{R}^h$-direction.
The IR skein algebra

The IR line defects correspond to framed oriented links in $\tilde{\mathcal{M}} = \tilde{\mathcal{C}} \times \mathbb{R}$. The OPE algebra is (twisted) $\mathfrak{gl}(1)$ skein algebra of $\tilde{\mathcal{M}}$, defined as the space of formal $\mathbb{Z}[q^{\pm 1}]$-linear combinations of framed oriented links (up to isotopy) in $\mathcal{M}$, modulo the following relations:

This skein algebra is isomorphic to the quantum torus.
The quantum trace map

The quantum trace map $F$ is a homomorphism from the UV skein algebra to the IR quantum torus algebra, physically it represents a $q$-deformed UV-IR map:

$$
\mathbb{L}(\zeta) \mapsto \sum_\gamma \overline{\Omega}(\mathbb{L}(\zeta), \gamma, u, q) X_\gamma(\zeta)
$$

$$
\overline{\Omega}(\mathbb{L}(\zeta), \gamma, u, q) := \text{Tr}_{\mathcal{H}_{\mathbb{L}, \gamma, u}}(-q)^{2J_3} q^{2l_3} e^{-\beta\{Q, Q^+\}} \in \mathbb{Z}[q, q^{-1}]
$$

[Gaiotto-Moore-Neitzke], [Fock-Goncharov], [Bonahon-Wong], [Córdova-Neitzke], [Cirafici-Del Zotto], [Coman-Gabella-Teschner], [Allegretti], [Gabella], [Ito-Okuda-Taki], [Galakhov-Longhi-Moore], [Neitzke-Y], [Cho-Kim-Kim-Oh], [Le], [Kim-Son], [Korinman-Quesney], [Goncharov-Shen], [Douglas-Sun]...
The quantum trace map

The quantum trace map $F$ is a homomorphism from the UV skein algebra to the IR quantum torus algebra, physically it represents a $q$-deformed UV-IR map:

$$\mathbb{L}(\zeta) \mapsto \sum_{\gamma} \Omega(\mathbb{L}(\zeta), \gamma, u, q) \chi_{\gamma}(\zeta)$$

$$\Omega(\mathbb{L}(\zeta), \gamma, u, q) := \text{Tr}_{\mathcal{H}_{\mathbb{L}, \gamma, u}}(-q)^{2J_3} q^{2l_3} e^{-\beta\{\mathcal{Q}, \mathcal{Q}^+\}} \in \mathbb{Z}[q, q^{-1}]$$

-Gaiotto-Moore-Neitzke, [Fock-Goncharov], [Bonahon-Wong], [Córdova-Neitzke], [Cirafici-Del Zotto],
-Coman-Gabella-Teschner, [Allegretti], [Gabella], [Ito-Okuda-Taki], [Galakhov-Longhi-Moore], [Neitzke-Y],
-[Cho-Kim-Kim-Oh],[Le],[Kim-Son],[Korinman-Quesney],[Goncharov-Shen],[Douglas-Sun]...

- When $M = \mathbb{R}^3$, $F$ computes a familiar link invariant:

$$F(L) = q^{Nw(L)} P_{\text{HOMFLY}}(L, a = q^N, z = q - q^{-1})$$

where $w(L)$ is the self-linking number of $L \subset M$.

It also works if $M = C \times \mathbb{R}$ and $L$ is contained in a 3-ball inside $M$.  [Bonahon-Wong],[Neitzke-Y],[Douglas-Sun]
The quantum trace map: an example

Example: \( g = \mathfrak{sl}(2), \ C \) is a four-punctured sphere.

\[
\tilde{C} = \{ \lambda : \lambda^2 + \phi_2 = 0 \} \subset T^* C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4 - 1)^2}dz^2.
\]
Properties of $\mathcal{X}_\gamma(\zeta)$

The VEVs of IR line defects $\mathcal{X}_\gamma(\zeta)$ have distinguished analytic properties: c.f. Bridgeland’s talks

- **Asymptotics**: As $\zeta \to 0$ along a ray in $\mathbb{C}^\times$:

  $$\mathcal{X}_\gamma(\zeta) \sim \exp \left( \frac{R}{\zeta} Z_\gamma \right),$$

  where $Z_\gamma = \oint_\gamma \lambda$ is central charge/classical period. c.f. Smith’s talk

- **Jumps**: Given a ray $\ell \subset \mathbb{C}^\times$ with phase $\theta_0$, then:

  $$\lim_{\text{arg}(\zeta) \to \theta_0^+} \mathcal{X}_\gamma(\zeta) = \mathcal{S}(\ell) \lim_{\text{arg}(\zeta) \to \theta_0^-} \mathcal{X}_\gamma(\zeta),$$

  $\mathcal{S}(\ell)$ consists of Kontsevich-Soibelman symplectomorphisms, determined by the Donaldson-Thomas invariants $\Omega(\gamma)$.

- **Reality**: 

  $$\overline{\mathcal{X}_\gamma(\zeta)} = \mathcal{X}_{-\gamma}(-1/\overline{\zeta}).$$
The Riemann-Hilbert problem

\( \mathcal{X}_\gamma(\zeta) \) are conjectured to solve a Riemann-Hilbert (RH) problem in \( \zeta \)-plane, formulated by Gaiotto-Moore-Neitzke; the solutions are given by certain TBA-like integral equations. This RH problem is related to the RH problem considered by Bridgeland via the conformal limit [Gaiotto].
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Barbieri-Bridgeland-Stoppa formulated a quantized RH problem and constructed solutions in certain example.

Q: Could we understand/generalize the construction from physics?
The conformal limit

Recall the Hitchin system depends on $R$ as the radius of the compactification circle. The conformal limit is given by [Gaiotto]

$$R \to 0, \quad \zeta \to 0, \quad \hbar = \zeta / R \text{ fixed}$$

The Hitchin section becomes the variety of opers $\mathcal{L}_\hbar$.

[Dumitrescu-Fredrickson-Kydonakis-Mazzeo-Mulase-Neitzke]

Example: Take $G_\mathbb{C} = SL(2, \mathbb{C})$, the family of opers is determined by a quadratic differential $\phi_2$ on $C$, locally written as:

$$\hbar^2 \partial_z^2 + P_2(z),$$

where $\phi_2 = P_2(z) dz^2$.

The spectral coordinates $\mathcal{X}_\gamma(\hbar)$ are important tools in exact WKB analysis for opers, due to the geometric reformulation of exact WKB via abelianization.

[Gaiotto-Moore-Neitzke],[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev]...
The Voros symbols $\mathcal{X}_\gamma(\hbar)$ and quantum periods

WKB ansatz for the Schrödinger equation:

$$\psi(z) = \exp \left( \frac{1}{\hbar} \int_{z_0}^{z} \lambda(z') dz' \right), \text{ where } \lambda^2 + P_2 + \hbar \partial_z \lambda = 0.$$ 

The quantum periods:

$$\Pi_\gamma(\hbar) := \int_\gamma \lambda^\text{formal}(\hbar) dz, \quad \gamma \in H_1(\tilde{\mathcal{C}}, \mathbb{Z}), \quad \lambda^\text{formal}_i(\hbar) = \sum_{n=0}^{\infty} \lambda_i^{(n)} \hbar^n$$

A natural way to resum quantum periods is Borel-resummation.

C.f. Pym's and Mariño's talk

Resurgent properties of quantum periods are controlled by 4d BPS spectrum. [Ito-Mariño-Shu],[Grassi-Gu-Mariño],[Ito-Shu]...

Relation of spectral coordinates $\mathcal{X}_\gamma(\hbar)$ to quantum periods:

- $\mathcal{X}_\gamma(\hbar)$ produces Borel-resummed quantum periods.

[Koike-Scháfke],[Nikolaev],[Allegretti],...

For higher order opers, this remains to be a conjecture.
Thank You and Stay Safe!