Line defects, UV-IR map, and exact WKB

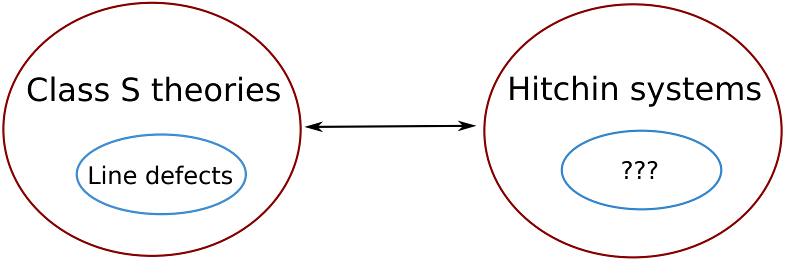
Fei Yan

Rutgers University

Donaldson-Thomas invariants and Resurgence,
Simons Collaboration meeting, Jan 11 - Jan 15, 2021

Jan 13th, 2021

Outlook



Based on review of work by Gaiotto-Moore-Neitzke, as well as some recent developments.

4d N=2 theories of class S

Class S theories $\mathcal{T}[\mathfrak{g}, C]$ are 4d N=2 supersymmetric theories originating from twisted compactification of a 6d $(2,0)$ theory of type $\mathfrak{g}$ ($\mathfrak{g} \in \{A, D, E\}$) on a Riemann surface C with appropriate decorations (punctures or twisted lines), in the zero area limit of C.

[Gaiotto],[Gaiotto-Moore-Neitzke]

4d N=2 theories of class S

Class S theories $\mathcal{T}[\mathfrak{g}, C]$ are 4d N=2 supersymmetric theories originating from twisted compactification of a 6d $(2,0)$ theory of type $\mathfrak{g}$ ($\mathfrak{g} \in \{A, D, E\}$) on a Riemann surface C with appropriate decorations (punctures or twisted lines), in the zero area limit of C.

[Gaiotto],[Gaiotto-Moore-Neitzke]

$\mathcal{T}[\mathfrak{g}, C]$ has a subspace of vacua $\mathcal{B}$ called the Coulomb branch, where the low energy effective theory is $U(1)^r$ gauge theory.

$$\mathcal{B} \subset \bigoplus_{k=1}^{r} H^0\left(C, K_C^{\otimes d_k}\left(\sum_i p_{d_k}^{(i)} z_i\right)\right)$$

[Gaiotto],[Gaiotto-Moore-Neitzke],[Chacaltana-Distler-Tachikawa],[Chacaltana-Distler-Trimm-Zhu] ...

A point in $\mathcal{B}$ corresponds to a branched covering $\widetilde{C} \to C$, where $\widetilde{C} \subset T^*C$ is the Seiberg-Witten curve.

Relation to Hitchin systems

Further compactifying $\mathcal{T}[\mathfrak{g}, C]$ on S^1_R, the low energy effective theory is a 3d $N = 4$ sigma model with target space $M_H(G, C)$.

Relation to Hitchin systems

Further compactifying $\mathcal{T}[\mathfrak{g}, C]$ on S^1_R, the low energy effective theory is a 3d $N = 4$ sigma model with target space $M_H(G, C)$.

Starting from 6d and changing the order of compactification on $C \times S^1_R$ [Gaiotto-Moore-Neitzke], $M_H(G, C)$ is identified with the moduli space of solutions to Hitchin's equations:

$$F_A + R^2 \left[\Phi, \bar{\Phi}\right] = 0,$$
$$\bar{\partial}_A \Phi = 0, \quad \partial_A \bar{\Phi} = 0.$$

$\partial + A$ is a G-connection in a top. trivial G-bundle $V \to C$, $\Phi \in \Omega^{1,0}(\mathrm{End}\, V)$ is the Higgs field.

(A, Φ) have specified boundary conditions at punctures of C. (regular: simple pole, irregular: higher order pole)

Today we take $\mathfrak{g} = \mathfrak{gl}(N)$ or $\mathfrak{sl}(N)$, $G = U(N)$ or $SU(N)$.

Relation to Hitchin systems

$M_H(G, C)$ is hyperkähler, has a $\mathbb{CP}^1$-worth of complex structures J_ζ. Different J_ζ expose different features of $M_H(G, C)$:

[Hitchin], [Simpson], [Biquard-Boalch], [Gaiotto-Moore-Neitzke]...

- $\zeta \in \mathbb{C}^\times$: Hitchin's equations indicate $\partial + \mathcal{A}$ is flat, with

$$\mathcal{A} := \frac{R}{\zeta}\Phi + A + R\zeta\bar{\Phi}$$

(M_H, J_ζ) diff. to a moduli space of flat $G_\mathbb{C}$-connections on C.

Relation to Hitchin systems

$M_H(G, C)$ is hyperkähler, has a $\mathbb{CP}^1$-worth of complex structures J_ζ. Different J_ζ expose different features of $M_H(G, C)$:

[Hitchin], [Simpson], [Biquard-Boalch], [Gaiotto-Moore-Neitzke]...

- $\zeta \in \mathbb{C}^\times$: Hitchin's equations indicate $\partial + \mathcal{A}$ is flat, with

$$\mathcal{A} := \frac{R}{\zeta}\Phi + A + R\zeta\bar{\Phi}$$

  (M_H, J_ζ) diff. to a moduli space of flat $G_{\mathbb{C}}$-connections on C.
- $\zeta = 0$: (M_H, J_0) diff. to moduli space of Higgs bundles M_{Higgs}, which is a complex integrable system: $M_{\text{Higgs}} \to \mathcal{B}$ with generic fiber being compact tori.

  The Seiberg-Witten curve $\widetilde{C}$ identified with spectral curve:

$$\widetilde{C} = \{(z \in C, \lambda \in T^*_z C) : \text{Det}\,(\Phi(z) - \lambda) = 0\} \subset T^*C$$

Line defects in class S theories

$\mathcal{T}[\mathfrak{g}, C]$ admits families of line defects $\mathbb{L}(\zeta)$ extending along $\mathbb{R}^t$-direction, where $\zeta \in \mathbb{C}^\times$ parametrizes preserved supercharges.

[Kapustin],[Kapustin-Saulina],[Drukker-Morrison-Okuda],[Drukker-Gaiotto-Gomis],[Drukker-Gomis-Okuda-Teschner] [Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Aharony-Seiberg-Tachikawa],[Moore-Royston-van den Bleeken]...

$\mathbb{L}(\zeta, \mathfrak{p}, \mathcal{R})$ depends on path $\mathfrak{p}$ on C, carrying representation $\mathcal{R}$ of $\mathfrak{g}$.

Line defects in class S theories

$\mathcal{T}[\mathfrak{g}, C]$ admits families of line defects $\mathbb{L}(\zeta)$ extending along $\mathbb{R}^t$-direction, where $\zeta \in \mathbb{C}^\times$ parametrizes preserved supercharges.
[Kapustin],[Kapustin-Saulina],[Drukker-Morrison-Okuda],[Drukker-Gaiotto-Gomis],[Drukker-Gomis-Okuda-Teschner]
[Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Aharony-Seiberg-Tachikawa],[Moore-Royston-van den Bleeken]...

$\mathbb{L}(\zeta, \mathfrak{p}, \mathcal{R})$ depends on path $\mathfrak{p}$ on C, carrying representation $\mathcal{R}$ of $\mathfrak{g}$.

- $\mathfrak{g} = A_1$, C only has regular punctures:
  $\mathfrak{p}$ is a non-self-intersecting closed curve on C.
- C contains irregular punctures:
  $\mathfrak{p}$ corresponds to integral laminations [Fock-Goncharov],[Gaiotto-Moore-Neitzke], collection of paths either closed or open with ends on marked points corres. to Stokes directions at irregular punctures.
- In general $\mathfrak{p}$ could contain junctions, where paths carrying different $\mathcal{R}_i$ meet, associated with certain $\mathfrak{g}$-invariant tensor.
  [Sikora],[Le],[Xie],[Saulina],[Coman-Gabella-Teschner],[Tachikawa-Watanabe],[Gabella]...

Line defects in class S theories

Upon circle compactification, the vacuum expectation values of $\mathbb{L}(\zeta)$ wrapping S^1_R are J_ζ-holomorphic functions on $M_{\text{flat}}(G_\mathbb{C}, C)$.

[Gaiotto-Moore-Neitzke]

Line defects in class S theories

Upon circle compactification, the vacuum expectation values of $\mathbb{L}(\zeta)$ wrapping S^1_R are J_ζ-holomorphic functions on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.

[Gaiotto-Moore-Neitzke]

- $\mathfrak{g} = A_1$, C has only regular punctures:

$$\begin{aligned}\langle \mathbb{L}(\zeta, \mathfrak{p}, \mathcal{R}) \rangle &= \text{Tr}_{\mathcal{R}} \text{Hol}_{\mathfrak{p}} \left(\frac{R\Phi}{\zeta} + A + R\zeta\bar{\Phi} \right) \\ &= \text{Tr}_{\mathcal{R}} \text{Hol}_{\mathfrak{p}} \mathcal{A}(\zeta).\end{aligned}$$

  The dependence on $\mathfrak{p}$ is only through its homotopy class.
- In general, compute parallel transport of $\mathcal{A}(\zeta)$ along paths, contract together via $\mathfrak{g}$-invariant tensors.

The UV-IR map for line defects

A useful way to study $\mathbb{L}(\zeta)$ in class S theories, is deforming to a point u on the Coulomb branch $\mathcal{B}$ and follow the defect into IR.

The UV-IR map for line defects

A useful way to study $\mathbb{L}(\zeta)$ in class S theories, is deforming to a point u on the Coulomb branch $\mathcal{B}$ and follow the defect into IR.

The IR limit of $\mathbb{L}(\zeta)$ is a superposition of supersymmetric line defects in the abelian theory, with integer coefficients in this superposition given by framed BPS index $\overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma, u)$.

[Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Cirafici-Del Zotto],[Coman-Gabella-Teschner],

[Moore-Royston-van den Bleeken],[Ito-Okuda-Taki],[Galakhov-Longhi-Moore],. . .

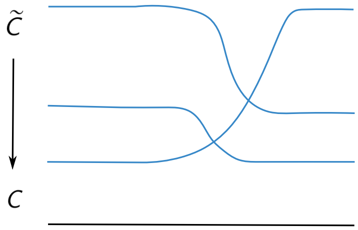
The UV-IR map for line defects:

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_{\gamma} \overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma, u) X_{\gamma}(\zeta)$$

$X_{\gamma}(\zeta)$ represent IR Wilson-'t Hooft lines with charge γ.

The IR line defects

Recall that, a point u on the Coulomb branch $\mathcal{B}$ corresponds to a branched covering $\widetilde{C} \to C$ (spectral curve/Seiberg-Witten curve):



Geometrically the IR line defect $X_\gamma(\zeta)$ correspond to loops $\tilde{\mathfrak{p}} \subset \widetilde{C}$ in class $\gamma \in H_1(\widetilde{C}, \mathbb{Z})$.

The IR line defects

Upon circle compactification, the VEV $\mathcal{X}_\gamma(\zeta)$ of $X_\gamma(\zeta)$ wrapping S^1_R are local Darboux coordinates on $M_{\mathrm{flat}}(G_{\mathbb{C}}, C)$:
Fock-Goncharov, complexified Fenchel-Nielsen, or more general spectral coordinates.

[Fock-Goncharov],[Fenchel-Nielsen],[Gaiotto-Moore-Neitzke],[Nekrasov-Rosly-Shatashvili],

[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev],[Jeong-Nekrasov],[Coman-Longhi-Teschner] ...

c.f. Bridgeland's talk and Coman's talk

The IR line defects

Upon circle compactification, the VEV $\mathcal{X}_\gamma(\zeta)$ of $X_\gamma(\zeta)$ wrapping S^1_R are local Darboux coordinates on $M_{\text{flat}}(G_{\mathbb{C}}, C)$:
Fock-Goncharov, complexified Fenchel-Nielsen, or more general spectral coordinates.

[Fock-Goncharov],[Fenchel-Nielsen],[Gaiotto-Moore-Neitzke],[Nekrasov-Rosly-Shatashvili],

[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev],[Jeong-Nekrasov],[Coman-Longhi-Teschner] ...

c.f. Bridgeland's talk and Coman's talk

- The holomorphic Poisson brackets are given by

$$\{\mathcal{X}_\gamma, \mathcal{X}_{\gamma'}\} = \langle \gamma, \gamma' \rangle \mathcal{X}_\gamma \mathcal{X}_{\gamma'}$$

- $\mathcal{X}_\gamma$ has nice asymptotic behavior as $\zeta \to 0$ and $\zeta \to \infty$; has discontinuities across "BPS rays" controlled by Kontsevich-Soibelman symplectomorphisms.
  $\mathcal{X}_\gamma$ are solutions to a certain Riemann-Hilbert problem.

  [Gaiotto-Moore-Neitzke],[Gaiotto],[Bridgeland],[Barbieri],[Bridgeland-Barbieri-Stoppa]...

The UV-IR map as the trace map

- $\langle \mathbb{L}(\zeta) \rangle$ are J_ζ-holomorphic trace functions on $M_{\mathrm{flat}}(G_{\mathbb{C}}, C)$.
- $\mathcal{X}_\gamma(\zeta) := \langle X_\gamma(\zeta) \rangle$ are Darboux-coordinates on $M_{\mathrm{flat}}(G_{\mathbb{C}}, C)$.

The UV-IR map as the trace map

- $\langle \mathbb{L}(\zeta) \rangle$ are J_ζ-holomorphic trace functions on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.
- $\mathcal{X}_\gamma(\zeta) := \langle X_\gamma(\zeta) \rangle$ are Darboux-coordinates on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.
- The UV-IR map then implies the trace map:

$$\text{Tr}_{\mathcal{R}} \text{Hol}_{\mathfrak{p}} \mathcal{A}(\zeta) = \sum_{\gamma} \underline{\overline{\Omega}}(\mathbb{L}(\zeta), \gamma) \mathcal{X}_\gamma(\zeta)$$

  $\mathcal{A}(\zeta)$: flat $G_{\mathbb{C}}$-connection on C, $\mathfrak{p}$: path on C.
- Alternatively we can understand $\mathcal{X}_\gamma(\zeta)$ as:

$$\mathcal{X}_\gamma(\zeta) = \text{Hol}_\gamma \nabla^{\text{ab}}(\zeta), \gamma \in H_1(\widetilde{C}, \mathbb{Z}).$$

  The UV-IR map is interpreted as a nonabelianization map.

  [Gaiotto-Moore-Neitzke],[Hollands-Neitzke]

Line defects OPE

The algebra structure on space of J_ζ-holomorphic functions corresponds to line defects operator products (OPE):

$$\langle \mathbb{L}_1(\zeta)\mathbb{L}_2(\zeta)\rangle = \langle \mathbb{L}_1(\zeta)\rangle\langle \mathbb{L}_2(\zeta)\rangle$$

This algebra structure admits a quantization via skein algebras.

[Reshetikhin-Turaev],[Turaev],[Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],

[Drukker-Gomis-Okuda-Teschner],[Tachikawa-Watanabe],[Coman-Gabella-Teschner],[Gabella]...

Line defects OPE

The algebra structure on space of J_ζ-holomorphic functions corresponds to line defects operator products (OPE):

$$\langle \mathbb{L}_1(\zeta)\mathbb{L}_2(\zeta)\rangle = \langle \mathbb{L}_1(\zeta)\rangle\langle \mathbb{L}_2(\zeta)\rangle$$

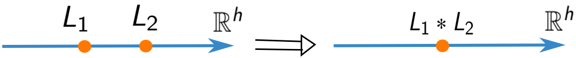
This algebra structure admits a quantization via skein algebras.

[Reshetikhin-Turaev],[Turaev],[Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],

[Drukker-Gomis-Okuda-Teschner],[Tachikawa-Watanabe],[Coman-Gabella-Teschner],[Gabella]...

Turning on Ω-deformation on a $\mathbb{R}^2$-plane, susy line defects sit along a fixed axis in $\mathbb{R}^3$ $\rightsquigarrow$ non-commutative associative OPE $*$

[Nekrasov-Shatashvili],[Gaiotto-Moore-Neitzke],[Ito-Okuda-Taki],[Yagi],[Oh-Yagi],...



Line defects OPE

- In the UV, non-commutative line defects OPE is complicated.
- In the IR, the OPE is given by quantum torus algebra:

$$X_{\gamma_1}(\zeta) * X_{\gamma_2}(\zeta) = (-q)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}(\zeta).$$

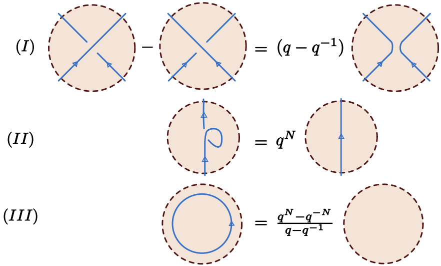
  $\langle , \rangle$ is the Dirac-Schwinger-Zwanziger pairing, identified with intersection pairing in $H_1(\widetilde{C}, \mathbb{Z})$.
  In particular,

$$X_{\gamma_1}(\zeta) * X_{\gamma_2}(\zeta) = (-q)^{2\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_2}(\zeta) * X_{\gamma_1}(\zeta)$$

  quantizes the Poisson algebra of coordinate functions $\mathcal{X}_\gamma$.
- This hints existence of a quantum trace map, embedding the UV skein algebra into a quantum torus algebra.

The UV skein algebra

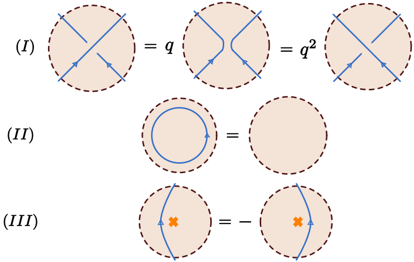
For example, taking $\mathfrak{g} = \mathfrak{gl}(2)$, the space of line defects equipped with OPE is described by the $\mathfrak{gl}(2)$ HOMFLY skein algebra of $M = C \times \mathbb{R}_h$, defined as the space of formal $\mathbb{Z}[q^{\pm 1}]$-linear combinations of framed oriented links (up to isotopy) in M, modulo the following relations: (N=2 in the figure)



The algebra structure is defined by "stacking" links along the $\mathbb{R}^h$-direction.

The IR skein algebra

The IR line defects correspond to framed oriented links in $\widetilde{M} = \widetilde{C} \times \mathbb{R}$. The OPE algebra is (twisted) $\mathfrak{gl}(1)$ skein algebra of $\widetilde{M}$, defined as the space of formal $\mathbb{Z}[q^{\pm 1}]$-linear combinations of framed oriented links (up to isotopy) in $\widetilde{M}$, modulo the following relations:



This skein algebra is isomorphic to the quantum torus.

The quantum trace map

The quantum trace map F is a homomorphism from the UV skein algebra to the IR quantum torus algebra, physically it represents a q-deformed UV-IR map:

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_{\gamma} \overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma, u, q) X_{\gamma}(\zeta)$$

$$\overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma, u, q) := \mathrm{Tr}_{\mathcal{H}_{\mathbb{L},\gamma,u}} (-q)^{2J_3} q^{2I_3} e^{-\beta\{\mathcal{Q},\mathcal{Q}^{+}\}} \in \mathbb{Z}[q, q^{-1}]$$

[Gaiotto-Moore-Neitzke],[Fock-Goncharov],[Bonahon-Wong],[Córdova-Neitzke],[Cirafici-Del Zotto], [Coman-Gabella-Teschner],[Allegretti],[Gabella],[Ito-Okuda-Taki],[Galakhov-Longhi-Moore],[Neitzke-Y], [Cho-Kim-Kim-Oh],[Le],[Kim-Son],[Korinman-Quesney],[Goncharov-Shen],[Douglas-Sun]...

The quantum trace map

The quantum trace map F is a homomorphism from the UV skein algebra to the IR quantum torus algebra, physically it represents a q-deformed UV-IR map:

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_{\gamma} \overline{\Omega}(\mathbb{L}(\zeta), \gamma, u, q) X_{\gamma}(\zeta)$$

$$\overline{\Omega}(\mathbb{L}(\zeta), \gamma, u, q) := \mathrm{Tr}_{\mathcal{H}_{\mathbb{L},\gamma,u}} (-q)^{2J_3} q^{2I_3} e^{-\beta\{\mathcal{Q},\mathcal{Q}^+\}} \in \mathbb{Z}[q, q^{-1}]$$

[Gaiotto-Moore-Neitzke],[Fock-Goncharov],[Bonahon-Wong],[Córdova-Neitzke],[Cirafici-Del Zotto], [Coman-Gabella-Teschner],[Allegretti],[Gabella],[Ito-Okuda-Taki],[Galakhov-Longhi-Moore],[Neitzke-Y], [Cho-Kim-Kim-Oh],[Le],[Kim-Son],[Korinman-Quesney],[Goncharov-Shen],[Douglas-Sun]...

- When $M = \mathbb{R}^3$, F computes a familiar link invariant:

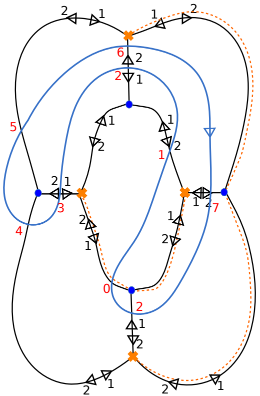
  $$F(L) = q^{Nw(L)} P_{\mathrm{HOMFLY}}(L, a = q^N, z = q - q^{-1}),$$

  where $w(L)$ is the self-linking number of $L \subset M$.
  It also works if $M = C \times \mathbb{R}$ and L is contained in a 3-ball inside M. [Bonahon-Wong],[Neitzke-Y],[Douglas-Sun]

The quantum trace map: an example

Example: $\mathfrak{g} = \mathfrak{sl}(2)$, C is a four-punctured sphere.

$$\widetilde{C} = \{\lambda : \lambda^2 + \phi_2 = 0\} \subset T^*C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4 - 1)^2} dz^2.$$



$$\begin{aligned}
&X_{\gamma_1+\mu_1-\mu_3} + X_{\gamma_2+\mu_1-\mu_3} + X_{\gamma_1+\gamma_2+\mu_1-\mu_3} + X_{-\gamma_2-\mu_1+\mu_3} + X_{\gamma_1+\mu_1+\mu_3} \\
&+ X_{\gamma_1-\gamma_2+\mu_1+\mu_3} + X_{2\gamma_1-3\gamma_2-\mu_2+2\mu_3-3\mu_4} \underline{-(q+q^{-1})}X_{2\gamma_1-2\gamma_2-\mu_2+2\mu_3-3\mu_4} \\
&+ X_{2\gamma_1-\gamma_2-\mu_2+2\mu_3-3\mu_4} + X_{\gamma_1-2\gamma_2-\mu_1+\mu_3-2\mu_4} + X_{\gamma_1-\gamma_2-\mu_1+\mu_3-2\mu_4} \\
&+ X_{\gamma_1+\mu_1+\mu_3-2\mu_4} \underline{-(q+q^{-1})}X_{2\gamma_1+\mu_1+\mu_3-2\mu_4} \underline{-(q+q^{-1})}X_{2\gamma_1-2\gamma_2+\mu_1+\mu_3-2\mu_4} \\
&+ X_{\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4} + \underline{(2+q^2+q^{-2})}X_{2\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4} + X_{\gamma_1-\mu_2-\mu_4} \\
&+ X_{\gamma_1-\gamma_2-\mu_2-\mu_4} + X_{\gamma_1+\mu_2-\mu_4} + X_{\gamma_1-\gamma_2+\mu_2-\mu_4} + X_{\gamma_1+2\mu_1+\mu_2-\mu_4} \\
&\underline{-(q+q^{-1})}X_{2\gamma_1+2\mu_1+\mu_2-\mu_4} + X_{2\gamma_1-\gamma_2+2\mu_1+\mu_2-\mu_4} + X_{\gamma_1+\gamma_2+2\mu_1+\mu_2-\mu_4} \\
&+ X_{2\gamma_1+\gamma_2+2\mu_1+\mu_2-\mu_4} + X_{\gamma_1-2\gamma_2-\mu_2+2\mu_3-\mu_4} + X_{\gamma_1-\gamma_2-\mu_2+2\mu_3-\mu_4}.
\end{aligned}$$

Properties of $\mathcal{X}_\gamma(\zeta)$

The VEVs of IR line defects $\mathcal{X}_\gamma(\zeta)$ have distinguished analytic properties: c.f. Bridgeland's talks

- Asymptotics: As $\zeta \to 0$ along a ray in $\mathbb{C}^\times$:
$$\mathcal{X}_\gamma(\zeta) \sim \exp\left(\frac{R}{\zeta} Z_\gamma\right),$$
where $Z_\gamma = \oint_\gamma \lambda$ is central charge/classical period. c.f. Smith's talk
- Jumps: Given a ray $\ell \subset \mathbb{C}^\times$ with phase θ_0, then:
$$\lim_{\arg(\zeta)\to\theta_0^+} \mathcal{X}_\gamma(\zeta) = \mathbb{S}(\ell) \lim_{\arg(\zeta)\to\theta_0^-} \mathcal{X}_\gamma(\zeta)$$
$\mathbb{S}(\ell)$ consists of Kontsevich-Soibelman symplectomorphisms, determined by the Donaldson-Thomas invariants $\Omega(\gamma)$.
- Reality:
$$\overline{\mathcal{X}_\gamma(\zeta)} = \mathcal{X}_{-\gamma}\left(-1/\bar{\zeta}\right).$$

The Riemann-Hilbert problem

$\mathcal{X}_\gamma(\zeta)$ are conjectured to solve a Riemann-Hilbert (RH) problem in ζ-plane, formulated by Gaiotto-Moore-Neitzke; the solutions are given by certain TBA-like integral equations. This RH problem is related to the RH problem considered by Bridgeland via the conformal limit [Gaiotto].

The Riemann-Hilbert problem

$\mathcal{X}_\gamma(\zeta)$ are conjectured to solve a Riemann-Hilbert (RH) problem in ζ-plane, formulated by Gaiotto-Moore-Neitzke; the solutions are given by certain TBA-like integral equations. This RH problem is related to the RH problem considered by Bridgeland via the conformal limit [Gaiotto].

Barbieri-Bridgeland-Stoppa formulated a quantized RH problem and constructed solutions in certain example.

Q: Could we understand/generalize the construction from physics?

The conformal limit

Recall the Hitchin system depends on R as the radius of the compactification circle. The conformal limit is given by [Gaiotto]

$$R \to 0, \ \zeta \to 0, \ \hbar = \zeta/R \text{ fixed}$$

The Hitchin section becomes the variety of opers $\mathcal{L}_\hbar$.

[Dumitrescu-Fredrickson-Kydonakis-Mazzeo-Mulase-Neitzke]

Example: Take $G_\mathbb{C} = SL(2, \mathbb{C})$, the family of opers is determined by a quadratic differential ϕ_2 on C, locally written as:

$$\hbar^2 \partial_z^2 + P_2(z),$$

where $\phi_2 = P_2(z)dz^2$.

The spectral coordinates $\mathcal{X}_\gamma(\hbar)$ are important tools in exact WKB analysis for opers, due to the geometric reformulation of exact WKB via abelianization.

[Gaiotto-Moore-Neitzke],[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev]...

The Voros symbols $\mathcal{X}_\gamma(\hbar)$ and quantum periods

WKB ansatz for the Schrödinger equation:

$$\psi(z) = \exp\left(\frac{1}{\hbar}\int_{z_0}^{z} \lambda(z')dz'\right), \text{ where } \lambda^2 + P_2 + \hbar\partial_z\lambda = 0.$$

The quantum periods:

$$\Pi_\gamma(\hbar) := \oint_\gamma \lambda^{\text{formal}}(\hbar)dz, \ \gamma \in H_1(\widetilde{C}, \mathbb{Z}), \ \lambda_i^{\text{formal}}(\hbar) = \sum_{n=0}^{\infty} \lambda_i^{(n)}\hbar^n$$

A natural way to resum quantum periods is Borel-resummation.
c.f. Pym's and Mariño's talk

Resurgent properties of quantum periods are controlled by 4d BPS spectrum. [Ito-Mariño-Shu],[Grassi-Gu-Mariño],[Ito-Shu]...

Relation of spectral coordinates $\mathcal{X}_\gamma(\hbar)$ to quantum periods:

- $\mathcal{X}_\gamma(\hbar)$ produces Borel-resummed quantum periods.

  [Koike-Schäfke],[Nikolaev],[Allegretti],...

  For higher order opers, this remains to be a conjecture.

Thank You and Stay Safe!