# GK geometry

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Based on work with Chris Couzens, Jerome Gauntlett and Dario Martelli.

- GK = Gauntlett-Kim.
- For mathematicians: natural Kähler geometry + PDE arising in string theory.
- For physicists: describes supersymmetric $\mathbf{AdS_3}$, $\mathbf{AdS_2}$ solutions in string/M-theory, latter including near horizon limits of extremal black holes.
- There is a close relation to Calabi-Yau cone geometry, and branes wrapped on Riemann surfaces.

Calabi-Yau cones:

- An odd-dimensional manifold $\mathbf{(X, g)}$ is Sasakian iff the cone metric $\mathbf{dr^2 + r^2 g}$ is Kähler, $\mathbf{C(X)} = \mathbb{R}_{>0} \times \mathbf{X}$.
- $\mathbf{X}$ naturally embedded at $\mathbf{r = 1}$.
- The *Reeb vector* $\xi = \mathbf{I(r\partial_r)}$ is holomorphic and Killing, unit length on $\mathbf{X}$.
- $\Rightarrow$ one-dimensional foliation $\mathcal{F}_\xi$ of $\mathbf{X}$.

- Define $\eta =$ one-form on $\mathbf{X}$ dual to $\xi$, then

$$\mathbf{g} = \eta^2 + \mathbf{g_T}\,,$$

where $\mathbf{d}\eta = \mathbf{2J}$, where $(\mathbf{g_T}, \mathbf{J})$ is a transverse Kähler metric and form.

- $\Rightarrow$ Sasakian geometry is sandwiched between two Kähler geometries.
- If orbits of $\xi$ all close, then $\mathbf{X} =$ circle orbibundle over a Kähler orbifold.

- The Kähler cone $\mathbf{C(X)}$ with $\mathbf{dim}_{\mathbb{C}}\,\mathbf{C(X)} = \mathbf{n}$ is Ricci-flat (Calabi-Yau) iff
  - $\mathbf{g} = \mathbf{g_X}$ is Einstein with $\mathbf{Ric_X} = \mathbf{(2n-2)g_X}$ (Sasaki-Einstein).
  - equivalently $\mathbf{g_T}$ is Einstein with $\mathbf{Ric_T} = \mathbf{2n\,g_T}$ (Kähler-Einstein).
- E.g. $\mathbf{C(X)} = \mathbb{C}^{\mathbf{n}} \setminus \{\mathbf{0}\}$, $\mathbf{X} =$ round sphere, $\mathbf{g_T} =$ Fubini-Study metric on $\mathbb{CP}^{\mathbf{n-1}}$.
- $\mathbf{C(X)}$ admits a holomorphic $\mathbf{(n,0)}$-form $\Omega$ with

$$\mathcal{L}_\xi \Omega = \mathbf{i n} \Omega\,,$$

iff $[\rho_{\mathbf{T}}] = \mathbf{2n[J]} \in \mathbf{H}^{1,1}(\mathcal{F}_\xi)$, $\rho_{\mathbf{T}} =$ transverse Ricci form.

In string/M-theory:

- $\mathbb{R}^{1,3} \times \mathbf{C(X)}$ and $\mathbb{R}^{1,2} \times \mathbf{C(X)}$ are string/M-theory solutions for $\mathbf{n = 3}$, $\mathbf{n = 4}$, respectively.
- Can put $\mathbf{N}$ D3-branes or $\mathbf{N}$ M2-branes at tip $\mathbf{r = 0}$ of the cones.
- There are corresponding $\mathbf{AdS_5} \times \mathbf{X}$ and $\mathbf{AdS_4} \times \mathbf{X}$ solutions, with $\mathbf{N}$ units of flux through $\mathbf{X}$.
- AdS/CFT: string/M-theory on these backgrounds is completely equivalent to the low-energy SCFT on the branes.
- Note Calabi-Yau cones exist in all dimensions, but applications to physics $\Rightarrow$ $\mathbf{n = 3}$, $\mathbf{n = 4}$.

GK geometry:

- Odd-dimensional manifold $(\mathbf{Y}, \mathbf{g})$, with a unit norm Killing vector $\xi \Rightarrow$ foliation $\mathcal{F}_\xi$.
- Again let $\eta =$ one-form dual to $\xi$, then

$$\mathbf{g} = \eta^2 + \mathrm{e}^{\mathbf{B}} \mathbf{g}_{\mathbf{T}} \,,$$

where $\mathbf{g}_{\mathbf{T}}$ is Kähler, and $\mathrm{e}^{\mathbf{B}} = \frac{1}{8}(\mathbf{n}-3)^2 \mathbf{R}$.

- $\Rightarrow$ need $\mathbf{n} \geq 4$ and transverse scalar curvature $\mathbf{R} = \mathbf{R}_{\mathbf{T}} > 0$.
- $\mathrm{d}\eta = \frac{1}{2}(\mathbf{n}-3)\rho$ (compare Sasakian $\mathrm{d}\eta = 2\mathbf{J}$).

- There is a natural closed two-form in the problem

$$\mathbf{F} = -\frac{2}{n-3}\mathbf{J} + \mathrm{d}(\mathrm{e}^{-B}\eta)\,.$$

- $*_Y \mathbf{F}$ determines the flux in the corresponding string/M-theory solutions.
- Bianchi identity $\mathrm{d}(\mathrm{e}^{(4-n)B} *_Y \mathbf{F}) = 0$ is equivalent to the Kähler PDE

$$\Box R \;=\; \frac{1}{2}R^2 - R_{ij}R^{ij}\,.$$

- We call a solution to this an on-shell GK geometry.

The metric cones $\mathbf{C(Y)}$:

- $\mathrm{d}\mathbf{r}^2 + \mathbf{r}^2 \mathbf{g_Y}$, where $\mathbf{r} > \mathbf{0}$.
- These are naturally complex, with holomorphic $\mathbf{(n, 0)}$-form $\Omega$ and

$$\mathcal{L}_\xi \Omega = \frac{2}{n-3} i \Omega \, ,$$

  where $\xi = \mathbf{I}(\mathbf{r}\partial_\mathbf{r})$.
- Not Kähler! (Physically, because of the flux.)
- However, $\mathbf{C(Y)}$ is in some sense a "Calabi-Yau cone".

- Integrating the PDE over $\mathbf{Y}$ implies

$$\int_{\mathbf{Y}} \eta \wedge \rho^2 \wedge e^{\mathbf{J}} = 0\,,$$

$\Rightarrow$ Sasakian geometries with $[\rho] \propto [\mathbf{J}] \in \mathbf{H}^{1,1}(\mathcal{F}_\xi)$ are not examples.

- Circle bundles over products of Kähler-Einstein manifolds are examples, with at least one factor having non-positive curvature (PDE is a trivial algebraic equation).
- Many classes of examples constructed using symmetry reduction:
  - cohomogeneity one
  - orthotoric Kähler

In string/M-theory:

- These describe supersymmetric $\mathbf{AdS_3} \times \mathbf{Y}$ and $\mathbf{AdS_2} \times \mathbf{Y}$ solutions of string/M-theory for $\mathbf{n = 4}$, $\mathbf{n = 5}$, respectively.
- They are sourced only by D3-brane and M2-brane flux, respectively:

$$\mathbf{F_5 = vol_{AdS_3} \wedge F + *_Y F}\,, \qquad \mathbf{G = vol_{AdS_2} \wedge F}\,.$$

- Flux quantization over codimension two cycles $\mathbf{S_A}$ in $\mathbf{Y}$:

$$\int_{\mathbf{S_A}} \boldsymbol{\eta \wedge \rho \wedge} \mathbf{e^J = N_A = fixed}\,.$$

(Mathematicians: think fixing the transverse Kähler class $\mathbf{[J] \in H^{1,1}(\mathcal{F}_\xi)}$, although this is not quite equivalent).

| Geometry | D3-branes | Field theory | M2-branes | Field Theory |
|---|---|---|---|---|
| Sasakian | $\mathbf{AdS_5 \times X_5}$ | $\mathbf{d = 4}$ | $\mathbf{AdS_4 \times X_7}$ | $\mathbf{d = 3}$ |
| GK | $\mathbf{AdS_3 \times Y_7}$ | $\mathbf{d = 2}$ | $\mathbf{AdS_2 \times Y_9}$ | $\mathbf{d = 1}$ |

There is a natural physical connection between the two rows:

- Wrap $\mathbf{d = 4}$ (D3-brane), $\mathbf{d = 3}$ (M2-brane) SCFTs on Riemann surface $\Sigma$ $\Rightarrow$ flow to $\mathbf{d = 2}$, $\mathbf{d = 1}$ SCFTs.
- Geometrically, $\mathbf{Y}$ = total space of $\mathbf{X}$ fibration over $\Sigma$:

$$\mathbf{X} \hookrightarrow \mathbf{Y} \to \Sigma\,.$$

- The $\mathbf{AdS_3}$, $\mathbf{AdS_2}$ solutions may arise as near horizon limits of extremal black strings/black holes in parent $\mathbf{AdS_5}$, $\mathbf{AdS_4}$, with horizon $= \Sigma$.

- Suggests the existence of large classes of GK geometries/black hole solutions.
- Classes of explicit examples have been constructed, but it's clear from physics there should be many more $\Rightarrow$ very rich.
- We recently attacked this following an idea due to Martelli-JFS-Yau.
- Assuming a solution exists, one can deduce some interesting properties.

Volume minimization (Martelli-JFS-Yau):

- In Sasakian geometry, fix a complex cone $\mathbf{C(X)}$, with holomorphic $\mathbf{(n,0)}$-form $\Omega$ and $\mathcal{L}_\xi \Omega = \mathbf{i n}\Omega$.
- In general the Reeb vector field $\xi$ can move: write $\xi = \sum_{i=1}^{s} \mathbf{b_i}\partial_{\varphi_i}$, $\mathbf{b_i} \in \mathbb{R}$, holomorphic torus action $\mathbf{U(1)^s}$.
- The Reeb vector $\xi$ for a Sasaki-Einstein metric minimizes the volume $\int_X \eta \wedge e^J$ over this space of vector fields.
- Mathematicians: a critical $\xi$ has zero transverse Futaki invariant = obstruction to existence of Kähler-Einstein metric. Related to K-stability.

GK geometries extremize the following supersymmetric action

$$\mathsf{S}(\xi, [\mathsf{J}]) \ = \int_{\mathsf{Y}} \eta \wedge \rho \wedge \mathrm{e}^{\mathsf{J}} \, ,$$

subject to the constraints

- $\displaystyle\int_{\mathsf{Y}} \eta \wedge \rho^2 \wedge \mathrm{e}^{\mathsf{J}} = 0$ ($\Leftrightarrow$ integral of PDE over **Y** is zero)
- $\displaystyle\int_{\mathsf{S}_\mathsf{A}} \eta \wedge \rho \wedge \mathrm{e}^{\mathsf{J}} = \mathsf{N}_\mathsf{A}$ (flux quantization through codimension 2 cycles $\mathsf{S}_\mathsf{A}$)

Note: depend only on $\xi$, and transverse Kähler class $[\mathsf{J}] \in \mathsf{H}^{1,1}(\mathcal{F}_\xi)$.

This geometric extremal problem is related to field theory extremal problems: **c**-extremization, **I**-extremization, and the attractor mechanism in black holes.

- For $\mathbf{AdS_3} \times \mathbf{Y_7}$ solutions, the dual $\mathbf{d = 2}$ SCFT has a central charge $\mathbf{c}$, and

$$\mathbf{c} = \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} S\,.$$

- For $\mathbf{AdS_2} \times \mathbf{Y_9}$ solutions, the partition function $\mathbf{Z}$ of the dual SCQM has

$$\log Z = \frac{4\pi L^9}{(2\pi)^8 \ell_p^9} S\,.$$

- For classes of extremal black holes in four dimensions, with near horizon $\mathbf{AdS_2}$ region, also

$$S_{\mathrm{BH}} = \frac{4\pi L^9}{(2\pi)^8 \ell_p^9} S\,.$$

I'll describe how to compute the action $\mathbf{S}$ for

$$\mathbf{X_5} \hookrightarrow \mathbf{Y_7} \rightarrow \boldsymbol{\Sigma},$$

where $\mathbf{X_5}$ = toric Sasakian 5-manifold, $\boldsymbol{\Sigma = \Sigma_g}$ = Riemann surface.

- Physically: D3-brane wrapping holomorphic curve $\boldsymbol{\Sigma_g} \subset$ Calabi-Yau 4-fold. Normal space to $\boldsymbol{\Sigma_g}$ is a Calabi-Yau 3-fold = $\mathbf{C(X_5)}$.
- $\mathbf{C(X_5)}$ = affine toric variety, specified by 3d convex rational polyhedral cone.
- Use $\mathbf{U(1)^3}$ action to fibre over $\boldsymbol{\Sigma_g} \Rightarrow$ Chern numbers $\mathbf{(n_1, n_2, n_3)} \in \mathbb{Z}^3$.
- Choose basis so holomorphic $\mathbf{(3,0)}$-form on $\mathbf{C(X_5)}$ has charges $\mathbf{(1,0,0)}$ under $\mathbf{U(1)^3}$. Then $\mathbf{n_1 = 2(1-g)}$.

Everything may be expressed in terms of the "master volume" of the fibre $\mathbf{X_5}$:

$$\mathcal{V} = \int_{X_5} \eta \wedge e^{\omega}, \qquad \text{where } \omega = \text{transverse Kähler form}.$$

May compute explicitly in terms of combinatorial data:

$$\mathcal{V} = \frac{(2\pi)^3}{2} \sum_{a=1}^{d} \lambda_a \frac{\lambda_{a-1}(\vec{v}_a, \vec{v}_{a+1}, \vec{b}) - \lambda_a(\vec{v}_{a-1}, \vec{v}_{a+1}, \vec{b}) + \lambda_{a+1}(\vec{v}_{a-1}, \vec{v}_a, \vec{b})}{(\vec{v}_{a-1}, \vec{v}_a, \vec{b})(\vec{v}_a, \vec{v}_{a+1}, \vec{b})}.$$

- Polyhedral cone has primitive inward pointing normal vectors $\vec{v}_a \in \mathbb{Z}^3$, $a = 1, \ldots, d$.
- Recall $\xi = \sum_{i=1}^{s} b_i \partial_{\varphi_i}$, so $\vec{b} \in \mathbb{R}^3$, and $(\cdot, \cdot, \cdot) = 3 \times 3$ determinant.
- $\lambda_a \in \mathbb{R}$ specify the transverse Kähler class $[\omega]$.

One can derive the following remarkable formula:

$$\mathbf{S} = -\mathbf{A}\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a} - 4\pi\sum_{i=1}^{3} n_i\frac{\partial\mathcal{V}}{\partial b_i}\,.$$

where **A** = Kähler class for $\Sigma_g$ (its area).

The constraints are:

- $0 = A\sum_{a,b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} - 2\pi n_1\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a} + 4\pi\sum_{a=1}^{d}\sum_{i=1}^{3} n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}$ (PDE),
- $\frac{2(2\pi\ell_s)^4 g_s}{L^4}N = -\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}$ (flux through fibre **N**, number of D3-branes),
- $\frac{2(2\pi\ell_s)^4 g_s}{L^4}M_a = \frac{A}{2\pi}\sum_{b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} + 2\sum_{i=1}^{3} n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}$ (baryonic fluxes).

- Pick your favourite affine toric variety $\mathbf{C(X_5)}$, specified by $\vec{\mathbf{v}}_\mathbf{a} \in \mathbb{Z}^3$.
- Take $\Sigma_\mathbf{g}$, and fibre $\mathbf{X_5}$ over it using $(\mathbf{n_1 = 2(1-g), n_2, n_3}) \in \mathbb{Z}^3$.
- Solve constraints for fixed fluxes: $\mathbf{N}$ = number of D3-branes, $\mathbf{M_a}$ = baryonic fluxes.

In practice this determines the transverse Kähler class data $\{\mathbf{A}, \lambda_\mathbf{a}\}$. Substituting back into the action, it is then a function of $\vec{\mathbf{b}} = (\mathbf{2, b_2, b_3})$:

$$\mathbf{S = S(b_2, b_3; \vec{v}_a, g, n_2, n_3, N, M_a)}\ .$$

- $\mathbf{S}$ is a quadratic function of the choice of vector field $(\mathbf{b_2, b_3})$!

Field theory dual:

- The $\mathbf{AdS_5} \times \mathbf{X_5}$ solution is dual to a $\mathbf{d = 4}$, $\mathcal{N} = \mathbf{1}$ quiver gauge theory.
- Dimensionally reducing this on $\Sigma_{\mathbf{g}}$ to $\mathbf{d = 2} \Rightarrow$ trial $\mathbf{c}$-function of Benini-Bobev, a function of the choice of R-symmetry.
- Action $\mathbf{S}$ is identical to this field theory function! Proof: Hosseini-Zaffaroni.

Some open issues:

- GK solutions necessarily extremize the action $\mathbf{S}$, but there are clear counterexamples to the converse, e.g. solutions with $\mathbf{c} < \mathbf{0}$.
- What are the necessary and sufficient conditions for solving the Kähler PDE $\Box \mathbf{R} = \frac{1}{2}\mathbf{R}^2 - \mathbf{R}_{ij}\mathbf{R}^{ij}$?
- Being at a critical point of $\mathbf{S}$ is analogous to having zero Futaki invariant in Kähler-Einstein geometry. Existence problem for Sasaki-Einstein manifolds is solved (Collins-Székelyhidi), generalizing K-stability of Chen-Donaldson-Sun.
- This is dually related to necessary and sufficient conditions for a superconformal fixed point in field theory.
- What types of geometries admit solutions? What characterizes e.g. the complex cone $\mathbf{C(Y)}$?

There is a similar construction for M2-branes. I'll focus on the simple case of

$$\mathbf{X_7} \hookrightarrow \mathbf{Y_9} \to \Sigma_{\mathbf{g}} \,,$$

where $\mathbf{X_7 = S^7}$. Twisting specified by $(\mathbf{n_1 = 2(1-g), n_2, n_3, n_4}) \in \mathbb{Z}^4$.

Following analogous procedure of imposing constraints, flux quantization $\Rightarrow$

$$\mathbf{S} = \mathbf{S}(\mathbf{b_1 = 1, b_2, b_3, b_4}) = -\frac{2\sqrt{2}\pi}{3}\mathbf{N}^{3/2}\sum_{\mathbf{i=1}}^{4}\mathbf{n_i}\frac{\partial \mathbf{F}}{\partial \Delta_{\mathbf{i}}} \,,$$

- $\mathbf{F} = \sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$ = prepotential for $\mathbf{d = 4}$ STU gauged supergravity!
- $\Delta_{\mathbf{i}} = -\frac{2}{\mathbf{N}}\frac{\partial \mathcal{V}}{\partial \lambda_{\mathbf{i}}}$ = certain linear functions of $\vec{\mathbf{b}}$.

- The M2-branes wrapped on $\Sigma_g$ are associated to certain extremal black hole solutions in $d = 4$, STU gauged supergravity Benini-Hristov-Zaffaroni.
- These have near horizon geometries that are $AdS_2 \times \Sigma_g$, that uplift to $D = 11$ solutions $AdS_2 \times Y_9$, with $S^7 \hookrightarrow Y_9 \to \Sigma_g$.
- Twisting parameters $n_i \in \mathbb{Z}^4$ are black hole magnetic charges under $U(1)^4$.
- The attractor mechanism in STU gauged supergravity is identical to extremizing our GK geometry action $S$!
- At the critical point for $(b_2, b_3, b_4)$, this determines $S_{BH}$.

Some interesting open questions:

- We've seen the attractor mechanism, that determines black hole entropy via an extremal problem, is equivalent to a geometric extremal problem.
- Is there a general geometric attractor mechanism for black holes in string/M-theory?
- In general this should also involve electric charges and angular momentum, not just magnetic charges.
- Large classes of supersymmetric black string/black hole solutions should exist, but are not known (I have described their near horizon geometry).