GK geometry

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Based on work with Chris Couzens, Jerome Gauntlett and Dario Martelli.
- GK = Gauntlett-Kim.

- For mathematicians: natural Kähler geometry + PDE arising in string theory.

- For physicists: describes supersymmetric $\text{AdS}_3$, $\text{AdS}_2$ solutions in string/M-theory, latter including near horizon limits of extremal black holes.

- There is a close relation to Calabi-Yau cone geometry, and branes wrapped on Riemann surfaces.
Calabi-Yau cones:

- An odd-dimensional manifold \((X, g)\) is Sasakian iff the cone metric \(dr^2 + r^2 g\) is Kähler, \(C(X) = \mathbb{R}_{>0} \times X\).
- \(X\) naturally embedded at \(r = 1\).
- The Reeb vector \(\xi = l(r \partial_r)\) is holomorphic and Killing, unit length on \(X\).
- \(\Rightarrow\) one-dimensional foliation \(F_\xi\) of \(X\).
Define $\eta$ = one-form on $X$ dual to $\xi$, then

$$g = \eta^2 + g_T,$$

where $d\eta = 2J$, where $(g_T, J)$ is a transverse Kähler metric and form.

⇒ Sasakian geometry is sandwiched between two Kähler geometries.

If orbits of $\xi$ all close, then $X = \text{circle orbibundle over a Kähler orbifold}$. 
The Kähler cone $\mathcal{C}(X)$ with $\text{dim}_\mathbb{C} \mathcal{C}(X) = n$ is Ricci-flat (Calabi-Yau) iff

$\nabla = \nabla_X$ is Einstein with $\text{Ric}_X = (2n - 2)\nabla_X$ (Sasaki-Einstein).

$\nabla_T$ is Einstein with $\text{Ric}_T = 2n \nabla_T$ (Kähler-Einstein).

E.g. $\mathcal{C}(X) = \mathbb{C}^n \setminus \{0\}$, $X = \text{round sphere}$, $\nabla_T = \text{Fubini-Study metric on } \mathbb{CP}^{n-1}$.

$\mathcal{C}(X)$ admits a holomorphic $(n, 0)$-form $\Omega$ with

$$\mathcal{L}_\xi \Omega = i\nabla \Omega,$$

iff $[\rho_T] = 2n[J] \in H^{1,1}(\mathcal{F}_\xi)$, $\rho_T = \text{transverse Ricci form}$. 
In string/M-theory:

- $\mathbb{R}^{1,3} \times C(X)$ and $\mathbb{R}^{1,2} \times C(X)$ are string/M-theory solutions for $n = 3$, $n = 4$, respectively.

- Can put $N$ D3-branes or $N$ M2-branes at tip $r = 0$ of the cones.

- There are corresponding $\text{AdS}_5 \times X$ and $\text{AdS}_4 \times X$ solutions, with $N$ units of flux through $X$.

- AdS/CFT: string/M-theory on these backgrounds is completely equivalent to the low-energy SCFT on the branes.

- Note Calabi-Yau cones exist in all dimensions, but applications to physics $\Rightarrow n = 3, n = 4$. 
GK geometry:

- Odd-dimensional manifold $(\mathcal{Y}, g)$, with a unit norm Killing vector $\xi$ ⇒ foliation $\mathcal{F}_\xi$.

- Again let $\eta = $ one-form dual to $\xi$, then

$$ g = \eta^2 + e^B g_T, $$

where $g_T$ is Kähler, and $e^B = \frac{1}{8} (n - 3)^2 R$.

- ⇒ need $n \geq 4$ and transverse scalar curvature $R = R_T > 0$.

- $d\eta = \frac{1}{2} (n - 3) \rho$ (compare Sasakian $d\eta = 2J$).
There is a natural closed two-form in the problem

\[ F = -\frac{2}{n-3} J + d(e^{-B}\eta). \]

\( \ast_Y F \) determines the flux in the corresponding string/M-theory solutions.

Bianchi identity \( d(e^{(4-n)B} \ast_Y F) = 0 \) is equivalent to the Kähler PDE

\[ \Box R = \frac{1}{2} R^2 - R_{ij}R^{ij}. \]

We call a solution to this an on-shell GK geometry.
The metric cones $\mathcal{C}(Y)$:

- $dr^2 + r^2 g_Y$, where $r > 0$.

- These are naturally complex, with holomorphic $(n, 0)$-form $\Omega$ and
  \[ \mathcal{L}_\xi \Omega = \frac{2}{n-3} i \Omega, \]
  where $\xi = I(r \partial_r)$.

- Not Kähler! (Physically, because of the flux.)

- However, $\mathcal{C}(Y)$ is in some sense a “Calabi-Yau cone”.
Integrating the PDE over \( \mathcal{Y} \) implies

\[
\int_{\mathcal{Y}} \eta \wedge \rho^2 \wedge e^J = 0,
\]

\( \Rightarrow \) Sasakian geometries with \([\rho] \propto [J] \in H^{1,1}(\mathcal{F}_\xi)\) are not examples.

Circle bundles over products of Kähler-Einstein manifolds are examples, with at least one factor having non-positive curvature (PDE is a trivial algebraic equation).

Many classes of examples constructed using symmetry reduction:

- cohomogeneity one
- orthotoric Kähler
In string/M-theory:

- These describe supersymmetric $\text{AdS}_3 \times Y$ and $\text{AdS}_2 \times Y$ solutions of string/M-theory for $n = 4$, $n = 5$, respectively.

- They are sourced only by D3-brane and M2-brane flux, respectively:

$$F_5 = \text{vol}_{\text{AdS}_3} \wedge F + *_Y F, \quad G = \text{vol}_{\text{AdS}_2} \wedge F.$$ 

- Flux quantization over codimension two cycles $S_A$ in $Y$:

$$\int_{S_A} \eta \wedge \rho \wedge e^J = N_A = \text{fixed}.$$ 

(Mathematicians: think fixing the transverse Kähler class $[J] \in H^{1,1}(\mathcal{F}_\xi)$, although this is not quite equivalent).
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There is a natural physical connection between the two rows:

- **Wrap** $d = 4$ (D3-brane), $d = 3$ (M2-brane) SCFTs on Riemann surface $\Sigma$ ⇒ flow to $d = 2$, $d = 1$ SCFTs.

- Geometrically, $Y = \text{total space of } X \text{ fibration over } \Sigma$:

  $X \hookrightarrow Y \rightarrow \Sigma$.

- The $\text{AdS}_3$, $\text{AdS}_2$ solutions may arise as near horizon limits of extremal black strings/black holes in parent $\text{AdS}_5$, $\text{AdS}_4$, with horizon $= \Sigma$. 
• Suggests the existence of large classes of GK geometries/black hole solutions.

• Classes of explicit examples have been constructed, but it’s clear from physics there should be many more $\Rightarrow$ very rich.

• We recently attacked this following an idea due to Martelli-JFS-Yau.

• Assuming a solution exists, one can deduce some interesting properties.
Volume minimization (Martelli-JFS-Yau):

• In Sasakian geometry, fix a complex cone $C(X)$, with holomorphic $(n, 0)$-form $\Omega$ and $\mathcal{L}_\xi \Omega = i n \Omega$.

• In general the Reeb vector field $\xi$ can move: write $\xi = \sum_{i=1}^{s} b_i \partial \varphi_i$, $b_i \in \mathbb{R}$, holomorphic torus action $U(1)^s$.

• The Reeb vector $\xi$ for a Sasaki-Einstein metric minimizes the volume $\int_X \eta \wedge e^J$ over this space of vector fields.

• Mathematicians: a critical $\xi$ has zero transverse Futaki invariant = obstruction to existence of Kähler-Einstein metric. Related to K-stability.
GK geometries extremize the following supersymmetric action

\[ S(\xi, [J]) = \int_Y \eta \wedge \rho \wedge e^J, \]

subject to the constraints

- \[ \int_Y \eta \wedge \rho^2 \wedge e^J = 0 \quad (\Leftrightarrow \text{integral of PDE over } Y \text{ is zero}) \]

- \[ \int_{S_A} \eta \wedge \rho \wedge e^J = N_A \quad (\text{flux quantization through codimension 2 cycles } S_A) \]

Note: depend only on \( \xi \), and transverse Kähler class \([J] \in H^{1,1}(F_\xi)\).
This geometric extremal problem is related to field theory extremal problems: $c$-extremization, $I$-extremization, and the attractor mechanism in black holes.

- For $\text{AdS}_3 \times Y_7$ solutions, the dual $d = 2$ SCFT has a central charge $c$, and

$$c = \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} S.$$ 

- For $\text{AdS}_2 \times Y_9$ solutions, the partition function $Z$ of the dual SCQM has

$$\log Z = \frac{4\pi L^9}{(2\pi)^8 \ell_p^9} S.$$ 

- For classes of extremal black holes in four dimensions, with near horizon $\text{AdS}_2$ region, also

$$S_{\text{BH}} = \frac{4\pi L^9}{(2\pi)^8 \ell_p^9} S.$$
I’ll describe how to compute the action $S$ for

$$X_5 \hookrightarrow Y_7 \rightarrow \Sigma,$$

where $X_5 =$ toric Sasakian 5-manifold, $\Sigma = \Sigma_g =$ Riemann surface.

- Physically: D3-brane wrapping holomorphic curve $\Sigma_g \subset$ Calabi-Yau 4-fold. Normal space to $\Sigma_g$ is a Calabi-Yau 3-fold $= C(X_5)$.
- $C(X_5) =$ affine toric variety, specified by 3d convex rational polyhedral cone.
- Use $U(1)^3$ action to fibre over $\Sigma_g \Rightarrow$ Chern numbers $(n_1, n_2, n_3) \in \mathbb{Z}^3$.
- Choose basis so holomorphic $(3, 0)$-form on $C(X_5)$ has charges $(1, 0, 0)$ under $U(1)^3$. Then $n_1 = 2(1 - g)$. 

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Everything may be expressed in terms of the “master volume” of the fibre $X_5$:

$$\mathcal{V} = \int_{X_5} \eta \wedge e^\omega,$$

where $\omega = \text{transverse Kähler form}$. May compute explicitly in terms of combinatorial data:

$$\mathcal{V} = \frac{(2\pi)^3}{2} \sum_{a=1}^{d} \lambda_a \frac{\lambda_{a-1}(\vec{v}_a, \vec{v}_{a+1}, \vec{b}) - \lambda_a(\vec{v}_{a-1}, \vec{v}_{a+1}, \vec{b}) + \lambda_{a+1}(\vec{v}_{a-1}, \vec{v}_a, \vec{b})}{(\vec{v}_{a-1}, \vec{v}_a, \vec{b})(\vec{v}_a, \vec{v}_{a+1}, \vec{b})}.$$

- Polyhedral cone has primitive inward pointing normal vectors $\vec{v}_a \in \mathbb{Z}^3$, $a = 1, \ldots, d$.
- Recall $\xi = \sum_{i=1}^{s} \vec{b}_i \partial \varphi_i$, so $\vec{b} \in \mathbb{R}^3$, and $(\cdot, \cdot, \cdot) = 3 \times 3$ determinant.
- $\lambda_a \in \mathbb{R}$ specify the transverse Kähler class $[\omega]$. 
One can derive the following remarkable formula:

\[
S = -A \sum_{a=1}^{d} \frac{\partial V}{\partial \lambda_a} - 4\pi \sum_{i=1}^{3} n_i \frac{\partial V}{\partial b_i}.
\]

where \( A = \text{Kähler class for } \Sigma_g \) (its area).

The constraints are:

1. \( 0 = A \sum_{a,b=1}^{d} \frac{\partial^2 V}{\partial \lambda_a \partial \lambda_b} - 2\pi n_1 \sum_{a=1}^{d} \frac{\partial V}{\partial \lambda_a} + 4\pi \sum_{a=1}^{d} \sum_{i=1}^{3} n_i \frac{\partial^2 V}{\partial \lambda_a \partial b_i} \) (PDE),

2. \( \frac{2(2\pi \ell_s)^4 g_s}{L^4} N = -\sum_{a=1}^{d} \frac{\partial V}{\partial \lambda_a} \) (flux through fibre \( N \), number of D3-branes),

3. \( \frac{2(2\pi \ell_s)^4 g_s}{L^4} M_a = \frac{A}{2\pi} \sum_{b=1}^{d} \frac{\partial^2 V}{\partial \lambda_a \partial \lambda_b} + 2 \sum_{i=1}^{3} n_i \frac{\partial^2 V}{\partial \lambda_a \partial b_i} \) (baryonic fluxes).
Pick your favourite affine toric variety \( C(X_5) \), specified by \( \vec{v}_a \in \mathbb{Z}^3 \).

Take \( \Sigma_g \), and fibre \( X_5 \) over it using \( (n_1 = 2(1 - g), n_2, n_3) \in \mathbb{Z}^3 \).

Solve constraints for fixed fluxes: \( N = \) number of D3-branes, \( M_a = \) baryonic fluxes.

In practice this determines the transverse Kähler class data \( \{A, \lambda_a\} \). Substituting back into the action, it is then a function of \( \vec{b} = (2, b_2, b_3) \):

\[
S = S(b_2, b_3; \vec{v}_a, g, n_2, n_3, N, M_a).
\]

\( S \) is a quadratic function of the choice of vector field \( (b_2, b_3) \)!
Field theory dual:

- The $\text{AdS}_5 \times X_5$ solution is dual to a $d = 4, \mathcal{N} = 1$ quiver gauge theory.

- Dimensionally reducing this on $\Sigma_g$ to $d = 2 \Rightarrow$ trial $c$-function of Benini-Bobev, a function of the choice of R-symmetry.

- Action $S$ is identical to this field theory function! Proof: Hosseini-Zaffaroni.
Some open issues:

- GK solutions necessarily extremize the action $S$, but there are clear counterexamples to the converse, e.g. solutions with $c < 0$.

- What are the necessary and sufficient conditions for solving the Kähler PDE $\Box R = \frac{1}{2} R^2 - R_{ij}R^{ij}$?

- Being at a critical point of $S$ is analogous to having zero Futaki invariant in Kähler-Einstein geometry. Existence problem for Sasaki-Einstein manifolds is solved (Collins-Székelyhidi), generalizing K-stability of Chen-Donaldson-Sun.

- This is dually related to necessary and sufficient conditions for a superconformal fixed point in field theory.

- What types of geometries admit solutions? What characterizes e.g. the complex cone $C(Y)$?
There is a similar construction for M2-branes. I'll focus on the simple case of

\[ X_7 \hookrightarrow Y_9 \rightarrow \Sigma_g, \]

where \( X_7 = S^7 \). Twisting specified by \( (n_1 = 2(1 - g), n_2, n_3, n_4) \in \mathbb{Z}^4 \).

Following analogous procedure of imposing constraints, flux quantization \( \Rightarrow \)

\[
S = S(b_1 = 1, b_2, b_3, b_4) = -\frac{2\sqrt{2}\pi}{3} N^{3/2} \sum_{i=1}^{4} n_i \frac{\partial F}{\partial \Delta_i},
\]

- \( F = \sqrt{\Delta_1\Delta_2\Delta_3\Delta_4} = \) prepotential for \( d = 4 \) STU gauged supergravity!
- \( \Delta_i = -\frac{2}{N} \frac{\partial V}{\partial \lambda_i} = \) certain linear functions of \( \vec{b} \).
The M2-branes wrapped on $\Sigma_g$ are associated to certain extremal black hole solutions in $d = 4$, STU gauged supergravity Benini-Hristov-Zaffaroni.

These have near horizon geometries that are $\text{AdS}_2 \times \Sigma_g$, that uplift to $D = 11$ solutions $\text{AdS}_2 \times Y_9$, with $S^7 \hookrightarrow Y_9 \twoheadrightarrow \Sigma_g$.

Twisting parameters $n_i \in \mathbb{Z}^4$ are black hole magnetic charges under $U(1)^4$.

The attractor mechanism in STU gauged supergravity is identical to extremizing our GK geometry action $S$!

At the critical point for $(b_2, b_3, b_4)$, this determines $S_{\text{BH}}$. 
Some interesting open questions:

- We’ve seen the attractor mechanism, that determines black hole entropy via an extremal problem, is equivalent to a geometric extremal problem.

- Is there a general geometric attractor mechanism for black holes in string/M-theory?

- In general this should also involve electric charges and angular momentum, not just magnetic charges.

- Large classes of supersymmetric black string/black hole solutions should exist, but are not known (I have described their near horizon geometry).