$G_2$, $\text{Spin}(7)$ BPS Equations and T-branes

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$G_2 \times \text{T-branes}$ \cite{1906.02212} Barbosa, Cveti\c{c}, Heckman, CL, Torres, Zoccarato

$\text{Spin}(7)$ \cite{1811.01959} Heckman, CL, Lin, Zoccarato

Cveti\c{c}, Heckman, Rocha\~is, Torres, Zoccarato

\cite{2003.13682}
Special holonomy $\Rightarrow$ important super-symmetric compactification of string theory

10D string theory $X_{10}$ (oriented spin)

want $X_{10} = \mathbb{R}^{1,3} \times Y_6$

holonomy is $SO(6)$

oriented compact spin manifold

"We are here!"
Sting theory is supersymmetric.

\[ S_8 \text{ (boson)} = \text{fermion} \]
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\( \rightarrow \) symmetry generated by some \( \varepsilon \) in spinor rep of \( SO(11,9) \) \( \text{[rotation of } X_{11,9} \text{]} \)

Want 4D spacetime to have susy

\[ \mathbb{E} = \mathbb{E}_{4D} \otimes \mathbb{E}_6 \]
\[ \text{SU}(4) \rightarrow \text{KSE:} \]
\[ \partial_6 \epsilon_6 + (\text{flux}) \epsilon_6 = 0 \]

If \( (\text{flux}) = 0 \) then \[ \partial_6 \epsilon_6 = 0 \]

"covariantly constant spinor".

\[ \text{SO}(1,9) \longrightarrow \text{SO}(1,3) \times \text{SO}(6) \]

\[ \mathbf{3} \oplus \mathbf{16} \longrightarrow (2,1,4) \oplus (1,2,\bar{4}) \]

\( \text{transform non-trivially under } \text{SO}(6) \)
If $Y_6$ is CY3

$SO(6) \rightarrow SU(3)$

4 $\rightarrow$ 3 $\oplus$ 1

Similar story for $G_2$, Spin(7)

$SO(7) \rightarrow G_2$

8 $\rightarrow$ 7 $\oplus$ 1

$SO(8) \rightarrow Spin(7)$

8 $\rightarrow$ 7 $\oplus$ 1

What if $(flux) \neq 0$?

$Y_6$ to be Kähler (hol $U(3)$) but turn on $(flux)$. 
\[ D_6 \equiv \epsilon_6 + (\text{flux}) \epsilon_6 = 0 \]

"SL(2,\mathbb{R}) flux" \(\rightarrow\) elliptically fibrad Calabi-Yau fourfold

"F-theory" \(\rightarrow\) 4d \(\mathcal{N}=1\)

\[ \pi: Y_4 \rightarrow \mathcal{B}_3 \]

\[ \hat{\text{Kähler}} \]
Interesting 4D \( N=1 \) physics

non-abelian gauge algebra \( g \)
matter fields in \( \mathfrak{g} \) \( R \) of \( g \)

\[
\text{gauge field} \quad (2,2; \text{adj})
\]

matter \(
\begin{cases} 
\text{left fermions} \quad (2,1; R) \\
\text{right fermions} \quad (1,2; R')
\end{cases}
\)

\# left and right chiral

-1 chiral spectrum.
What do physicists work from $G_2$- compactifications?

1) supersymmetry ✓
2) non-abelian gauge group.
3) chiral matter.

M-theory on a smooth $G_2$-manifold:
abelian gauge group
no charged matter

not the physics we want for the real world.
Singularity:

\[
\begin{cases}
\text{codim 4} & \rightarrow \text{non-abelian gauge} \\
\text{codim 6} & \rightarrow \text{non-chiral matter} \\
\text{codim 7} & \rightarrow \text{chiral matter}
\end{cases}
\]

TCS - singularity from CY3 "building blocks"
not in TCS construction?
Maxim: (ORM)

When one encounters an obstacle, what to do?

1) Mathematician: attack the obstacle until it is gone.
2) Physicist: slip past the obstacle and get to their destination.
Instead of building global geometry

we use dual gauge theory description.
Pantev-Wijnholt: partial topological twist of 7D super-Yang-Mills

on $M_3$ associated inside $G_2$

$\rightarrow$ BPS equation (PW system)

vector multiplet: $A$ gauge field

$\Phi$ scalar

$\Psi$ fermion.
$SO(1,6) \times SU(2)_R \longrightarrow SO(1,3) \times SU(2)_{M_3} \times SU(2)_R$

\[ A = (7, 1) \]

\[ \phi = (1, 3) \]

\[ (2, 2; 1, 1) \oplus (1, 1; 3, 1) \]

\[ (1, 1; 1, 3) \]

\[ \text{diagonal } SU(2) \]

$A, \phi$ are adjoint-valued one-forms on $M_3$. Combine them $A = A + i\phi$. 
Introduce new structures:

\[ D_{\mathbf{A}} = \frac{d}{dt} \mathbf{A} \quad F = [D_{\mathbf{A}}, D_{\mathbf{\bar{A}}}^\dagger] \]

\[ D_{\mathbf{\bar{A}}} = \frac{d}{dt} \mathbf{\bar{A}} \quad F = [D_{\mathbf{A}}, D_{\mathbf{\bar{A}}}] \]

In terms of the original field (\( \mathbf{A}, \phi \)):

\[ \ln \mathbf{F}_{ij} = \mathbf{F}_{ij} - [\phi_i, \phi_j] \quad [i, j = 1, \ldots, 3] \]
SUSY requires $F_-, D$-terms.

$$F_{ij} = 0, \quad g^{ij} D_{ij} = 0$$

moduli: span of vacua

$$\{ F_{ij} = 0 \} \leftrightarrow \{ \mathcal{G}_C \}$$

$$\{ F_{ij} = 0, \quad g^{ij} D_{ij} = 0 \} \leftrightarrow \{ \mathcal{G}_C \}$$
\( \text{Spin}(7) \rightarrow \text{SYM of 4-manifold: } M_8 \)

\[
\text{SO}(1,3) \times \text{SO}(4) \times \text{SO}(2)_\mathbb{R}
\]

\[
\text{SO}(2)_\mathbb{R} \times \text{SO}(2)_\mathbb{R}
\]

diagonal of these.

\( \phi_{SD} \) self-dual two-form

BPS equations: \( D_A \phi = 0 \quad F_{SD} + \phi \times \phi = 0 \)
\[ M_4 = M_3 \times S^1 \]

\[ \phi_{SD} = \phi_{PW} \wedge dt + *_3 \phi_{PW} \]

\[ F - [\phi_{PW}, \phi_{PW}] + *_3 (\partial_t A - d_3 A_t) = 0 \]

\[ D_A \phi_{PW} + *_3 D_t \phi_{PW} = 0 \]

\[ D_A *_3 \phi_{PW} = 0 \]

\[ A_t = \delta_t A = \partial_t \phi \]

\[ \text{PW equation} = 0 \]
Consider fluxed solutions of PW system. 

\[ (F \neq 0) \quad \text{[fluxless solutions] } \]

[Brown, Cizel, Hubner, Schütz, Namiki].

Consider \( M_3 \) locally, \( \Sigma \subset M_3 \)

local patch \( M_3 = \Sigma \times I \).
Let \( t \) be a coordinate. \( x_a = (x_1, x_2) \) coordinates. \( \Sigma \)

\[
\begin{align*}
F_{ab} - [\phi_a, \phi_b] &= 0 \\
D_a \phi_b - D_b \phi_a &= 0 \\
g^{ab} D_a \phi_b + g^{tt} D_t \phi_b &= 0
\end{align*}
\]

\[
F_{ta} - [\phi_t, \phi_a] = 0
\]

\[
D_t \phi_c - D_c \phi_t = 0
\]

\[
\text{“almost” the Hitchin equations.}
\]

\underline{Solution:}

Power series expansion in \( t \).

\[
A_i(t, x_a) = \sum_{j=0} \Lambda_i^{(j)}(x_a) t^j
\]
Pick a gauge: 

\[ \mathbf{A}_t^{(j)} = 0. \]

\[ \sum_{\nu = 0}^{\infty} \mathbf{G}_{ab}^{(j)} t^{\nu} = 0 \]

\[ \sum_{\nu = 0}^{\infty} \mathbf{H}_{ab}^{(j)} t^{\nu} = 0 \]

And 3 more equations which are linear in either \( \mathbf{A}^{(j+1)} \), \( \mathbf{\phi}^{(j+1)} \).

\[ j = 0 \quad \text{we get} \quad \mathbf{F}_{ob}^{(1)} - [\mathbf{\phi}_c^{(0)}, \mathbf{\phi}_b^{(0)}] = 0 \]

Assume \( \mathbf{A}_a^{(0)}, \mathbf{\phi}_a^{(0)} \) on such that these equations are solved.

\[ \Rightarrow \text{solved at all orders in } t. \]

\[ \mathbf{\phi}_a^{(0)} \text{ sets the trajectory of the solution} \]
Non-abelian background \((SU(3))\)

\[
\phi = \begin{bmatrix}
  i g_1 \partial_1 & - v_2 \epsilon e^{*} & \bar{e} e^{*} \partial_2 & 0 \\
  v_2 e^{*} \partial_2 & i g_2 \partial_2 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
  a & 0 & 0 & 0 \\
 0 & -a & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
a = \frac{i}{2} \left( \partial_{\bar{z}} f d\bar{z} - \partial_{\bar{z}} f d\bar{z} \right)
\]

\[\text{counting zero-modes.} \]

\[\Rightarrow \text{observe chiral matter fields.} \]

**Conclude**

7-brane configuration for sols to PW system

behaves like chiral matter

\[\Rightarrow \text{bypassed difficulty with codim 7 singularity.} \]