Title: Branes in the moduli space of Higgs bundles

Fixed data: \( C \), compact Riemann surface of genus \( g \) (possibly with punctures)

\[ G = \text{SU}(n), \quad G_\alpha = \text{SU}(n, \mathbb{C}) \]

\[ E \to C \] complex vector bundle of rank \( n \)

\[ \to \] Hitchin moduli space

\( M \) has a rich geometric structure

#1 \( M \) has a noncompact hyperkähler metric metric \( g_{\mathbb{C}} \)

\[ \to \] have a CP\(^2\) of Kähler manifolds \( M = (M, g_{\mathbb{C}}, I_\mathbb{C}) \)

\[ M_{g=0} \] is \( G_\alpha \)-Higgs bundle moduli space

\[ M_{g=0} \times \] moduli space of flat \( G_\alpha \) connection

\[ \boxed{\text{Def: A Higgs bundle is a pair} \ (\Omega E, \Phi)} \]

\[ \text{satisfying} \ \Omega \Phi = 0. \]

\[ \text{#2 In its avatar as the Higgs bundle moduli space,} \]

\( M \) is an algebraic completely integrable system

\[ M \]

\[ \text{\textbullet} \]
**Def** A solution of Hitchin equations is \((\mathfrak{g}, \mathbf{Y})\) plus a hermitian metric satisfying \(F_D^{-1} + [\mathbf{Y}, \mathbf{Y}^\dagger] = 0\).

**Branes in Hitchin moduli space**

Motivation: homological mirror symmetry & Hitchin moduli space

\[ M(C, G) \text{ is mirror to } M(C, {}^L G) \]

\[ \sim \text{ Langlands dual} \]

\[ M(C, G) \quad M(C, {}^L G) \]

Non-abelian then are dual abelian varieties.

**Def** A brane is an object in one of the following categories:

- on symplectic \(A\)-side, an \(A\)-brane is an object in the Fukaya category
- on complex \(B\)-side, a \(B\)-brane is an object in the derived category of coherent sheaves
The category of coherent sheaves.

**Def** (Approximate)
- an A-brane is a Lagrangian submanifold
- a B-brane is a holomorphic submanifold

Natural questions:
- Given an A-brane in $(M\mathcal{C}, \mathcal{C})$, what is the mirror B-brane in $(\mathcal{M}, \mathcal{C})$?
- Construct examples of submanifolds which are Lagrangian/holomorphic with respect to $\mathcal{I}/\mathcal{W}_{\mathcal{I}}$, $\mathcal{I}/\mathcal{W}_{\mathcal{I}}$, $\mathcal{K}/\mathcal{I}$.

4 types: $(B, A, A)$, $(B, B, B)$, $(A, A, B)$, $(A, B, A)$.

A few methods for constructing:
- reps of $\pi_1(C)$ which extend to reps of $\pi_1(M^3)$
- define an $(A, B, A)$ brane.
real structure "G-Higgs bundle"

\[ \text{SL}(n, \mathbb{C}) \]

\[ \text{SL}(n, \mathbb{R}) \quad \text{SO}(n, \mathbb{C}) \quad \text{SU}(n) \]

\[ A = \overline{A} \quad A^T = A^T \quad A^{-1} = \overline{A}^T \]

Famous \((B, A, A)\) brane: Hitchin section

Component of \(\pi_1(C) \to (P)\text{SL}(n, \mathbb{R})\)
containing Fuchsian representations

Solution of Hitchin's equations also must respect involution
so easy to write down

e.g. \(\Sigma = (E, \Theta) = K_e^{1/2} \oplus K_e^{-1/2} \quad K_e \) hol cotangent

\[ q = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix} \quad 1: K_e^{1/2} \to K_e^{-1/2} \otimes K_e \otimes K_e^{1/2} \]

\[ q_2: K_e^{-1/2} \to K_e^{1/2} \otimes K_e \otimes K_e \]

\[ q: \begin{pmatrix} 0 \\ \Theta K_e \end{pmatrix} \quad K_e^{-1/2} \otimes K_e^2 \]

\[ q_2 = \det(q)e \in H^0(K_e^2) \]

The hermitian metric \(h\) solving Hitchin's equation
is diagonal (on in rank 2)

\[ h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \lambda \in \mathbb{R}. \]
Many results about $M$ are easier to prove for the Hitchin section.

**ONE EXAMPLE:** Asymptotic Geometry of the Hitchin moduli space.

**Conjecture (Bavesto-Moore-Netzke)**

$$g_{Z^2}^2 - g_{sf} \sim e^{-t}$$

Exponential decay was first proved on the Hitchin section [Dumas-Netzke] before being extended off.

One more interesting brane: "opers" (Holomorphic Schrödinger operators). Central object in geometric Langlands correspondence.

Given a local hol coordinate on $C$

$$g_z = \frac{dz}{P(z)dz}$$

$K \in C^X$

$\leadsto$ differential operator

$$L_K = \frac{\partial^2}{\partial z^2} - \frac{P}{k^2}$$

Can make $L_K$ globally defined

Interpret $L_K : K_c^{-1/2} \longrightarrow K_c^{3/2}$ (rather than...
Interpret $L_\kappa$ as holomorphic flat $SU(2,\mathbb{C})$ connection:

$$L_\kappa f = 0 \iff \left( d + \frac{1}{\kappa} \left( 0 \ 1 \right) dz \right) (f_{\kappa}^f) = 0$$

$\nabla^\text{oper}_\kappa$

Natural holomorphic bundle? $E_\kappa$ is an extension

$$0 \to K_c^{1/2} \to E_\kappa \to K_c^{-1/2} \to 0$$

Given Higgs bundle $(\mathcal{E}, \mathcal{D})$, define the following family of flat connections

$$\nabla_{\mathcal{Y}, R} = \frac{R}{\mathcal{Y}} \mathcal{Y} + D_{(\mathcal{D}_E, h^2)}$$

Where $h_R$ solves Hitchin's equation $f_1((E, \mathcal{D}_E), \mathcal{Y})$.

\[ \text{Thm} \]

Fix $\kappa = \frac{\mathcal{Y}}{R}$ $(\mathcal{D}_E, \mathcal{Y})$ in Hitchin section

$$\nabla_\kappa = \lim_{R \to \infty} \nabla_{\mathcal{Y}, R} \text{ exists and }$$

$$\nabla_\kappa \text{ on } (E, \nabla_\kappa^0, 1) \text{ is gauge equivalent to } \nabla^\text{oper}_\kappa \text{ on } E_\kappa.$
c.f. conjecture of Gaiotto "Open and TBA"

\[ \text{Thm} \] (Collier-Wentworth) extend this off of Hitchin and away from open

\[ Y-\text{NAHC} : M_{\text{Higgs}} \to M \text{ flat-connection} \]

Conformal limits: \[ M_{\text{Higgs}} \to M \text{ flat-connections} \]

Viewing opens inside Hitchin moduli, questions about their asymptotic geometry.