# Title: Branes in the moduli space of Higgs bundles

Fixed data: $C$, compact Riemann surface of genus $g$
(possibly w/ punctures)

$$G = SU(n), \quad G_{\mathbb{C}} = SL(n,\mathbb{C})$$

$E \longrightarrow C$ complex vector bundle of rank $n$

$\rightsquigarrow$ Hitchin moduli space

$\mathcal{M}$ has a rich geometric structure

#1 $\mathcal{M}$ has a noncompact hyperkähler metric $g_{L^2}$

$\rightsquigarrow$ have a $\mathbb{CP}^1_\zeta$ of Kähler manifolds $\mathcal{M}_\zeta = (\mathcal{M}, g_{L^2}, I_\zeta)$

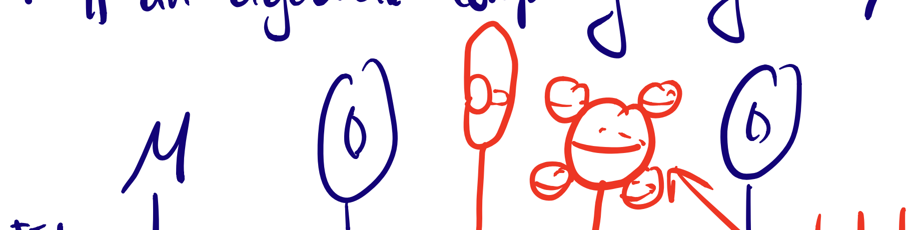
$\mathcal{M}_{\zeta=0}$ is $G_{\mathbb{C}}$-Higgs bundle moduli space

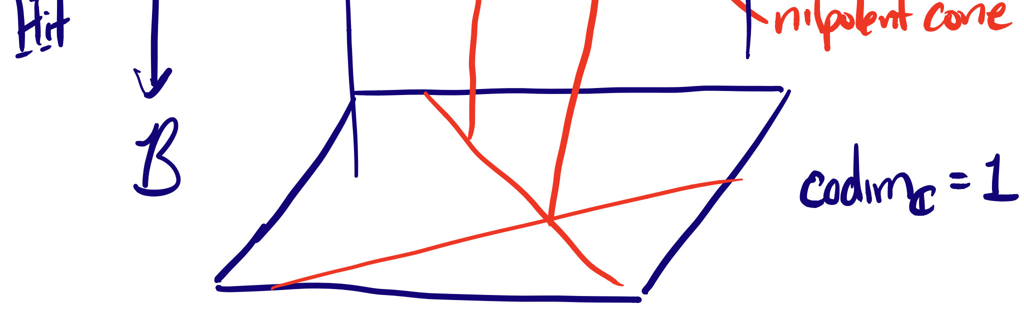
$\mathcal{M}_{\zeta\in\mathbb{C}^\times}$ is moduli space of flat $G_{\mathbb{C}}$ connections

$\boxed{\text{Def}}$ A <u>Higgs bundle</u> is a pair $(\bar{\partial}_E, \varphi)$
satisfying $\bar{\partial}_E \varphi = 0$. $\nwarrow \Omega^{1,0}(C, \mathrm{End}_0 C)$

$\uparrow$ traceless

#2 In its avatar as the Higgs bundle moduli space,
$\mathcal{M}$ is an algebraic completely integrable system






$\boxed{\text{Def}}$ A solution of Hitchin's equations is $(\bar\partial_E, \varphi)$ plus a hermitian metric satisfying $F_D^{\perp} + [\varphi, \varphi^{\dagger_h}] = 0$.

## Branes in Hitchin moduli space

Motivation: homological mirror symmetry for Hitchin moduli space

$$\mathcal{M}(C,G) \text{ is mirror to } \mathcal{M}(C, {}^L G)$$

← Langlands dual

$$\begin{tikzcd} \mathcal{M}(C,G) \arrow[dr] & & \mathcal{M}(C,{}^L G) \arrow[dl] \\ & B & \end{tikzcd}$$

Nonsingular fibers are dual abelian varieties.

$\boxed{\text{Def}}$ A <u>brane</u> is an object in one of the following categories:

- on symplectic / A-side, an <u>A-brane</u> is an object in Fukaya category
- on complex / B-side, a B-brane is an object in the derived

category of coherent sheaves.

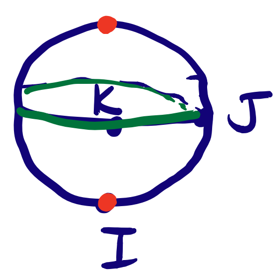
$\boxed{\text{Def}}$ (Approximate)

- an A-brane is a Lagrangian submanifold
- a B-brane is a holomorphic submanifold

Natural questions:

- given A-brane in $(\mathcal{M}(C,G), \omega_I)$ what is the mirror B-brane in $(\mathcal{M}(C, {}^LG), I)$?
- construct examples of submanifolds which are Lagrangian/holomorphic with respect to $I/\omega_I$ $J/\omega_J$ $K/\omega_K$

4 types: (B, A, A) $\longleftrightarrow$



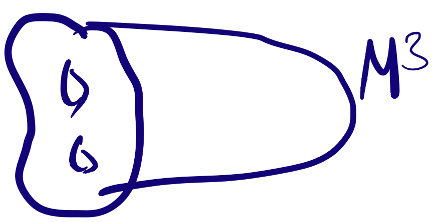

(B, B, B)

mirror symm action on types

(A, A, B)

(A, B, A).

A few methods for constructing:

- 


reps of $\pi_1(C)$ which extend to reps of $\pi_1(M^3)$ define an (A,B,A) brane

∘ real structures "G-Higgs bundle"

$$SL(n,\mathbb{C})$$

$$\begin{array}{ccc} SL(n,\mathbb{R}) & SO(n,\mathbb{C}) & SU(n) \\ A=\bar{A} & A^{-1}=A^{T} & A^{-1}=\bar{A}^{T} \end{array}$$

Famous $(B,A,A)$ brane: Hitchin section

Component of $\pi_1(C) \longrightarrow (P)SL(n,\mathbb{R})$
containing Fuchsian representations

Solution of Hitchin's equations also must respect involution
so easy to write down

e.g. $\mathcal{E}=(E,\bar\partial) = K_C^{1/2} \oplus K_C^{-1/2}$ $\quad K_C$ hol cotangent

$$\varphi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}.$$

$1: K_C^{1/2} \longrightarrow K_C^{-1/2} \otimes K_C \simeq K_C^{1/2}$

$q_2: K_C^{-1/2} \longrightarrow K_C^{1/2} \otimes K_C \simeq K_C^{-1/2} \otimes \underset{\substack{\cup \\ q_2}}{\underline{K_C^{2}}}$

$$\varphi: \mathcal{E} \longrightarrow \mathcal{E} \otimes \underline{K_C}$$

$$q_2 = \det(\varphi) \in H^0(K_C^2)$$

The hermitian metric $h$ solving Hitchin's equation
is diagonal (on in rank 2)

$$h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \lambda \in \mathbb{R}.$$

⇝ Many results about $\mathcal{M}$ are easier to prove for the Hitchin section.

ONE EXAMPLE: Asymptotic Geometry of the Hitchin moduli space.

Conjecture (Gaiotto-Moore-Neitzke)

$$g_{L^2} - g_{sf} \sim e^{-\varepsilon t}$$


Exponential decay was first proved on the Hitchin section [Dumas–Neitzke] before being extended off.

One more interesting brane: "opers" (Holomorphic Schrödinger operators)

Central object for geometric Langlands correspondence.

Given $z$ local hol coordinate on $C$

$q_2 = P(z)dz^2$

$\hbar \in \mathbb{C}^\times$

⇝ differential operator $L_\hbar = \frac{\partial^2}{\partial z^2} - \frac{P^2}{\hbar^2}$

Can make $L_\hbar$ globally defined

- interpret $L_\hbar : K_C^{-1/2} \longrightarrow K_C^{3/2}$ (rather than function)

• only allow coord changes like $w = \frac{az+b}{cz+d}$ ("fixing $\mathbb{CP}^1$ struct on $\mathbb{C}$")

Interpret $L_\hbar$ as holomorphic flat $SL(2,\mathbb{C})$ connection:

$$L_\hbar f = 0 \iff \underbrace{\left( d + \frac{1}{\hbar}\begin{pmatrix} 0 & 1 \\ P & 0 \end{pmatrix} dz \right)}_{\nabla_\hbar^{oper}} \begin{pmatrix} f \\ -\hbar f' \end{pmatrix} = 0$$

Natural holomorphic bundle? $\mathcal{E}_\hbar$ is an extension

$$0 \longrightarrow K_C^{1/2} \longrightarrow \mathcal{E}_\hbar \longrightarrow K_C^{-1/2} \longrightarrow 0$$

Given Higgs bundle $((E,\bar\partial_E),\varphi)$ define the following family of flat connections

$$\nabla_{\zeta,R} = \frac{R}{\zeta}\varphi + D_{(\bar\partial_E, h_R)} + R\,\zeta\,\varphi^{\dagger_{h_R}}$$

where $h_R$ solves Hitchin's equations for $((E,\bar\partial_E), R\varphi)$.

$\boxed{\text{Thm}}$ Fix $\hbar = \frac{\zeta}{R}$ $(\bar\partial_E,\varphi)$ in Hitchin section
(DFKMMN)

$$\nabla_\hbar = \lim_{R\to 0} \nabla_{\zeta,R} \text{ exists and}$$

$\nabla_\hbar$ on $(E, \nabla_\hbar^{0,1})$ is gauge equivalent to oper $\nabla_\hbar^{oper}$ on $\mathcal{E}_\hbar$.

c.f. conjecture of Gaiotto "Opers and TBA"

$\boxed{\text{Thm}}$ (Collier-Wentworth) extend this off of Hitchin and away from opers

$$\hbar\text{-NAHC}: \mathcal{M}^{\text{Higgs}} \longrightarrow \mathcal{M}^{\text{flat connection}}$$

$$\text{conformal limits}: \mathcal{M}^{\text{Higgs}} \longrightarrow \mathcal{M}^{\text{flat connections}}$$

Viewing opers inside Hitchin moduli, questions about their asymptotic geometry.