GEOMETRIC UNIFICATION OF HIGGS BUNDLE VACUA

September 16th 2020
BHV SYSTEM

Higgs bundle on a Kähler surface $S$

Higgs field is a complex adjoint valued $(2,0)$-form

Supersymmetry equations

$$\bar{\partial}_A \phi = 0$$
$$F_{(0,2)} = 0$$

Equations are in unitary frame

出现在 F-theory 紧缩化中的 Calabi-Yau 四维面

Kähler form

Holomorphic structure makes construction of solutions feasible (even non-abelian)
PW SYSTEM

Higgs bundle on a three-manifold $Q$

Higgs field is an adjoint valued 1-form

Supersymmetry equations are

$$D_A \phi = 0 \quad D_A \star \phi = 0$$

$$F = [\phi, \phi]$$

Can be written in terms of complexified connection $\mathcal{A} = A + i\phi$ as

$$\mathcal{F} = 0 \quad D_A \star \phi = 0$$

Appears in M-theory compactifications on a $G_2$-manifold
**SPIN(7) SYSTEM**

Higgs bundle on a *four-manifold* $M$

Higgs field is an adjoint valued self-dual two form

Supersymmetry equations are

\[ D_A \phi = 0 \]
\[ F_{SD} + \phi \times \phi = 0 \]

$F_{SD}$ is the self-dual part of the curvature of the bundle

Cross product is

\[ (\phi \times \phi)_{ij} = \frac{1}{4} [\phi_{ik}, \phi_{jl}] g^{kl} \]

This system is very similar to the Vafa-Witten system

Spin(7) system includes other Higgs bundles as solutions

[Heckman, Lawrie, Ling, GZ '18]

[Vafa, Witten '94]
PW TO SPIN(7)

To relate the two consider the case of Spin(7) on a four-manifold $M = Q \times S^1$.

Then write the SD forms as

$$\phi = \hat{\phi} \wedge dt + \star_3 \hat{\phi} \quad \hat{\phi} \in \Omega^1(Q)$$

The Spin(7) supersymmetry equations become

$$F - [\hat{\phi}, \hat{\phi}] + \star_3 (D_tA - d_3 A_t) = 0 \quad D_A \hat{\phi} + \star_3 D_t \hat{\phi} = 0$$

$$D_A \star_3 \hat{\phi} = 0$$

One recovers PW if $A_t = \partial_t A = \partial_t \hat{\phi} = 0$

PW is the dimensional reduction of Spin(7) along the additional direction
BHV TO SPIN(7)

We take the four manifold to be a Kähler manifold $S$.

In a Kähler manifold $S$, two forms admit a Hodge decomposition:

$$\Omega^2_+(S) \simeq H^{(2,0)}(S) \oplus H^{(0,2)}(S) \oplus H^{(1,1)}_{n.p.}(S)$$

The Spin(7) equations become:

$$F_{(0,2)} - \frac{i}{2} \phi_{(1,1)} \times \phi_{(0,2)}^\dagger = 0 \quad \bar{\partial}_A \phi_{(2,0)} - \frac{i}{2} \partial_A \phi_{(1,1)} = 0$$

$$J \wedge F = \frac{i}{2} [\phi_{(2,0)}, \phi_{(0,2)}^\dagger]$$

BHV is recovered for configurations with the $(1,1)$-component set to zero.
$BHV \subset Spin(7) \supset PW$
PW TO BHV INTERPOLATIONS

We work with a four manifold $M = \Sigma \times \mathbb{R} \times S^1$

Calling $t$ the coordinate on $\mathbb{R}$ we have

\[ |\phi_{(1,1)}| \sim e^{\lambda t} \quad t \to -\infty \]
\[ |\partial_\theta \psi| \sim e^{-\lambda_2 t} \quad |A_\theta| \sim e^{-\lambda_1 t} \quad t \to +\infty \]

For $t \to -\infty$ solution approaches BHV

For $t \to +\infty$ solution approaches PW

In the middle there is a Spin(7) solution

This is the Higgs bundle version of GCS construction of Spin(7) manifolds

[Braun, Schäfer-Nameki '18]
To give a concrete example, take $\Sigma \simeq T^2$

Take abelian solutions $\phi \times \phi = 0$ with no flux

$$\phi = \phi_{BHV} + \phi_{PW}$$

$\phi_{BHV} = \omega_\Sigma \wedge \rho(w) + h \cdot c.$

holomorphic on $\Sigma$

three poles on the cylinder

$$\phi_{PW} = \partial_z f dz \wedge dw + \partial_{\bar{z}} f d\bar{z} \wedge d\bar{w} + \frac{i}{2} \partial_t f (dz \wedge d\bar{z} + dw \wedge d\bar{w})$$

$$\partial_u f = \text{Re} \left[ f_1(u) \frac{\tanh(u) + 1}{2} \right] \quad \partial_v f = \text{Re} \left[ f_2(v) \frac{\coth(v) + 1}{2} \right]$$

$u = t + ix$

$v = t + iy$
PW TO PW INTERPOLATIONS

Take the four manifold $M = Q \times \mathbb{R}$

System will interpolate between different PW solutions (physically we build an interface)

Take $Q = T^3$

To build solution: take a pair of coordinates $(x, y)$ inside the $T^3$

Build a PW solution on this $T^2_{(x,y)} \times \mathbb{R}$

$$\phi_{(x,y)} = \text{Re} \left[ f_1(t + ix) \frac{\tanh(t + ix) + 1}{2} + f_2(t + iy) \frac{\tanh(t + iy) + 1}{2} \right]$$

Build a different PW solution with another $T^2 \subset T^3$, and combine them
ZERO MODES OF SPIN(7)

Zero modes equations come from taking infinitesimal variations around a solution

\[ A = \langle A \rangle + a \quad \phi = \langle \phi \rangle + \phi \]

Zero mode equations for Spin(7) are

\[ D_A a + \star D_A a + \phi \times \phi = 0 \]
\[ D_A \phi - [\phi, a] = 0 \]

Solutions are identified via gauge transformations

\[ a \simeq a + D_A \xi \]
\[ \phi \sim \phi + [\phi, \xi] \]

This can be characterised in terms of a complex

\[ 0 \to \Omega^0(adE) \xrightarrow{\delta_0} \Omega^1(adE) \oplus \Omega^2_+(adE) \xrightarrow{\delta_1} \Omega^2_+(adE) \oplus \Omega^3(adE) \to 0 \]

Space of zero modes is

\[ T\mathcal{M}_{\text{Spin}(7)} \simeq \frac{\ker \delta_1}{\im \delta_0} \]
ZERO MODES OF SPIN(7)

Where are zero modes localised?

Take abelian solutions (with no flux)

Use spectral cover methods: for $\mathfrak{a}_{n+1}$ in the fundamental rep

$$\det(\phi - v \mathbb{I}_n) = 0 \quad \forall \in \Omega^2_+(M)$$

Modes are localised where sheets intersect: this happens in codimensions 2 and 3
CONCLUSIONS

We studied Higgs bundles that appear in Spin(7) compactifications of M-theory.

This system provides a unification of other known Higgs bundles.

It is possible to build interpolations between BHV and PW solutions.

Moreover it is possible to build interfaces between PW solutions.

Thank you!