Magnetic Quivers for Singular HyperKähler Spaces

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10 January 2022
PLAN OF THE TALK

I - Introduction: two questions and one (partial) answer.

II - The concept of magnetic quivers

III - Examples

IV - Generalizations and unknown territory.
I- Introduction:
Two questions and one (partial) answer
SYMPLECTIC SINGULARITIES

- $X$ is normal affine variety over $\mathbb{C}$.

- $X$ has symplectic singularities if there is a holomorphic symplectic form $\omega$ on $X$ smooth whose pullback extends to a holomorphic 2-form $\Omega$ on any resolution $Y \to X$. [Beauville 99]
SYMPLECTIC SINGULARITIES

- $X$ is a normal affine variety over $\mathbb{C}$.

- $X$ is (has) symplectic singularities if there is a holomorphic symplectic form $\omega$ on $X_{smooth}$ whose pullback extends to a holomorphic 2-form $\Omega$ on any resolution $Y \to X$. [Beauville 99]

- $X$ is a conical symplectic singularity (CSS) if it has a good $\mathbb{C}^*$-action ($\mathcal{O}[X] = \bigoplus_{i \in \mathbb{N}} R_i$ with $R_0 = \mathbb{C}$) with respect to which $\omega$ is homogeneous. [Namikawa 11]
EXAMPLES OF CSS

- Normal nilpotent orbit closures
- Nakajima quiver varieties
- Conical hyperKähler quotients
- Higgs branch of supersymmetric QFT with 8 supercharges
- Coulomb branch of "good" 3d $\mathcal{N}=4$ theories

Examples in other talks today.
EXAMPLES OF CSS

- Normal nilpotent orbit closures
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- Examples in other talks today.

Question 1: is there a UNIFORM description of CSS?
For $X$ a CSS there exists a finite stratification:

- Gives the structure of the CSS
- In physics, characterizes how theories are connected to each other.
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- Gives the structure of the CSS
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**Question 2:** How can this Hasse diagram be computed explicitly?
Magnetic Quivers

A partial answer is given by Magnetic Quivers and the quiver subtraction algorithm.
II - The concept of Magnetic Quiver
DEFINITIONS

- Provisional definition: Quiver = connected finite graph with nodes labeled by positive integers, with balance $\geq 0$.

Example:

```
1 -- 2 -- 2 -- 1
```

DEFINITIONS

- Provisional definition: Quiver = connected finite graph with nodes labeled by positive integers, with balance \( \geq 0 \).

Example:

\[
\begin{array}{c}
1 & \rightarrow & 2 & \rightarrow & 2 & \rightarrow & 1 \\
\end{array}
\]

- Map \( \mathcal{C} \)

[Cremonesi, Hanany, Zaffaroni 13]  
[Nakajima 15] [Bullimore, Dimofte, Gaiotto 15]  
[Braverman, Finkelberg, Nakajima 16]

Quiver \( \rightarrow \) 3d \( \mathcal{N}=4 \) gauge theory \( \rightarrow \) 3d \( \mathcal{N}=4 \) IR SCFT \( \rightarrow \) Coulomb branch

\( \mathcal{C} \)
**Definitions**

*Given a CSS $X$, a quiver $Q$ is a magnetic quiver* for $X$ if $\mathcal{C}(Q) = X$.

*(more generally, $\bigcup_{i=1}^{\hat{N}} \mathcal{C}(Q_i) = X$)*
DEFINITIONS

- Given a CSS $X$, a quiver $Q$ is a magnetic quiver for $X$ if $\mathcal{C}(Q) = X$.

  *(more generally, $\bigcup_{i=1}^{n} \mathcal{C}(Q_i) = X$)*

- Quiver subtraction is an algorithm
  \[
  \{ \text{Quivers} \} \xrightarrow{S} \{ \text{Hasse diagrams} \}
  \]

[AB, Cabrera, Grimminger, Hanany, 
Spierling, Zajac, Zhong, 19]
EXAMPLE OF QUIVER SUBTRACTION

*See my June 2020 talk*
EXAMPLE OF QUIVER SUBTRACTION

* See my June 2020 talk
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EXAMPLE OF QUIVER SUBTRACTION

*See my June 2020 talk*
III - Examples

A - HyperKähler quotients (quiver varieties)
B - Wreathed quivers
C - Quasi-minimal singularities
D - Higgs branch of 4d $\mathcal{N}=2$ SCFTs
QUIVER VARIETIES : $\text{SL}/\text{GL}$. 

\[ Higgs \left( \begin{array}{c} \cdots \nabla_{N_i} \cdots \nabla_{N_n} \\ k_1 \cdots k_n \end{array} \right) = \text{HK quotient by } \text{GL}(k_1) \times \cdots \times \text{GL}(k_n). \]

When \( k_{i-1} + k_{i+1} + N_i \geq 2k_i \), magnetic quiver well known.
QUIVER VARIETIES: SL/GL.

\[ \text{Higgs} \left( \begin{array}{c|ccc} 0 & N_1 & \cdots & N_n \\ \hline k_1 & & & \\ \hline & k_n & & \\ \end{array} \right) = \text{HK quotient by } \prod_{k_i} \text{GL}(k_i) \times \cdots \times \text{GL}(k_n). \]

When \( k_{i-1} + k_{i+1} + N_i \geq 2k_i \), magnetic quiver well known.

Generalizations

1) Drop the \((*)\) condition

2) Replace some \( \text{GL}(k_i) \) by \( \text{SL}(k_i) \) in \((***)\)
**Quiver Varieties: ** $SL/GL$

$$\text{Higgs} \left( \begin{array}{c} \square \text{N}_1 \\ 0 \vdots \square \text{N}_n \\ k_1 \vdots \k_n \end{array} \right) = \text{HK quotient by } GL(k_1) \times \cdots \times GL(k_n).$$

When $k_{i-1} + k_{i+1} + \text{N}_i \geq 2k_i$, magnetic quiver well known.

**Generalizations**

1) Drop the $(\ast)$ condition

2) Replace some $GL(k_i)$ by $SL(k_i)$ in $(\ast\ast)$

**Answer**: the Brane Locking algorithm

[AB, Grimminger, Hanany, Kalveks, Zhong 21]
THE BRANE LOCKING ALGORITHM

$GL(5)$ $GL(5)$

#MQ = 1
THE BRANE LOCKING ALGORITHM

$GL(5) \quad SL(5)$

$\# MQ = 2$
THE BRANE LOCKING ALGORITHM

\[ \# \text{MQ} = 2 \]
THE BRANE LOCKING ALGORITHM

\[ 2 \quad 3 \]
\[ SL(5) \quad SL(5) \]

\[ \# MQ = 4 \]
THE BRANE LOCKING ALGORITHM

Questions:

- Cross check the results using other methods?
- What is $\# MQ$ in general? ($= \#$ irreducible components)
- What is the dimension?
- Physics: is brane locking part of string theory?
WREATHED and FOLDED QUIVERS

Quiver $Q$ with automorphism (sub-)group $\Gamma$.

$$C(\Gamma\text{-wreathed } Q) = C(Q)/\Gamma$$

[AB, Hanany, Miketa 20]

$$C(\Gamma\text{-folded } Q) = C(Q)^\Gamma$$

[Cremonei, Fedito, Hanany, Mekareeya 14]
[Nakajima, Weekes 13]

This is a generalization of the map $C$

$\rightarrow$ New possibilities for $C^{-1}$
<table>
<thead>
<tr>
<th>Initial</th>
<th>Discretely Gauged</th>
<th>Folded</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image0.png" alt="Initial Graph" /></td>
<td><img src="image1.png" alt="Discretely Gauged Graph" /></td>
<td><img src="image2.png" alt="Folded Graph" /></td>
</tr>
<tr>
<td>$\mu_2 t^2$</td>
<td>$(\mu_1 + \mu_2) t^2 + \mu_1^2 t^4$</td>
<td>$\mu_2 t^2$</td>
</tr>
<tr>
<td>$(\mu_1 + \mu_2) t^2 + (2\mu_1 + \mu_2) t^2 + \mu_2 t^4$</td>
<td>$\mu_2 t^2 + (\mu_1^2 + \mu_2) t^4 + 2\mu_1^3 t^6 - \mu_6 t^{12}$</td>
<td>$\mu_2 t^2$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$B_3$</td>
<td>$G_2$</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$G_2$</td>
<td>$G_2$</td>
</tr>
</tbody>
</table>

The expressions represent the polynomials associated with each graph, where $\mu_1$ and $\mu_2$ are parameters, and $G_2$ indicates the symmetry group for each graph.
Example: Higgs (\[0^6 \atop \sim \text{SL}(3)] \, \sim \text{SL}(3) = \text{SL}(3) \times \mathbb{Z}_2$

$\mathbb{Z}_2$-wreathed quiver:

Check using refined Hilbert series computation:

Wendt's integration formula $\leftrightarrow$ Wreathed Monopole formula

[Anis-Tamargo, AB, Pini 21]

[Wendt 01]
QUASI-MINIMAL SINGULARITIES

$\equiv$ Slices in affine Grassmannians that are not $\tilde{G}_{\text{min}}$ or $\mathbb{C}^2/\Gamma$ ($\Gamma \subset \text{SU}(2)$)

List:

\[
\begin{cases}
ac_n = & E \\
aq_2 = & E \\
cq_2 = & E 
\end{cases}
\]
Higgs Branch of 4d $\mathcal{N}=2$ SCFTs

4d SCFTs can be organized according to the rank, i.e. the dim of their Coulomb branch.

[Argyres, Lotito, Lu, Martone 16]
[Apruzzi, Giacomelli, Schäfer-Namiki 20]

<table>
<thead>
<tr>
<th>Rank 1 SCFT</th>
<th>Magnetic quiver</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_5$</td>
<td><img src="C5" alt="Diagram" /></td>
</tr>
<tr>
<td>$C_3 \times A_1$</td>
<td><img src="C3xA1" alt="Diagram" /></td>
</tr>
<tr>
<td>$C_2 \times U_1$</td>
<td><img src="C2xU1" alt="Diagram" /></td>
</tr>
<tr>
<td>$A_3$</td>
<td><img src="A3" alt="Diagram" /></td>
</tr>
<tr>
<td>$A_1 \times U_1$</td>
<td><img src="A1xU1" alt="Diagram" /></td>
</tr>
<tr>
<td>$A_2$</td>
<td><img src="A2" alt="Diagram" /></td>
</tr>
</tbody>
</table>

[AB, Grimminger, Hanany, Speiling, Zafir, Zhang 20]
Rank 2 uses all types of quivers introduced above, and all kinds of transverse slices.

<table>
<thead>
<tr>
<th>#</th>
<th>$d_{NB}$</th>
<th>$f$</th>
<th>Quiver</th>
<th>$d_{NB}$</th>
<th>$f$</th>
<th>Quiver</th>
<th>UR</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>23</td>
<td>$\text{su}(6)_{16} \times \text{su}(2)_9$</td>
<td><img src="image1" alt="Quiver 1" /></td>
<td>41</td>
<td>19</td>
<td>$\text{su}(5)_{18}$</td>
<td><img src="image2" alt="Quiver 2" /></td>
</tr>
<tr>
<td>34</td>
<td>13</td>
<td>$\text{su}(4)_{12} \times \text{su}(2)_7 \times \text{u}(1)$</td>
<td><img src="image3" alt="Quiver 3" /></td>
<td>45</td>
<td>6</td>
<td>$\text{su}(3)_{12} \times \text{u}(1)$</td>
<td><img src="image4" alt="Quiver 4" /></td>
</tr>
<tr>
<td>35</td>
<td>11</td>
<td>$\text{su}(3)<em>{10} \times \text{su}(3)</em>{10} \times \text{u}(1)$</td>
<td><img src="image5" alt="Quiver 5" /></td>
<td>46</td>
<td>3</td>
<td>$\text{su}(2)_{10} \times \text{u}(1)$</td>
<td><img src="image6" alt="Quiver 6" /></td>
</tr>
<tr>
<td>36</td>
<td>8</td>
<td>$\text{su}(3)_{16} \times \text{su}(2)_6 \times \text{u}(1)$</td>
<td><img src="image7" alt="Quiver 7" /></td>
<td>47</td>
<td>32</td>
<td>$\text{sp}(12)_{11}$</td>
<td>See Table 7</td>
</tr>
<tr>
<td>37</td>
<td>6</td>
<td>$\text{su}(2)_8 \times \text{su}(2)_8 \times \text{u}(1)^2$</td>
<td><img src="image8" alt="Quiver 8" /></td>
<td>48</td>
<td>8</td>
<td>$\text{sp}(4)_{5} \times \text{so}(4)_8$</td>
<td>?</td>
</tr>
<tr>
<td>38</td>
<td>2</td>
<td>$\text{u}(1)^2$</td>
<td><img src="image9" alt="Quiver 9" /></td>
<td>49</td>
<td>14</td>
<td>$\text{sp}(8)_7$</td>
<td>See Table 7</td>
</tr>
<tr>
<td>39</td>
<td>29</td>
<td>$\text{sp}(14)_9$</td>
<td><img src="image10" alt="Quiver 10" /></td>
<td>50</td>
<td>4</td>
<td>$\text{sp}(4)_{13/3}$</td>
<td>?</td>
</tr>
<tr>
<td>40</td>
<td>17</td>
<td>$\text{su}(2)_8 \times \text{sp}(10)_7$</td>
<td><img src="image11" alt="Quiver 11" /></td>
<td>51</td>
<td>28</td>
<td>$\text{sp}(8)<em>{10} \times \text{su}(2)</em>{20}$</td>
<td><img src="image12" alt="Quiver 12" /></td>
</tr>
<tr>
<td>41</td>
<td>15</td>
<td>$\text{su}(2)_5 \times \text{sp}(8)_7$</td>
<td><img src="image13" alt="Quiver 13" /></td>
<td>52</td>
<td>14</td>
<td>$\text{sp}(4)<em>9 \times \text{su}(2)</em>{16} \times \text{su}(2)_{18}$</td>
<td><img src="image14" alt="Quiver 14" /></td>
</tr>
<tr>
<td>42</td>
<td>11</td>
<td>$\text{sp}(8)_6 \times \text{u}(1)$</td>
<td><img src="image15" alt="Quiver 15" /></td>
<td>53</td>
<td>7</td>
<td>$\text{su}(2)<em>7 \times \text{su}(2)</em>{14} \times \text{u}(1)$</td>
<td><img src="image16" alt="Quiver 16" /></td>
</tr>
<tr>
<td>43</td>
<td>6</td>
<td>$\text{sp}(6)_5$</td>
<td><img src="image17" alt="Quiver 17" /></td>
<td>54</td>
<td>6</td>
<td>$\text{su}(2)_6 \times \text{su}(2)_8$</td>
<td><img src="image18" alt="Quiver 18" /></td>
</tr>
<tr>
<td>55</td>
<td>2</td>
<td>$\text{su}(2)_5$</td>
<td><img src="image19" alt="Quiver 19" /></td>
<td>56</td>
<td>2</td>
<td>$\text{su}(2)_{10}$</td>
<td><img src="image20" alt="Quiver 20" /></td>
</tr>
</tbody>
</table>

[AB, Grimminger, Martone, Zaffan 21]
Conjecture: \( \forall r \geq 2, \exists 4d \ N=2 \ SCFT \) with rank \( r \) such that its HB does not admit a MQ in the class introduced above ("unitary")

Example

\[ X = Higgs_{\text{twisted}} \begin{pmatrix} \text{A3} \\ \text{class S} \end{pmatrix} \]

\( \dim_H X = 11 \)

\( \text{Isom} \ X = \mathfrak{su}(2) \oplus \mathfrak{u}(3) \oplus \mathfrak{u}(1) \)

\( \rightarrow \) needs to go to orthosymplectic quivers
IV - Conclusion:
Where does it end?
- \( E \) is not injective (ie MQ not unique)
- nor surjective

Ex: \[ 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \] \( \rightarrow \) \( C_{\text{min}}(E_6) \)
S needs additional input: what is the list of elementary slices to subtract?

Very recent addition to the list: infinite family $\mathcal{Z}(d)$ ($d \geq 4$) of isolated CSS (with trivial local fundamental group)

[Blánsky, Bonnafé, Fu, Juteau, Levy, Sommers 21]
- $S$ needs additional input: what is the list of elementary slices to subtract?

Very recent addition to the list: infinite family $Z(d)$ ($d \geq 4$) of isolated CSS (with trivial local fundamental group)

[Bellamy, Bonnafé, Fu, Juteau, Levy, Sommers 21]

- $C^{-1}$ closely related to recent progress in string theory. To be continued...