Magnetic Quivers for Singular HyperKähler Spaces

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PLAN OF THE TALK

I - Introduction : two questions and one (partial) answer.

II - The concept of magnetic quivers

III - Examples

IV - Generalizations and unknown territory.

I- Introduction :

Two questions and one (partial) answer

SYMPLECTIC SINGULARITIES

- X = normal affine variety over $\mathbb{C}$.

- X is (has) symplectic singularities if there is a holomorphic symplectic form ω on X_{smooth} whose pullback extends to a holomorphic 2-form Ω on any resolution $Y \to X$. [Beauville 99]

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- X is a conical symplectic singularity (CSS) if it has a good $\mathbb{C}^*$-action $\left(\mathbb{C}[X] = \bigoplus_{i \in \mathbb{N}} R_i \text{ with } R_0 = \mathbb{C} \right)$ with respect to which ω is homogeneous.
[Namikawa 11]

EXAMPLES OF CSS

- Normal nilpotent orbit closures
- Nakajima quiver varieties
- Conical hyperKähler quotients
- Higgs branch of supersymmetric QFT with 8 supercharges
- Coulomb branch of "good" 3d $\mathcal{N}=4$ theories.
- Examples in other talks today.

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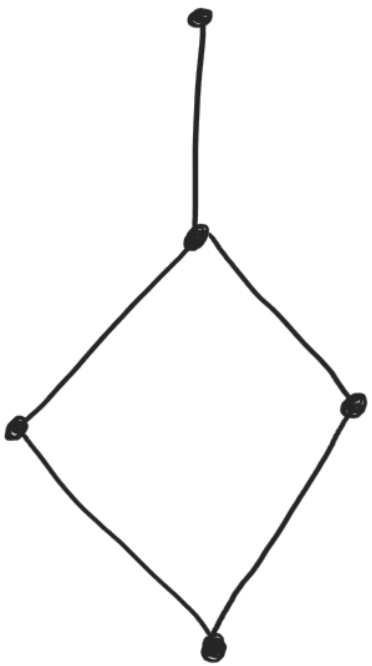
Question 1: is there a UNIFORM description of CSS?

STRATIFICATION & HASSE DIAGRAM

[Kaledin 03]

For X a CSS there exists a finite stratification:



$$\boxed{\begin{array}{c}\{\text{CSS}\}\\ \big\downarrow \\ \{\text{Hasse diagrams}\}\end{array}}$$

- Gives the structure of the CSS
- In physics, characterizes how theories are connected to each other.

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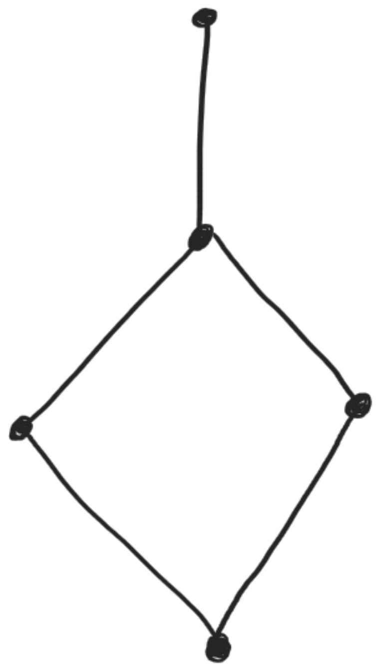


$$\boxed{\begin{array}{c}\{\text{CSS}\}\\ \big\downarrow \\ \{\text{Hasse diagrams}\}\end{array}}$$

- Gives the structure of the CSS
- In physics, characterizes how theories are connected to each other.

Question 2 : How can this Hasse diagram be computed explicitly?

MAGNETIC QUIVERS

A partial answer is given by Magnetic Quivers and the quiver subtraction algorithm.

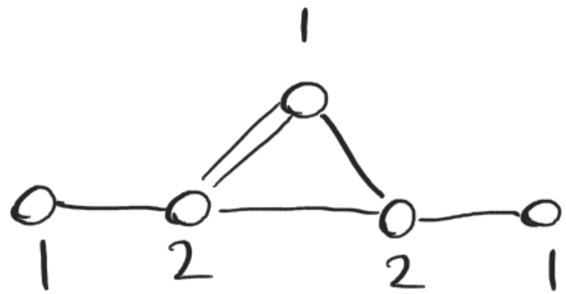
$$
\begin{tikzcd}
 & \{\mathrm{CSS}\} \arrow[dd] \arrow[dl, dashed, bend right, "\text{“}\mathcal{C}^{-1}\text{”}"'] \\
 & \\
\{\text{Quivers}\} \arrow[uur, "\mathcal{C}"'] \arrow[r, "\mathcal{S}"'] & \{\text{Hasse diagrams}\}
\end{tikzcd}
$$

II - The concept of Magnetic Quiver

DEFINITIONS

- Provisional definition : Quiver = connected finite graph with nodes labeled by positive integers, with balance ≥ 0.

Example :



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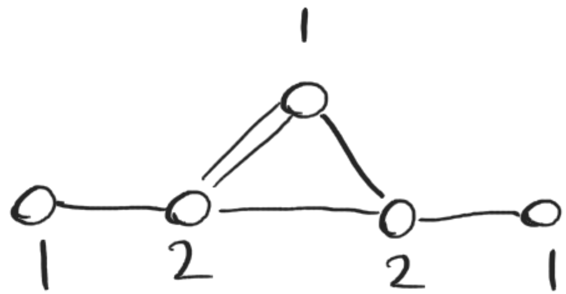

- Map $\mathcal{C}$ [Cremonesi, Hanany, Zaffaroni 13] [Nakajima 15] [Bullimore, Dimofte, Gaiotto 15] [Braverman, Finkelberg, Nakajima 16]

$$\begin{tikzcd}
\text{Quiver} \arrow[r] \arrow[rrr, bend right, "\mathcal{C}"'] & \begin{array}{c}\text{3d } \mathcal{N}=4 \\ \text{gauge} \\ \text{theory}\end{array} \arrow[r] & \begin{array}{c}\text{3d } \mathcal{N}=4 \\ \text{IR} \\ \text{SCFT}\end{array} \arrow[r] & \begin{array}{c}\text{Coulomb} \\ \text{branch}\end{array}
\end{tikzcd}$$

DEFINITIONS

- Given a CSS X, a quiver Q is a magnetic quiver* for X if $\mathcal{C}(Q) = X$

*(more generally, $\bigcup_{i=1}^{N} \mathcal{C}(Q_i) = X$)

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- Given a CSS X, a quiver Q is a magnetic quiver* for X if $\mathcal{C}(Q) = X$

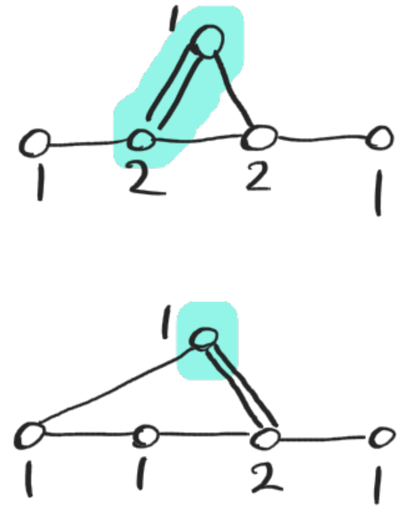
*(more generally, $\bigcup_{i=1}^{N} \mathcal{C}(Q_i) = X$)

- Quiver subtraction is an algorithm

$$\{\text{Quivers}\} \xrightarrow[\mathcal{S}]{} \{\text{Hasse diagrams}\}$$

[AB, Cabrera, Grimminger, Hanany, Sperling, Zajac, Zhong, 19]

EXAMPLE OF QUIVER SUBTRACTION



*See my June 2020 talk

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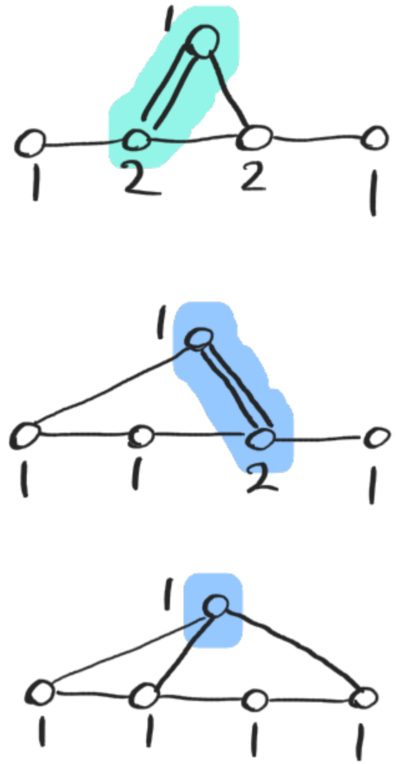

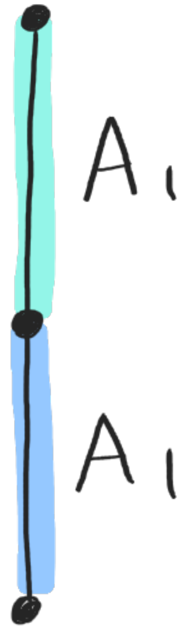

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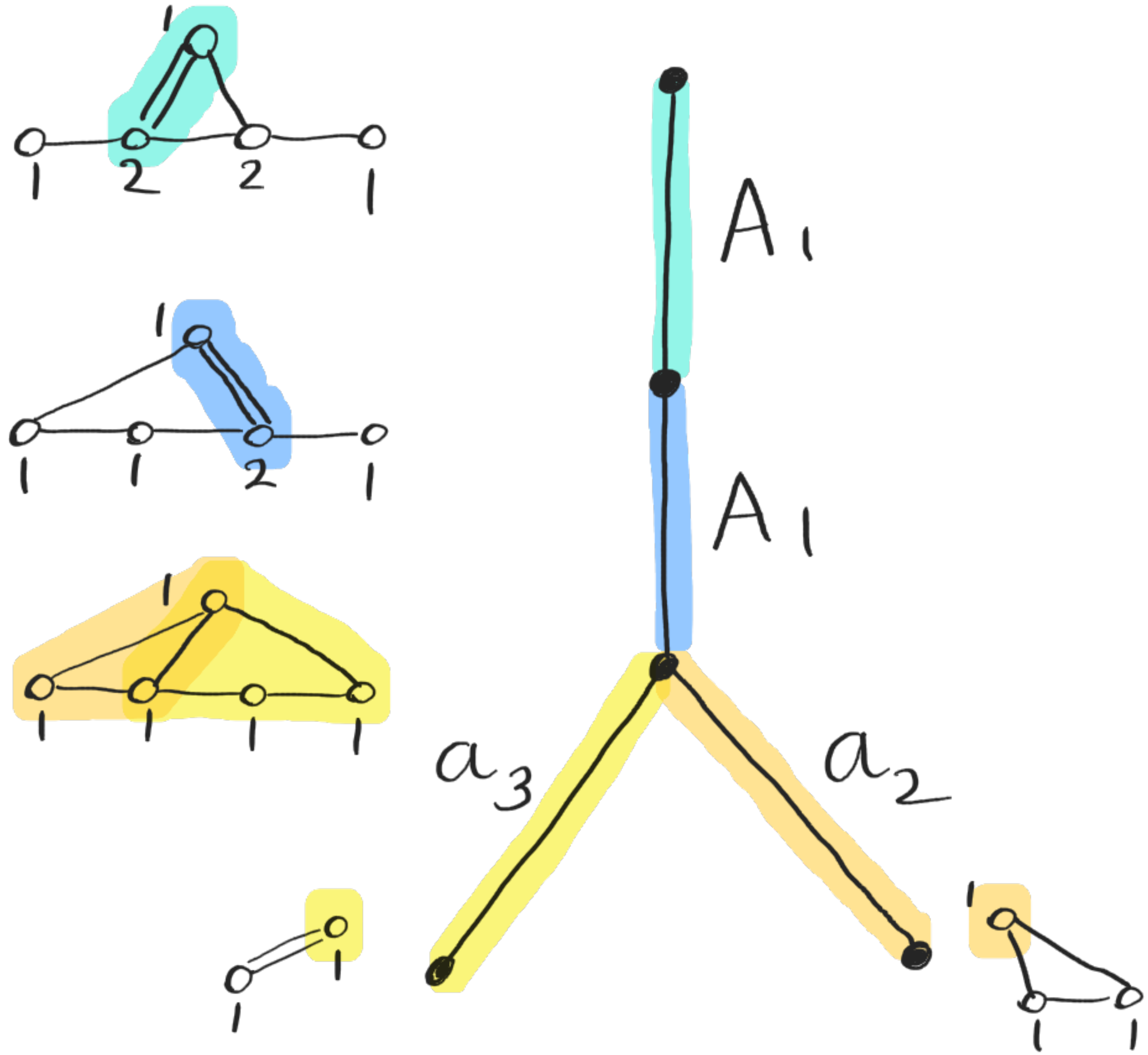
EXAMPLE OF QUIVER SUBTRACTION



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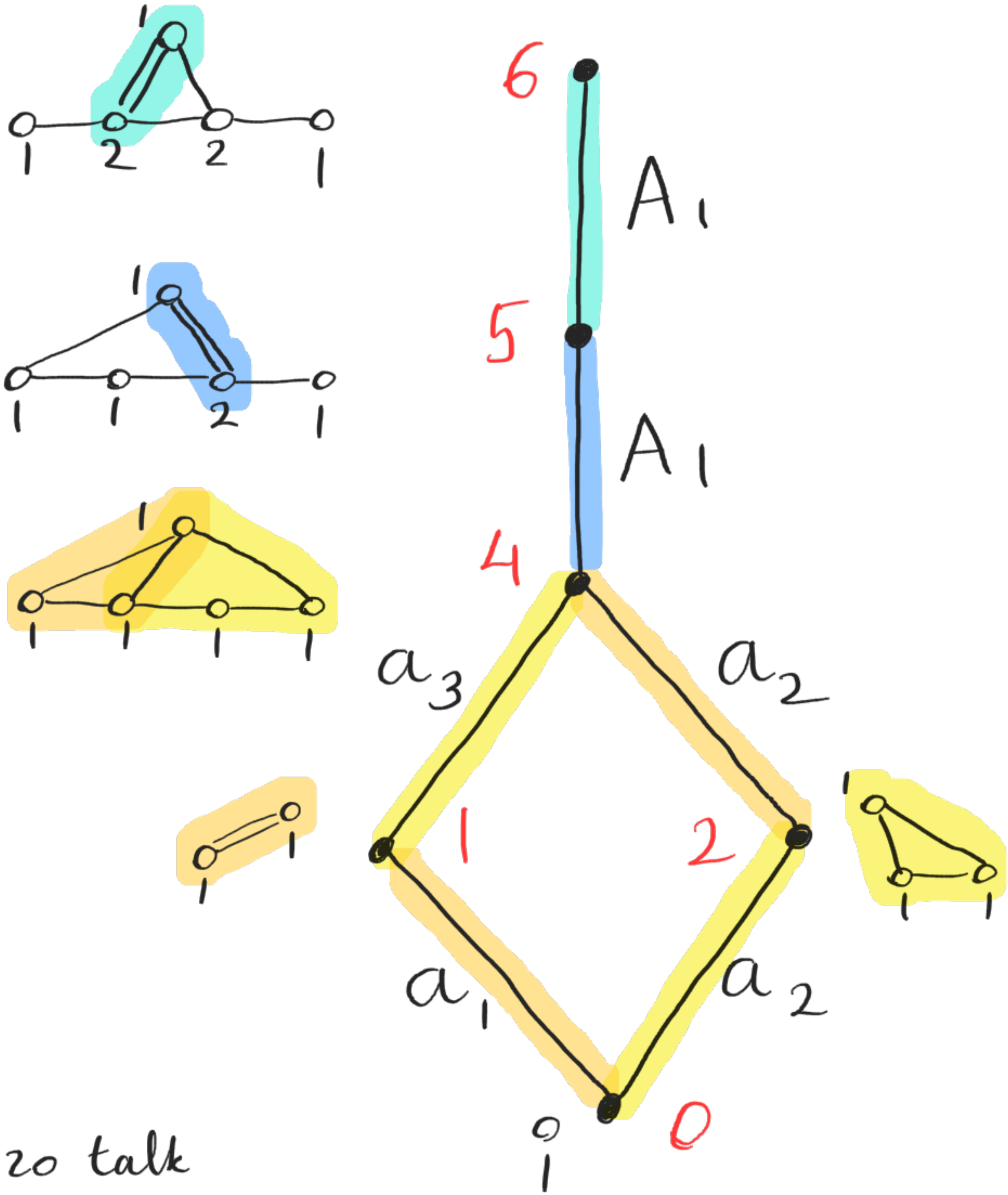

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III - Examples

A - HyperKähler quotients (quiver varieties)

B - Wreathed quivers

C - Quasi - minimal singularities.

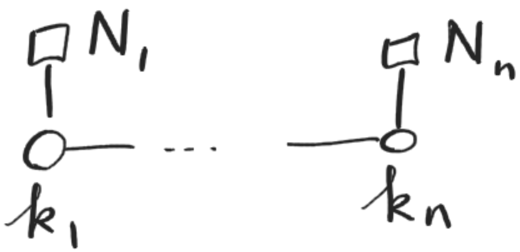
D - Higgs branch of 4d $\mathcal{N}=2$ SCFTs

QUIVER VARIETIES : SL / GL.

$$\mathcal{H}\mathrm{iggs}\left(\begin{array}{ccc} \square\, N_1 & & \square\, N_n \\ | & & | \\ \underset{k_1}{\circ} & \!\!-\!\!-\; \cdots \;-\!\!-\!\! & \underset{k_n}{\circ}\end{array}\right) = \text{HK quotient by } GL(k_1)\times\cdots\times GL(k_n).$$

When $k_{i-1}+k_{i+1}+N_i \geqslant 2k_i$, magnetic quiver well known.

QUIVER VARIETIES : SL / GL.

$\mathcal{H}$iggs (

) = HK quotient by (**) $GL(k_1) \times \cdots \times GL(k_n)$.

When $k_{i-1} + k_{i+1} + N_i \geqslant 2k_i$, magnetic quiver well known.

(*)

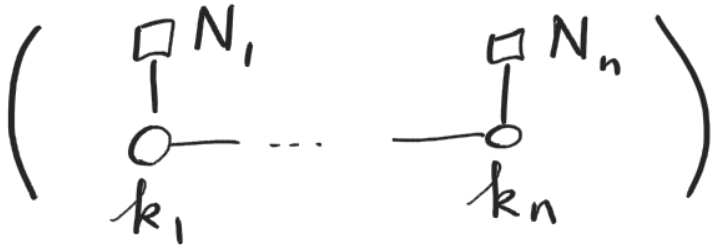
Generalizations

1) Drop the (*) condition

2) Replace some $GL(k_i)$ by $SL(k_i)$ in (**)

QUIVER VARIETIES : SL / GL.

Higgs



= HK quotient by (**) $GL(k_1) \times \cdots \times GL(k_n)$.

When $k_{i-1} + k_{i+1} + N_i \geq 2k_i$ (*), magnetic quiver well known.

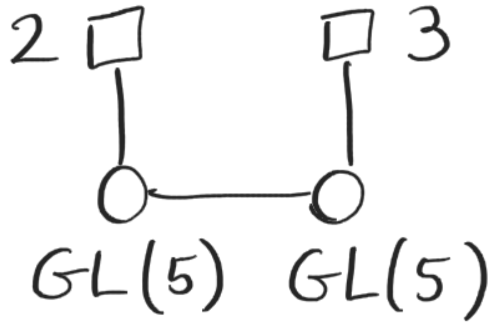
Generalizations

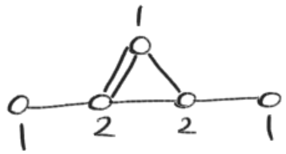
1) Drop the (*) condition
2) Replace some $GL(k_i)$ by $SL(k_i)$ in (**)

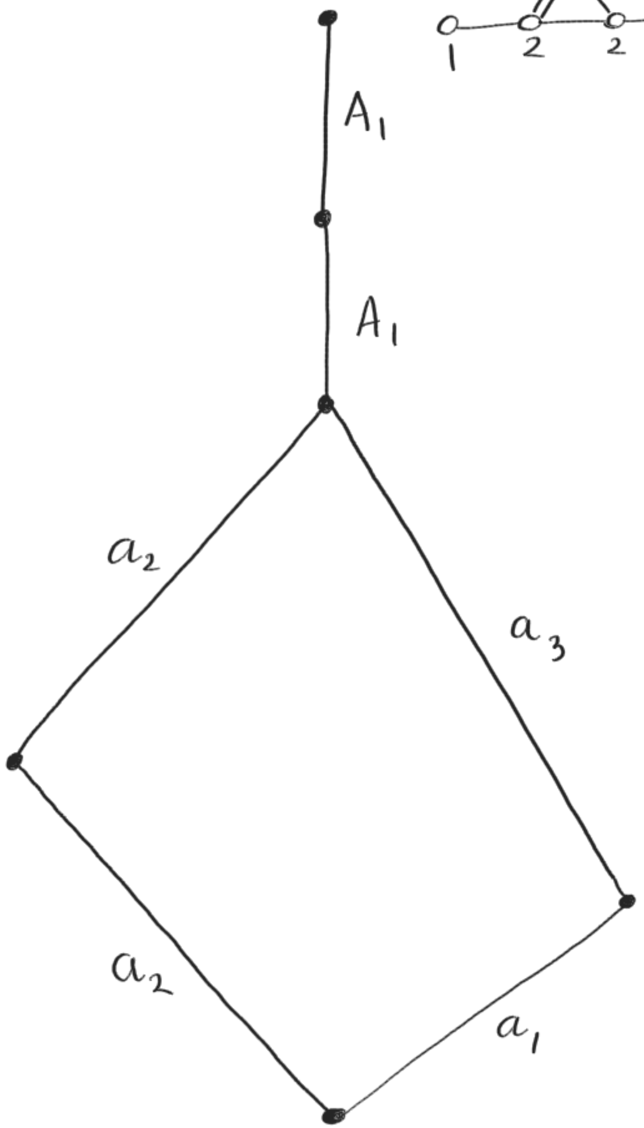
Answer : the Brane Locking algorithm

[AB, Grimminger, Hanany, Kalveks, Zhong 21]

THE BRANE LOCKING ALGORITHM

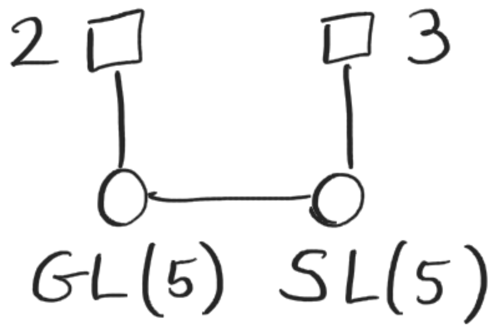


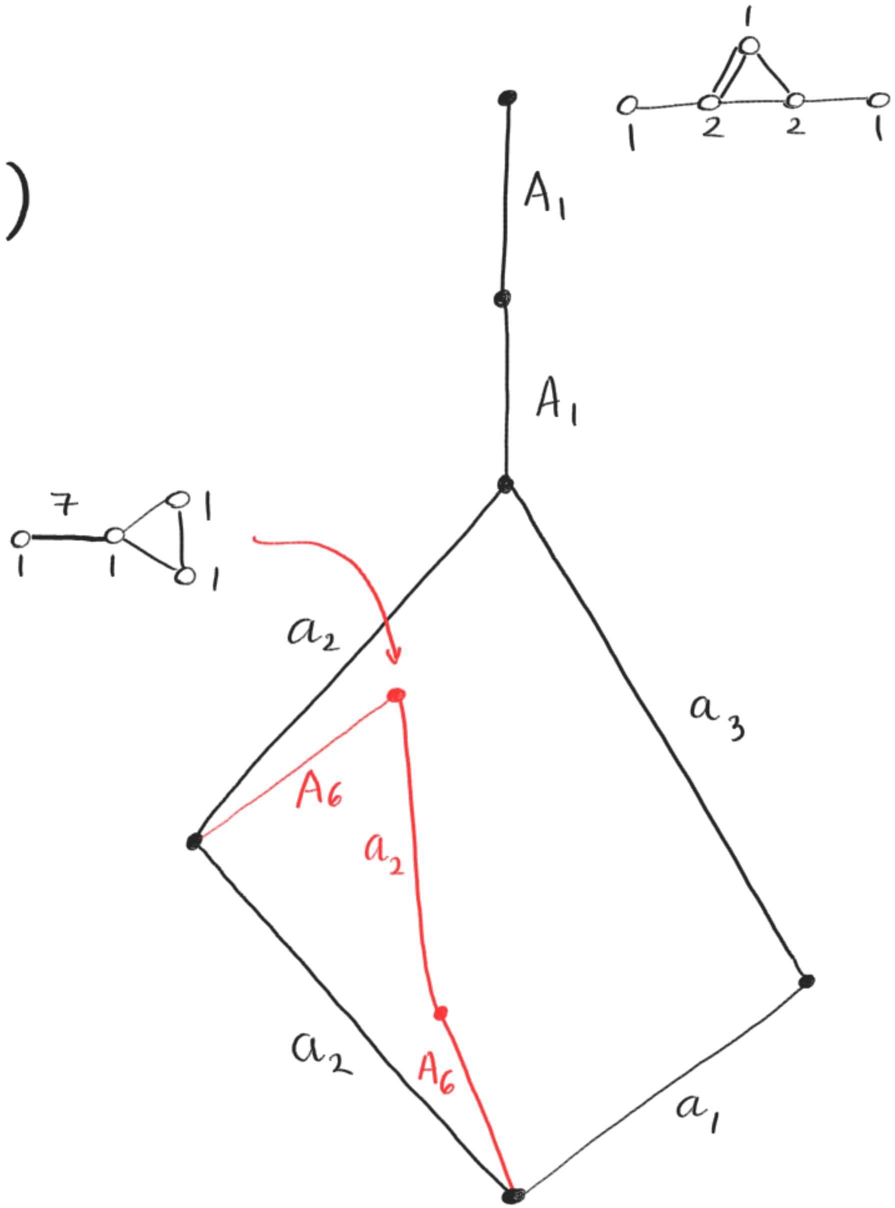




$\# MQ = 1$

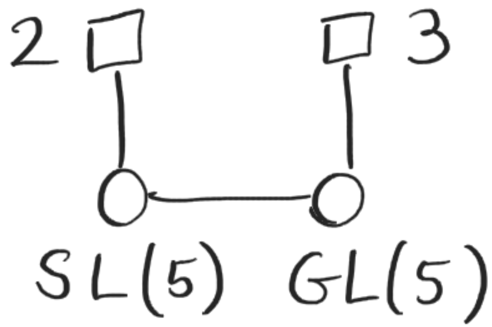
THE BRANE LOCKING ALGORITHM

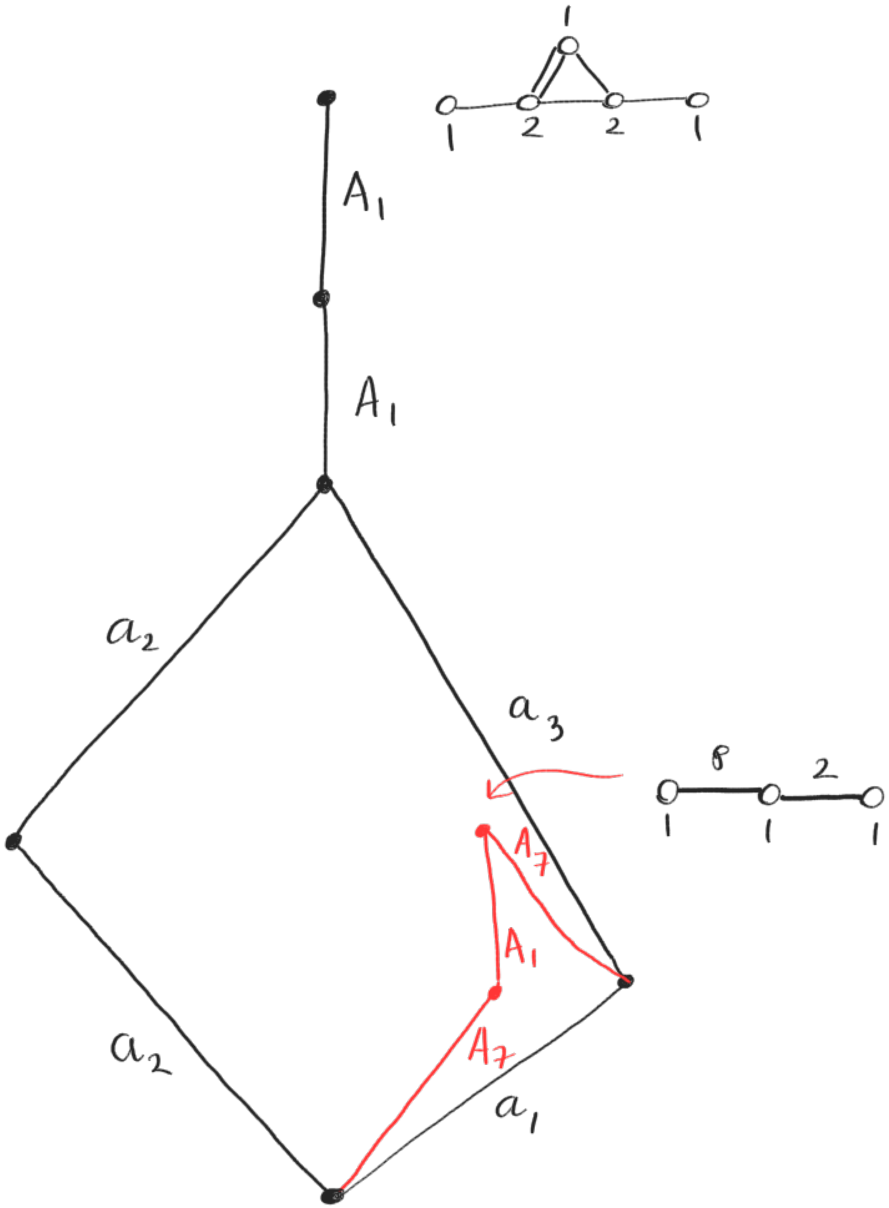




$\# MQ = 2$

THE BRANE LOCKING ALGORITHM





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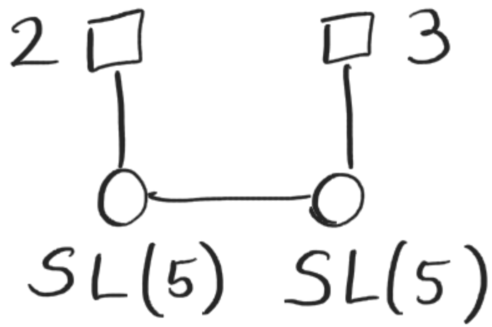
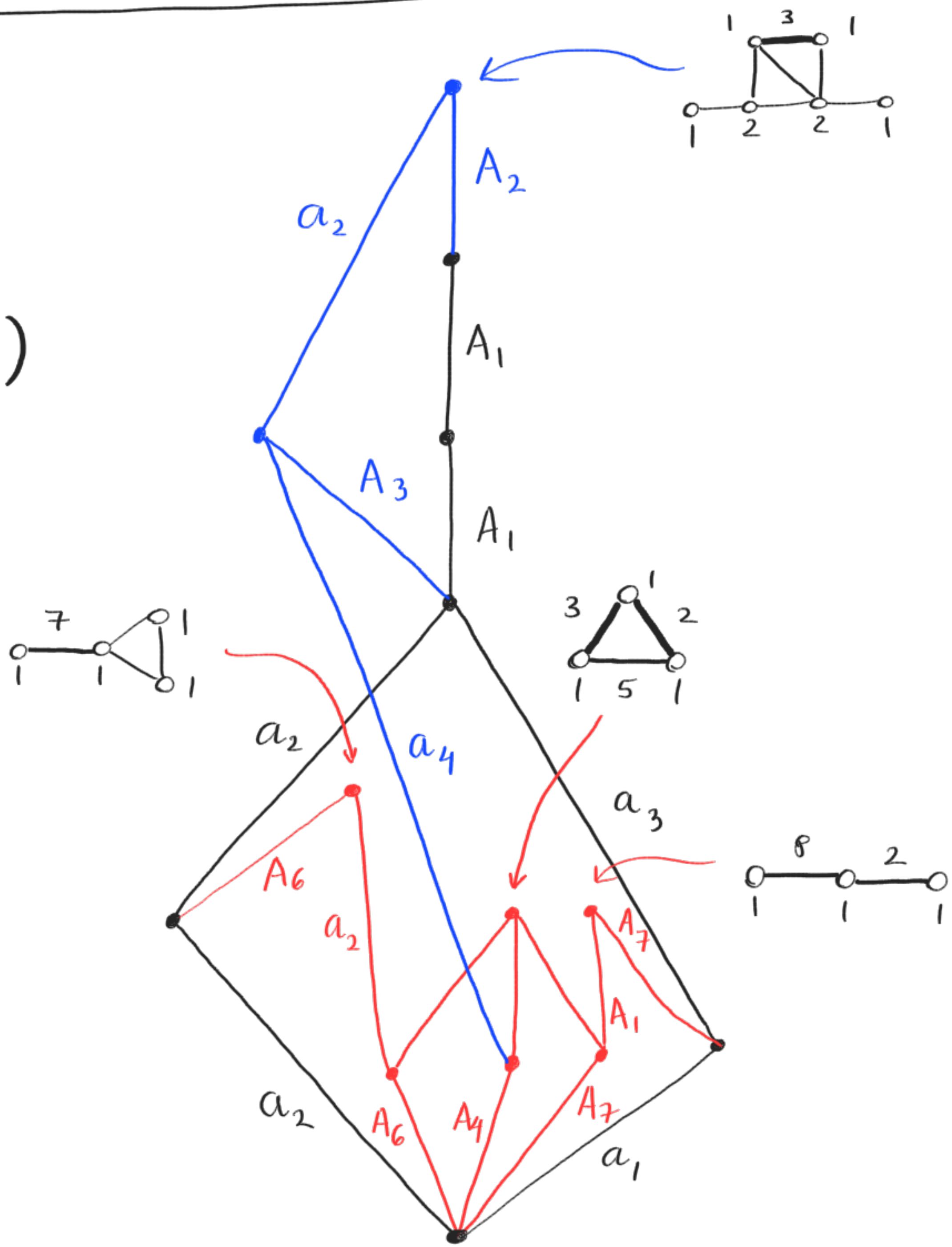




#MQ = 4

THE BRANE LOCKING ALGORITHM

Questions :

- Cross check the results using other methods ?
- What is $\#$MQ in general ?
  $(= \#$ irreducible components$)$
- What is the dimension ?
- Physics : is brane locking part of string theory ?

WREATHED and FOLDED QUIVERS

Quiver Q with automorphism (sub-) group Γ.

$$\mathcal{C}(\Gamma\text{-wreathed } Q) = \mathcal{C}(Q)/\Gamma$$

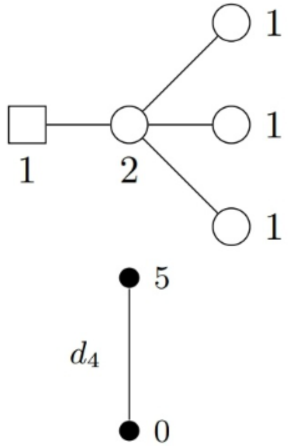
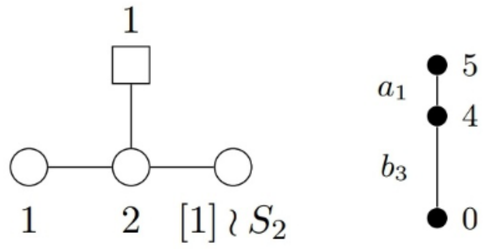
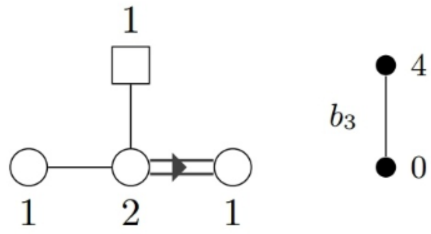
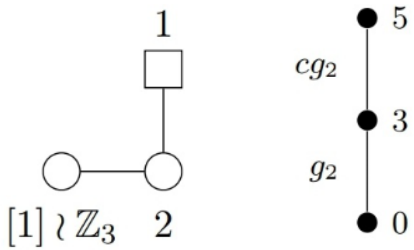
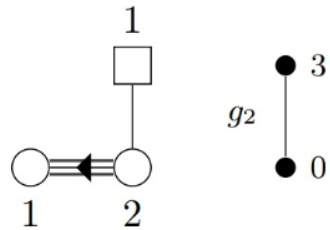
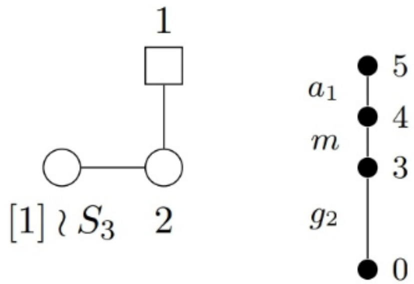
[AB, Hanany, Miketa 20]

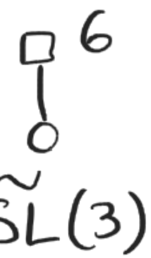
$$\mathcal{C}(\Gamma\text{-folded } Q) = \mathcal{C}(Q)^{\Gamma}$$

[Cremonesi, Ferlito, Hanany, Mekareeya 14]
[Nakajima, Weekes 19]

This is a generalization of the map $\mathcal{C}$

↳ New possibilities for $\mathcal{C}^{-1}$

Initial	Discretely Gauged	Folded
 $\mu_2 t^2$ D_4 $(\mu_1+\mu_2)t^2$ B_3 $(2\mu_1+\mu_2)t^2+\mu_2 t^4$ G_2	 $\mu_2 t^2+\mu_1^2 t^4$ B_3 $(\mu_1+\mu_2)t^2+\mu_1^2 t^4+\mu_1\mu_2 t^6+\mu_2^2 t^8-\mu_1^2\mu_2^2 t^{12}$ G_2	 $\mu_2 t^2$ B_3 $(\mu_1+\mu_2)t^2$ G_2
	 $\mu_2 t^2+(\mu_1^2+\mu_2)t^4+2\mu_1^3 t^6-\mu_1^6 t^{12}$ G_2	 $\mu_2 t^2$ G_2
	 $\mu_2 t^2+\mu_1^2 t^4+\mu_1^3 t^6+\mu_2^2 t^8+\mu_1^3\mu_2 t^{10}-\mu_1^6\mu_2^2 t^{20}$ G_2	

Example : Higgs (

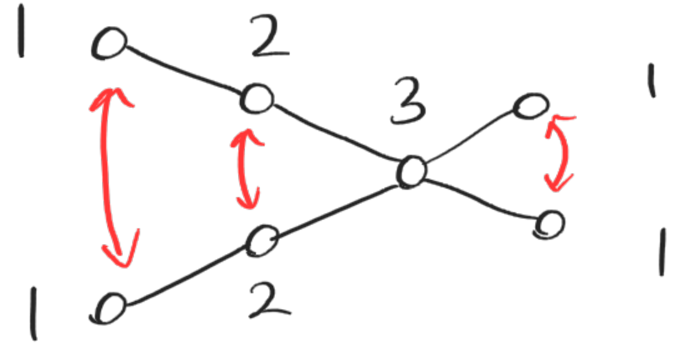
)

$\widetilde{SL}(3) = SL(3) \rtimes \mathbb{Z}_2$

outer automorphism.

[Arias-Tamargo, AB, Pini 21]

MQ = $\mathbb{Z}_2$ - wreathed quiver :



Check using refined Hilbert series computation :

Wendt's integration formula $\longleftrightarrow$ Wreathed Monopole formula

[Wendt 01]

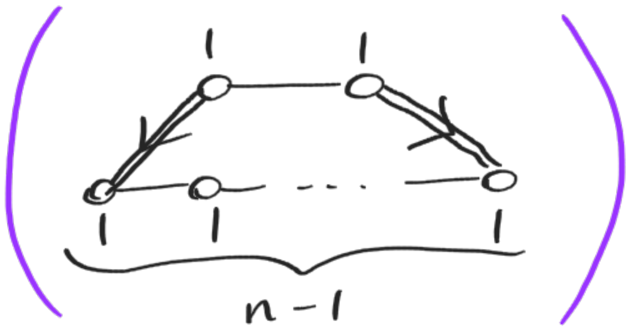
QUASI-MINIMAL SINGULARITIES

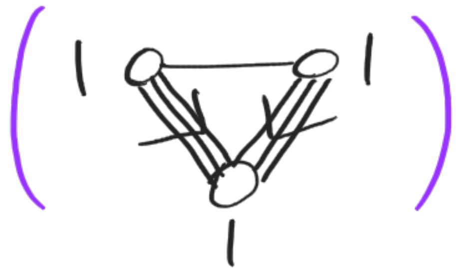
[Malkin, Ostrik, Vybornov 03]

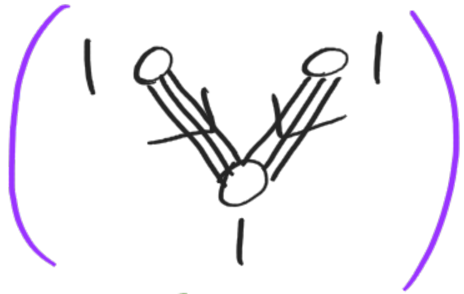
$\equiv$ Slices in affine Grassmannians that are

<u>not</u> $\overline{\mathcal{O}}_{min}$ or $\mathbb{C}^2/\Gamma$ $\quad (\Gamma \subset SU(2))$

List:

- $ac_n = \mathcal{C}$ 

- $ag_2 = \mathcal{C}$ 

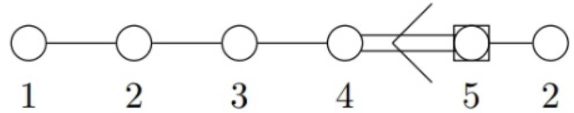
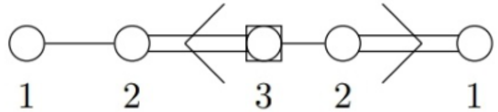
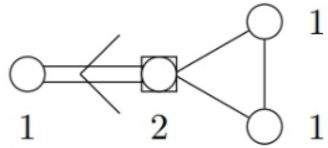
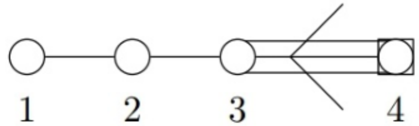
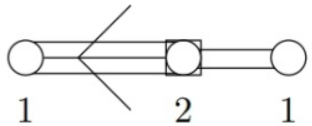
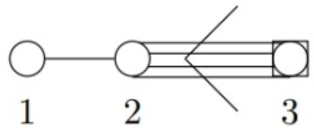
- $cg_2 = \mathcal{C}$ 

[Braverman, Finkelberg, Nakajima 16]
[AB, Grimminger, Hanany, Sperling, Zhong 21]

HIGGS BRANCH OF 4d $\mathcal{N}=2$ SCFTs

4d SCFTs can be organized according to the rank, i.e. the $\dim_{\mathbb{C}}$ of their Coulomb branch.

[Argyres, Lotito, Lu, Martone 16]
[Apruzzi, Giacomelli, Schäfer-Nameki 20]

Rank 1 SCFT	Magnetic quiver
C_5	
$C_3 \times A_1$	
$C_2 \times U_1$	
A_3	
$A_1 \times U_1$	
A_2	

[AB, Grimminger, Hanany, Sperling, Zafrir, Zhong 20]

HIGGS BRANCH OF 4d $\mathcal{N}=2$ SCFTs

Rank 2 uses **all** types of quivers introduced above, and **all** kinds of transverse slices

#	d_{HB}	$\mathfrak{f}$	Quiver		
33	23	$\mathfrak{su}(6)_{16} \times \mathfrak{su}(2)_9$	1 2 3 4 5 6 3	4	1
34	13	$\mathfrak{su}(4)_{12} \times \mathfrak{su}(2)_7 \times \mathfrak{u}(1)$	1 2 3 4 3 1	4	1
35	11	$\mathfrak{su}(3)_{10} \times \mathfrak{su}(3)_{10} \times \mathfrak{u}(1)$	1 2 3 3 2 1	4	1
36	8	$\mathfrak{su}(3)_{10} \times \mathfrak{su}(2)_6 \times \mathfrak{u}(1)$	1; 1 2 3 2	4	1
37	6	$\mathfrak{su}(2)_8 \times \mathfrak{su}(2)_8 \times \mathfrak{u}(1)^2$	1; 1 2 2 1	4	1
38	2	$\mathfrak{u}(1)^2$	1; 1 1	3	1
39	29	$\mathfrak{sp}(14)_9$	1 2 3 4 5 6 7 2	4	1
40	17	$\mathfrak{su}(2)_8 \times \mathfrak{sp}(10)_7$	1 2 3 4 5 2 1	4	1
41	15	$\mathfrak{su}(2)_5 \times \mathfrak{sp}(8)_7$	B_1 C_1 B_0; B_1 C_1 B_2 C_2 B_3 C_3 D_2 C_1	?	2
42	11	$\mathfrak{sp}(8)_6 \times \mathfrak{u}(1)$	1; 1 2 3 4 1	4	1
43	6	$\mathfrak{sp}(6)_5$	1 2 3 1	3	1

#	d_{HB}	$\mathfrak{f}$	Quiver	UR	
44	19	$\mathfrak{su}(5)_{16}$	1 2 3 4 5	6	1
45	6	$\mathfrak{su}(3)_{12} \times \mathfrak{u}(1)$	1 2 3 1	6	1
46	3	$\mathfrak{su}(2)_{10} \times \mathfrak{u}(1)$	1 2 1	5	1
47	32	$\mathfrak{sp}(12)_{11}$	See Table 7	?	
48	8	$\mathfrak{sp}(4)_5 \times \mathfrak{so}(4)_8$	?	?	
49	14	$\mathfrak{sp}(8)_7$	See Table 7	?	
50	4	$\mathfrak{sp}(4)_{13/3}$	?	?	
51	28	$\mathfrak{sp}(8)_{13} \times \mathfrak{su}(2)_{26}$	1 3 5 7 9 4	4	1 4
52	14	$\mathfrak{sp}(4)_9 \times \mathfrak{su}(2)_{16} \times \mathfrak{su}(2)_{18}$	1 3 5 4 2	4	1 4
53	7	$\mathfrak{su}(2)_7 \times \mathfrak{su}(2)_{14} \times \mathfrak{u}(1)$	2; 1 3 2	4	1 4
54	6	$\mathfrak{su}(2)_6 \times \mathfrak{su}(2)_8$	1 2 4	5	1 4
55	2	$\mathfrak{su}(2)_5$	1 2	5	1
56	2	$\mathfrak{su}(2)_{10}$	1 2	?	1

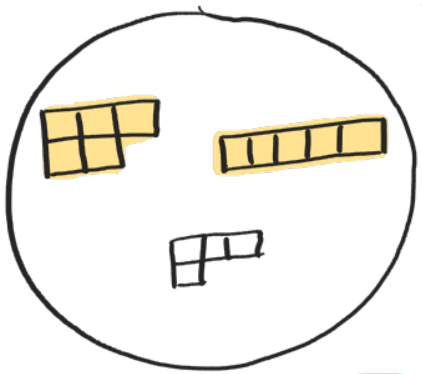
[AB, Grimminger, Martone, Zafrir 21]

HIGGS BRANCH OF 4d $\mathcal{N}=2$ SCFTs

Conjecture: $\forall\, r \geqslant 2$, $\exists$ 4d $\mathcal{N}=2$ SCFT with rank r such that its HB does **not** admit a MQ in the class introduced above ("unitary")

Example

$X = \mathcal{H}\text{iggs}\Big($ twisted A_3 class S



$\Big)$

$\dim_{\mathbb{H}} X = 11$

$\mathrm{Isom}\, X = \mathfrak{su}(2) \oplus \mathfrak{sp}(3) \oplus \mathfrak{u}(1)$

$\rightsquigarrow$ needs to go to **orthosymplectic quivers**

IV – Conclusion :

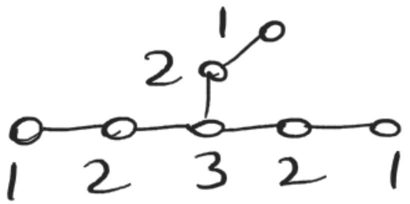
Where does it end ?

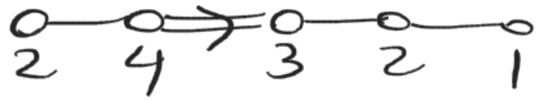
$$
\begin{tikzcd}
 & \{CSS\} \arrow[dl, dashed, bend right, "``\mathcal{C}^{-1}\text{''}"'] \arrow[d] \\
\{\text{Quivers}\} \arrow[ur, "\mathcal{C}"'] \arrow[r, "\mathcal{S}"'] & \{\text{Hasse diagrams}\}
\end{tikzcd}
$$

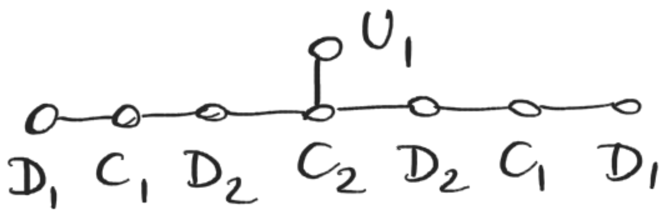
- $\mathcal{C}$ is not injective (ie MQ not unique)

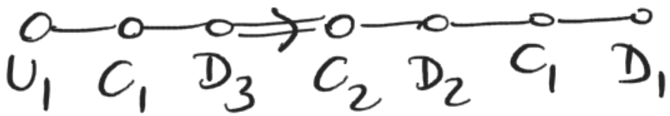
  nor surjective

Ex :









$$\xrightarrow{\mathcal{C}} \mathcal{O}_{min}(E_6)$$

- $\mathcal{S}$ needs additional input : what is the list of elementary slices to subtract ?

Very recent addition to the list : infinite family $Z(d)$ $(d \geq 4)$ of isolated CSS (with trivial local fundamental group)

[Bellamy, Bonnafé, Fu, Juteau, Levy, Sommers 21]

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- $\mathcal{C}^{-1}$ closely related to recent progress in string theory. To be continued...