Quaternion-Kähler Manifolds via algebraic torus action on projective contact manifolds

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motivation: LeBrun–Salamon conjecture

- The conjecture in Riemannian differential geometry: positive quaternion-Kähler manifolds are symmetric spaces (Wolf spaces).
- The conjecture in complex algebraic geometry: every Fano complex contact manifold is homogeneous
- The twistor space of positive quaternion-Kähler manifold is a Fano contact manifold with Einstein metric; hence the group of its automorphisms is reductive [this will be our blanket assumption]

main results

Let X be a Fano contact manifold of dimension $2n + 1$ with reductive group of automorphisms.

Theorem (A)

If $n \leq 4$ then X is homogeneous.

Theorem (B)

Suppose that the group of automorphims of X is of rank $\geq \max\{2, (n-3)/2\}$ then X is homogeneous.

Corollary

Let $\mathcal{M}$ be a positive quaternion-Kähler manifold of dimension $4n$. If $n \leq 4$ or $\mathcal{M}$ admits a faithful action of a torus of rank $\geq \max\{2, (n-3)/2\}$ then $\mathcal{M}$ is symmetric.

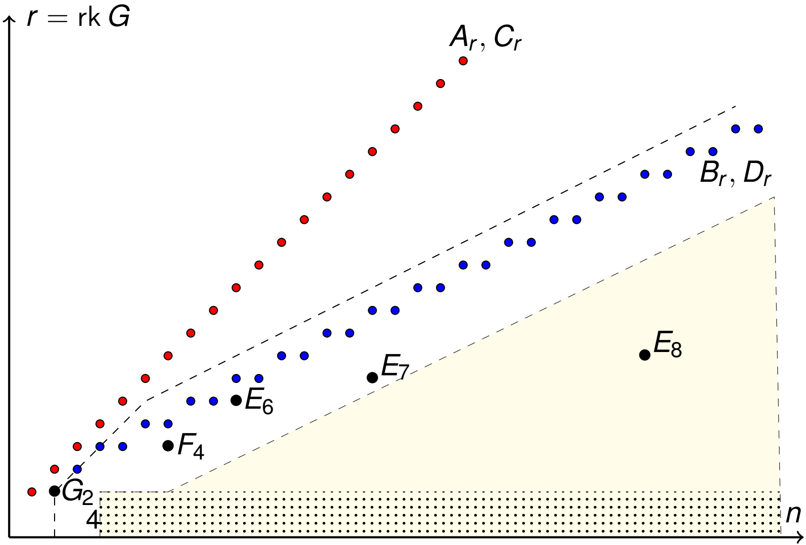
Fano contact manifolds

- Let L be an ample line bundle on a complex manifold X, $\dim X = 2n+1$, a contact form $\theta \in \mathrm{H}^0(X, \Omega_X \otimes L)$ is such that $(d\theta)^{\wedge n} \wedge \theta$ nowhere vanishes; this implies $-K_X = (n+1)L$
- Let F be the kernel of $\theta : TX \to L$ then $d\theta$ defines nondegenerate skew-symmetric pairing:

$$d\theta : F \times F \to L$$

- LeBrun-Salamon conjecture: Every Fano contact manifold is the closed orbit in the projectivisation of adjoint representation of a simple algebraic group G.
  Recall: $\mathrm{rk}\, G$ is the rank of a maximal algebraic torus in G

ranks of simple groups and dim of adjoint



earlier results

Partial results on contact and quaternion-Kähler manifolds:

- small dimension: Hitchin, Poon, Salamon, Druel, Herrera[2],
- big torus, L has many sections: Bielawski, Fang, Beauville,
- may assume $\operatorname{Pic} X = \mathbb{Z} \cdot L$, otherwise

  $$(X, L) = (\mathbb{P}^{2n+1}, \mathcal{O}(2)) \text{ or } (\mathbb{P}(T\mathbb{P}^{n+1}), \mathcal{O}(1))$$

  which the case of C_n and A_n adjoint orbit; LeBrun, Salamon, based on classification of large index Fano's
- contact form $\theta : TX \to L$ yields $\mathrm{H}^0(X, L) \hookrightarrow \mathrm{H}^0(X, TX)$; sections of $L \Longrightarrow$ fields tangent to contact automorphims.

torus in action

- Let $H = (\mathbb{C}^*)^r$ denote an algebraic torus with $M = \mathbb{Z}^r$ the lattice of characters; r denotes the rank of H and M

- Assume that H acts almost faithfully on (X, L), with given linearization $\mu : H \times L \to L$
- Decomposition of the set of fixed points into connected components
$$X^H = Y_1 \sqcup \cdots \sqcup Y_s$$

- Decomposition of space of sections into eigenspaces
$$\mathrm{H}^0(X, L) = \bigoplus_{u \in M} \mathrm{H}^0(X, L)_u$$

polytopes of sections and fixed points

- By $\widetilde{\Gamma}(X, H, L, \mu) \subset M$ we denote the eigenvalues of the action of H on $\mathrm{H}^0(X, L)$ and
$$\Gamma = \Gamma(X, H, L, \mu)$$
is their convex hull in $M_{\mathbb{R}}$.
- By $\widetilde{\Delta}(X, H, L, \mu) \subset M$ we denote the set of the characters $\mu(Y_i)$ of the action of H on on fibers of L over Y_i's and
$$\Delta = \Delta(X, H, L, \mu)$$
is their convex hull in $M_{\mathbb{R}}$.
- A connected component $Y \subset X^H$ is called *extremal* if $\mu(Y)$ is a vertex of Δ.

main technical results

Theorem (C)

Let (X, L) *be a contact Fano manifold,* $\dim X = 2n + 1$*, and* $\operatorname{Pic} X = \mathbb{Z}L$*. Suppose that*

- *the group* G *of contact automorphisms of* X *is reductive of rank* $r \geq 2$*, and*
- *the action of a maximal torus* $H \subset G$ *on* X *has isolated points as extremal fixed point components.*

Then G *is simple and* (X, L) *is the closed orbit in the projectivization of the adjoint representation of* G*.*

main techniques in the proof

- discrete methods for $(\mathbb{C}^*)^r$ action:
  - grid data, grids
  - downgrading and restriction
- birational geometry of $\mathbb{C}^*$ action:
  - adjunction, adjoint morphism
  - algebraic bordism, Cremona transformations
- other tools:
  - Białynicki-Birula decomposition
  - equivariant cohomology

grid data

Consider (X, L) with an action of H, linearization μ and fixed points components $X^H = \bigsqcup_0^s Y_i$; the grid data is:

- isomorphism class of Y_i for $i = 1, \ldots, s$
- for every Y_i take isomorphism classes of summands of the decomposition of the conormal bundle

  $$N^*_{Y_i/X} = \bigoplus N^*_{\nu^i_j} : \quad \nu^i_j \in M$$

  weights $\nu^i_j \in M$ make the *compass* of the action
- linearization of the action on L over fixed point components

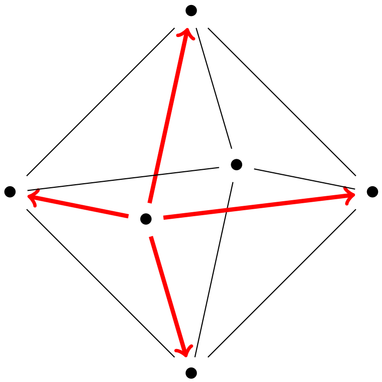
  $$\mu : \{Y_i\} \to M$$

Lefschetz-Riemann-Roch: grid data determines equivariant cohomology of the line bundle $L^{\otimes m}$.

example: grid of a quadric $(\mathbb{Q}^4, \mathcal{O}(1))$

Take $\mathbb{Q}^4 = \{x_1x_2 + x_3x_4 + x_5x_6 = 0\} \subset \mathbb{P}^5$ with $(\mathbb{C}^*)^3$ action

$$(t_1, t_2, t_3) \cdot (x_1, \ldots, x_6) = (t_1x_1, t_1^{-1}x_2, \ldots, t_3x_5, t_3^{-1}x_6)$$



downgrading and reduction

Given a sequence of tori $H_1 \hookrightarrow H \to H_2 = H/H_1$ and the associated sequence of lattices of characters

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

Then H_2 acts on components of X^{H_1} and for every connected component $Y_i \subset X^{H_1}$ we get

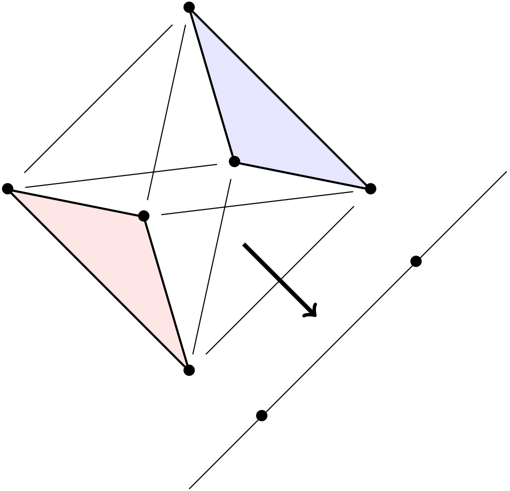
$$Y_i^{H_2} = X^H \cap Y_i$$

Moreover the grid data for the action of H_2 on (Y_i, L_{Y_i}) is just restriction of the original grid data for (X, L) to Y_i.

example: downgrading the action on $\mathbb{Q}^4$

Take $\mathbb{Q}^4 = \{x_1x_2 + x_3x_4 + x_5x_6 = 0\} \subset \mathbb{P}^5$ with $\mathbb{C}^*$ action

$$t \cdot (x_1, \ldots, x_6) = (tx_1, t^{-1}x_2, \ldots, tx_5, t^{-1}x_6)$$



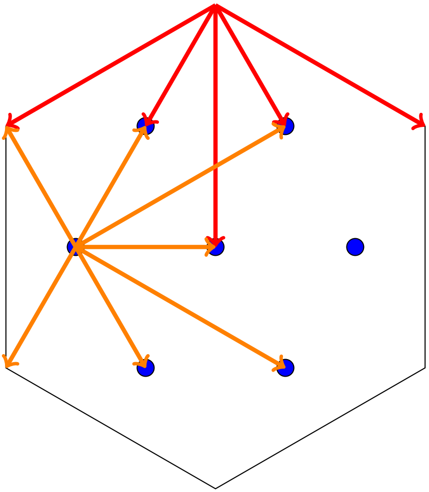
plan of the proof

1. reduce theorems A and B to theorem C: for B use downgrading, induction on faces, and non-vanishing for small dimensional Fano's, BB-decomposition is crucial; for A use estimates on $\dim \mathrm{H}^0(X, L)$ by Salamon;

2. to prove theorem C show $\Delta(X, L) = \Gamma(X, L)$, hence Δ is a root polytope for a group of A_2/G_2 type; use biration geometry to understand the fixed point components and their normal, hence recover the grid data;

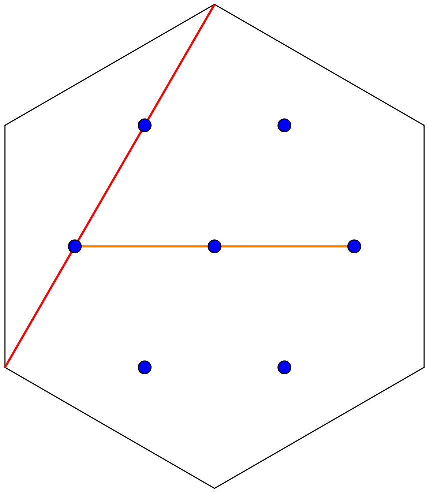
3. use Lefschetz-Riemann-Roch to recover that (X, L) is the closed orbit in the projectivisation of the adjoint representation.

analysing A_2/G_2 type, compasses



$d\theta : F \times F \to L$ determines symmetries in compass

analysing A_2/G_2 type, reductions



reduction to small bandwidth, equalized $\mathbb{C}^*$ actions

small bandwidth equalized $\mathbb{C}^*$ actions

Consider $\mathbb{C}^*$ action on (X, L) with linearization μ

- the bandwidth of the action $= \mu_{\mathsf{max}} - \mu_{\mathsf{min}}$
- equalized means that compass contains only ± 1
- equalized $\Longrightarrow$ bandwidth $= \deg L$ on a general orbit

Theorem (D)

Take equalized $\mathbb{C}^*$ action of bandwidth 3 on (Z, L) with isolated extremal points. Then the pair (Z, L) is one of the following:

1. *a scroll over $\mathbb{P}^1$,*
2. *$Z = \mathbb{Q}^{m-1} \times \mathbb{P}^1$,*
3. *Z is homogeneous of type $C_3(3)$, $A_5(3)$, $D_6(6)$, $E_7(7)$.*

Corollary

Downgrading of $(\mathbb{C}^*)^2$ action on contact (X, L): case 2 implies grid data for adjoint varieties of type B_n and D_n, case 3 implies types F_4, E_6, E_7, E_8.

birational geometry

- take $\mathbb{C}^*$ action on (X, L) with fixed components point components Y^+ and Y^- where a generic orbit has limit at 0, ∞, respectively,
- if $Y^\pm$ are divisors, then we get birational map $Y^+ \dashrightarrow Y^-$,
- if the action is equalized then blowing-up $Y^\pm$ (possbily not divisors) we get divisors of fixed points,
- in situation of Theorem D(3) we get a Cremona quadratic transformation which is resolved by a single blow-up (classical stuff)