Plan: (1) RH construction of HK metrics, in general
(2) The case of moduli spaces of Higgs bundles
(3) The evidence it works

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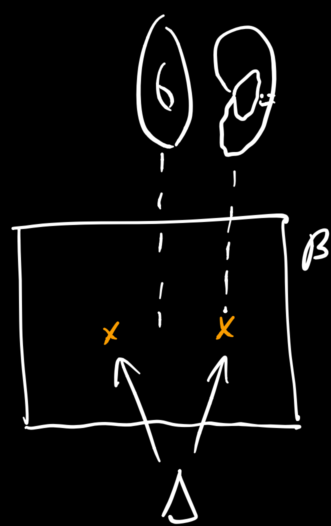
(1) RH construction [Gaiotto-Moore-N, Kontsevich-Soibelman, Joyce, Gross-Siebert, Fock-Goncharov, Hitchin-Karlhede-Lindström-Roček, _]

Inputs

a) complex integrable system $\mathcal{M}^\vee$

hol. symplectic $\mathcal{M}^\vee \xrightarrow{\pi} \mathcal{B} \ni u$, $\pi^{-1}(u) \to u$ Lagrangian, generically torus

$\mathcal{B}' = \mathcal{B} \setminus \Delta$



b) "DT-type/BPS invariants"

local system Γ of lattices over $\mathcal{B}'$, $\Gamma_u = H_2(\mathcal{M}^\vee; \pi^{-1}(u)) \xrightarrow{\partial} H_1(\pi^{-1}(u))$

$\Omega : \Gamma \to \mathbb{Z}$

obeying conditions: 1) Kontsevich-Soibelman WCF
(determined by?) 2) local conditions around Δ

[e.g. at generic part $\Delta_0 \subset \Delta$
have 1 "vanishing cycle" γ_0
impose $\Omega(n\gamma_0) = \begin{cases} 1 & \text{if } n = \pm 1 \\ 0 & \text{otherwise} \end{cases}$]

c) "B-field" $B \in H^{1,1}(\mathcal{M}^\vee)$, $\int_{\text{fiber}} B = 0$ $B = 0$ for rest of talk

Output

Recipe for a HK metric g_{RH} on $\mathcal{M}$ = SYZ mirror to $\mathcal{M}^\vee$ (fiberwise dual)

Recall [HKLR]

$$\left(\text{HK space } \mathcal{M}\right) \longleftrightarrow \left(\begin{array}{l} \text{twistor space } Z \overset{\text{diffeo}}{\simeq} \mathcal{M} \times \overset{\zeta}{\mathbb{CP}^1} \text{ with:} \\ \cdot\ \mathbb{C} \text{ structure, proj. to } \mathbb{CP}^1 \text{ is hol.} \\ \cdot\ \text{fiberwise hol. sympl. structure} \\ \quad [\mathcal{O}(2)\text{-twisted}] \\ \cdot\ \text{real structure} \\ \quad \text{covering } \zeta \mapsto -\frac{1}{\bar\zeta} \end{array}\right)$$

Construct Z by giving local hol. Darboux coords $\mathcal{X}_\gamma$ on $\mathcal{M} \times \mathbb{CP}^1$
$\gamma \in \Gamma$

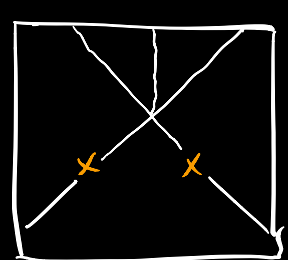
as solutions of integral equations:

$$\mathcal{X}_\gamma(\zeta) = \zeta^{-1} Z_\gamma + i\theta_\gamma + \zeta \bar{Z}_\gamma + \frac{1}{4\pi i} \sum_{\mu \in \Gamma_u} \Omega(\mu) \langle \gamma, \mu \rangle \int_{\zeta' \in \mathbb{R}_- Z_\mu} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left(1 \pm e^{\mathcal{X}_\mu(\zeta')}\right)$$

$Z_\gamma = \int_\gamma \omega \in \mathbb{C}$

coords. on torus fiber of $\mathcal{M}$
$\left(\theta_\gamma = \oint_\gamma A\right)$

Assume such $\mathcal{X}_\gamma(\zeta)$ exist. Then the $\mathcal{X}_\gamma(\zeta)$ are piecewise smooth, jump at some codim-1 "walls":



$\mathcal{B}$

Use them as hol. Darboux coords on Z

$\rightsquigarrow$ HK metric on $\mathcal{M}$.
g_{RH}

Asymptotics:

$$g_{RH} = g^{sf} + \mathcal{O}(e^{-2M}) \qquad M = \min\{|Z_\gamma| : \Omega(\gamma) \neq 0\}$$

[flat on torus fibers: $\omega_\circ = \frac{1}{2}\langle dZ, d\bar{Z}\rangle - \langle d\theta, d\theta\rangle$
Kähler form in $\mathbb{C}$ str $\zeta = 0$]

So "as we go far from Δ, $g \simeq g^{sf}$ exponentially" [cf. Gross-Wilson]

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② Higgs bundles

C compact Riemann surface
G " Lie gp

A G-Higgs bundle is: over C
- hol $G_{\mathbb{C}}$-bundle $E \to C$
- $\varphi \in H^0(\mathrm{End}\, E \otimes K_C)$

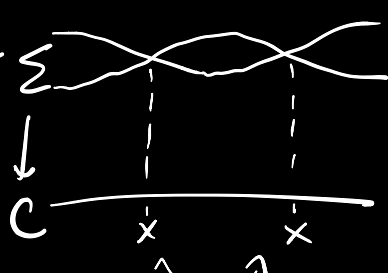
$\mathcal{M}^\vee = \mathcal{M}_H(G, C)$ moduli space of [semistable] $G_{\mathbb{C}}$-Higgs bundles

Thm [Hitchin - Simpson - Donaldson - Corlette]
- Given $(E, \varphi) \in \mathcal{M}^\vee$ $\exists!$ Hermitian metric h in E such that
$$F_{D_h} + [\varphi, \varphi^{\dagger_h}] = 0 \qquad (D_h = \text{Chern connection})$$
- The resulting L^2 metric g_{L^2} on $\mathcal{M}_H$ is hyperkähler.

Want to study $\mathcal{M} = \mathcal{M}_H(G^\vee, C)$ using RH method.
Specialize to $G = SU(2)$, $G^\vee = PSU(2)$.

a) $\mathcal{M}^\vee$ is a complex int. sys:
$$\begin{array}{ccc} \mathcal{M}^\vee & & (E,\varphi) \\ \downarrow \pi & & \downarrow \\ \phi_2 \ni H^0(C, K_C^2) = \mathcal{B} & & \phi_2 = -\frac{1}{2}\mathrm{Tr}\,\varphi^2 \end{array}$$

Spectral curve $\Sigma_{\phi_2} = \{\lambda^2 + \phi_2 = 0\} \subset T^*C$
$$\begin{array}{c} \Sigma_{\phi_2} \\ p \downarrow \\ C \end{array}$$



$$\Gamma_{\phi_2} = H_1^{odd}(\Sigma_{\phi_2}, \mathbb{Z})$$

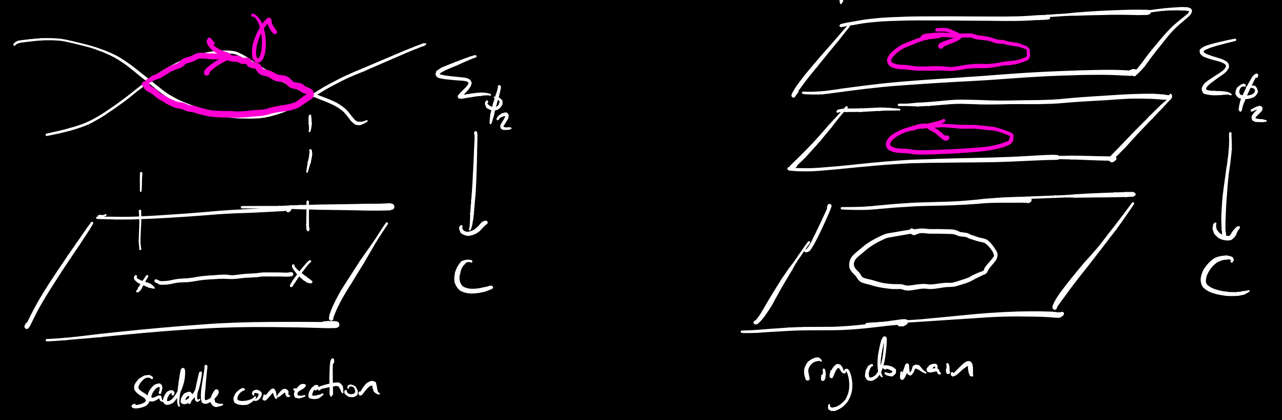
zeroes of ϕ_2

b) DT-type/BPS invariants: for $\phi_2 \in \mathcal{B}$

have $|\phi_2|$ flat metric on C, singular at zeroes of ϕ_2:

consider finite-length geodesics on C.

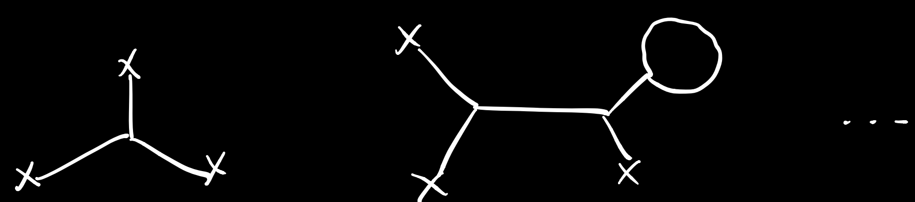
Each has an associated class $\gamma \in \Gamma_{\phi_2} = H_1^{odd}(\Sigma_{\phi_2}, \mathbb{Z})$



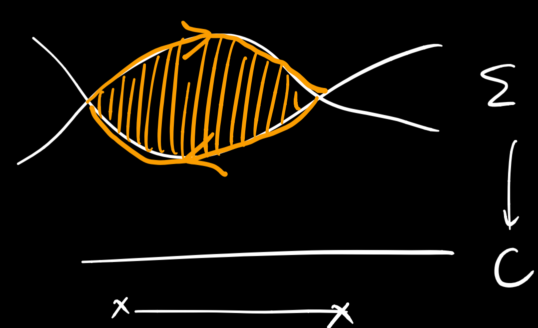
$\Omega(\gamma) = (\#\text{ saddle conn. with class } \gamma) - 2 \cdot (\#\text{ ring domains with class } \gamma)$

Conj $g_{L^2} = g_{RH}$.

Rk Similar story for $G = SU(N)$ but with more elaborate def. of $\Omega(\gamma)$



Rk $\Omega(\gamma)$ should also be understood as count of slag chains w/ ∂ on Σ



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③

a) <u>Asymptotics:</u> $G = SU(2)$

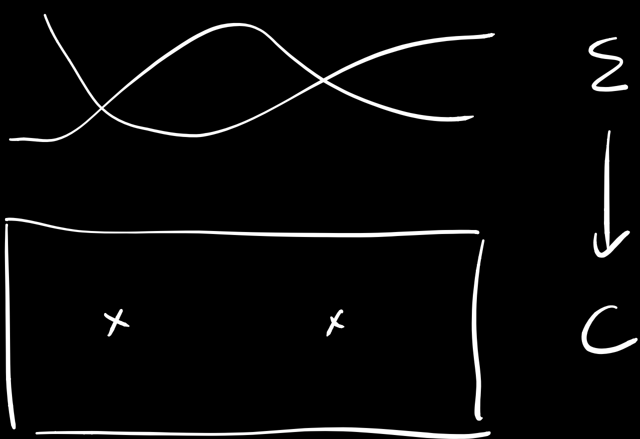
$$g_{RH} = g^{sf} + \mathcal{O}(e^{-2M})$$

$M = 2\times$ length of shortest saddle connection

<u>Thm</u> [Mazzeo-Swoboda-Weiss-Witt, Dumas-N, Fredrickson] $G = SU(2)$

$$g_{L^2} = g^{sf} + \mathcal{O}(e^{-cM})$$

<u>Pf idea</u>



g^{sf} is "L^2 metric for $G = U(1)$ Higgs bundles over Σ"

and sol^{n}s of $SU(2)$ Hitchin eq. for pt on $\mathcal{B}$

are very close to pushforward of sol^{n} of $U(1)$ eq from Σ

<u>except</u> near branch pts.

b) <u>Numerics:</u>

$$G = SU(2), \quad C = \mathbb{CP}^1, \quad E = \mathcal{O}\oplus\mathcal{O}, \quad \varphi = \begin{pmatrix} 0 & 1 \\ z^3 - c & 0 \end{pmatrix} dz \qquad c \in \mathbb{C}$$

Higgs bundles w/ wild singularity at $z = \infty$

$$g = g(c)\,|dc|^2 \qquad g(c) = g(|c|)$$

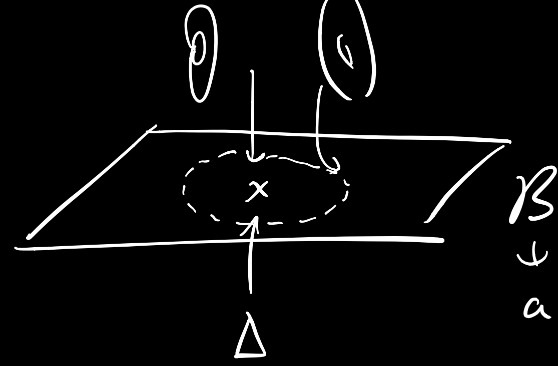
Ex T' rank 2

monodromy

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$\uparrow \ \uparrow$

$\gamma_e \ \gamma_m$



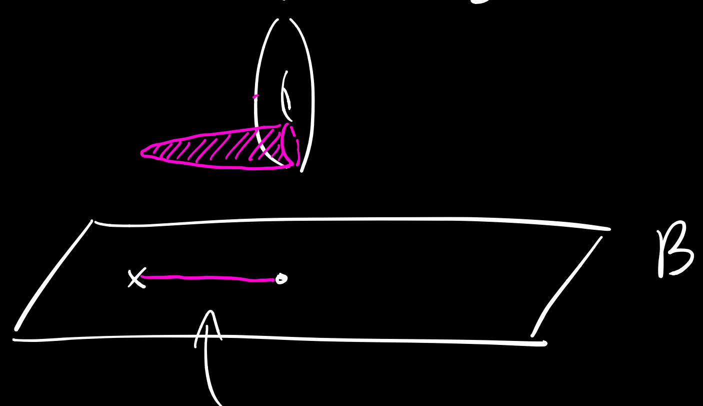
$$\Omega(\pm\gamma_e) = 1 \qquad Z_{\gamma_e} = a$$

$$\Omega(\gamma) = 0 \text{ all other } \gamma \qquad Z_{\gamma_m} = \frac{1}{2\pi i} a \log a$$

$$g_{RH} = g_{OV} \quad \text{(Ooguri-Vafa)}$$

Another enum. intp. of $\Omega(\gamma)$: they count sLag chains inside $\mathcal{M}^\vee$

[K. Chan]



split attractor flow [Denef]

[K-S]
