Twistor spaces and hyperkähler metrics

Roger Bielawski
Leibniz Universität Hannover

Simons Collaboration, June 2020

HK manifolds and twistor spaces

A *hyperkähler* manifold is a Riemannian manifold M equipped with a fibrewise action of quaternions $\mathbb{H}$ on the tangent bundle, parallel w.r.t. the Levi-Civita connection. Any unit imaginary quaternion defines an integrable complex structure on M, and the metric is Kähler w.r.t. all of them.

A non-metric version of this is a *hypercomplex* manifold, i.e. a manifold equipped a fibrewise action of quaternions on TM such that each unit imaginary quaternion defines an integrable complex structure. On a hypercomplex manifold there is a unique torsion-free connection, called the *Obata connection*, such that the action of $\mathbb{H}$ is parallel.

To a hyperkähler or a hypercomplex manifold M we can associate its twistor space Z. It is a complex manifold, diffeomorphic to $M \times S^2$, where the complex structure at $(m, \zeta) \in M \times S^2$ is $I_\zeta \oplus I_{\mathbb{P}^1}$, and I_ζ denotes the complex structure M corresponding to a unit imaginary quaternion $\zeta \in \mathbb{P}^1$.

HK manifolds and twistor spaces

Z has a natural holomorphic projection $\pi : Z \to \mathbb{P}^1$ and an antiholomorphic involution $\sigma : Z \to Z$ given by the antipodal map on S^2. Any $m \in M$ defines a σ-invariant section of π. Its normal bundle splits into line bundles of degree 1.

This data is enough to recover M as a hypercomplex manifold: it is a complete and maximal family of σ-invariant sections of $Z \xrightarrow{\pi} \mathbb{P}^1$ with normal bundle isomorphic to $\mathcal{O}(1)^{\oplus 2n}$. With respect to any complex structure I_ζ, M is biholomorphic to the fibre $Z_\zeta = \pi^{-1}(\zeta)$.

A hyperkähler metric on M induces one more piece of data on Z: an $\mathcal{O}(2)$-valued complex symplectic form Ω along the fibres of π. In terms of the standard basis i, j, k of $\mathbb{H}$, and the corresponding Kähler forms $\omega_1, \omega_2, \omega_3$, the form Ω is given by

$$\Omega = (\omega_2 + i\omega_3) + 2\omega_1\zeta - (\omega_2 - i\omega_3)\zeta^2.$$

Let us look at some examples.

HK manifolds and twistor spaces

1. If M is the flat $\mathbb{R}^{4n} \simeq \mathbb{H}^n$, then Z is the total space of the vector bundle $\mathcal{O}(1)^{\oplus 2n}$ on $\mathbb{P}^1$.

2. Let $M = S^1 \times \mathbb{R}^3$ with a flat metric such that the circumference of the circle is $1/c$. Then Z is the total space of a principal $\mathbb{C}^*$-bundle on $\mathrm{T}\mathbb{P}^1$, with transition function $\exp\{-c\eta/\zeta\}$ (here ζ is the affine coordinate on $\mathbb{P}^1\backslash\{\infty\}$ and η the induced fibre coordinate on $\mathrm{T}(\mathbb{P}^1\backslash\{\infty\})$). The associated line bundle is denoted by L^c, i.e. $Z \simeq L^c\backslash\{\textit{zero section}\}$.

Z can be also viewed as a quotient of the twistor space of $\mathbb{R}^4$, i.e. of $\mathcal{O}(1) \oplus \mathcal{O}(1) \simeq \mathbb{P}^3\backslash\mathbb{P}^1$, by $\mathbb{Z}$.

3. Let M be $\mathbb{R}^4$ with the Taub-NUT metric (with mass parameter c). Then

$$Z = \{(x,y,z) \in L^c(1) \oplus L^{-c}(1) \oplus \mathcal{O}(2);\ xy = z\}.$$

4. The twistor space a Gibbons-Hawking gravitational instanton (a.k.a. ALE-space of type A_k) is a resolution of the variety

$$\{(x,y,z) \in \mathcal{O}(k) \oplus \mathcal{O}(k) \oplus \mathcal{O}(2); xy = \prod_{i=0}^{k}(z - p_i)\},$$

where $p_i(\zeta) = a_i\zeta^2 + 2b_i\zeta + c_i$ are fixed sections of $\mathcal{O}(2)$.

5. Similarly, the twistor space of an ALF-space of type A_k is obtained by putting L^c-twists, i.e. Z is a resolution of

$$\{(x,y,z) \in L^c(k) \oplus L^{-c}(k) \oplus \mathcal{O}(2); xy = \prod_{i=0}^{k}(z - p_i)\}.$$

There are many questions one can ask about twistor spaces. For example, Z will often contain other Kodaira moduli spaces of σ-invariant sections with normal bundle $\simeq \mathcal{O}(1)^{\oplus 2n}$, so we get several hyperkähler or pseudo-hyperkähler manifolds from Z.
Secondly, one can ask what happens to the differential geometry of M at points, where the normal bundle of a section jumps.

Thirdly, one can consider all sections (with normal bundle $\mathcal{O}(1)^{\oplus 2n}$), not just σ-invariant. This is a natural complexification of M, but it has its own natural geometry - a complex analogue of hyperkähler geometry (vide Jardim-Verbitsky).

On this manifold of all sections one can then consider different real structures. For example, the submanifold of sections invariant under an antiholomorphic involution, which covers conjugation on $\mathbb{P}^1$, is a so-called *hypersymplectic* manifold (Hitchin, Dancer-Swann, Röser), i.e. a pseudo-Riemannian manifold with geometry based on split quaternions.

Finally, one can also consider not just sections, but also curves of higher degree in Z, and the geometry of their moduli space.

In the remainder of the talk I will want to address most of these questions.

Example. Let Z be the twistor space of the Calabi-Eguchi-Hanson instanton (i.e. A_1-ALE space): a resolution of

$$\{(x, y, z) \in \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2); xy = (z - p_1)(z - p_2)\},$$

for a pair of quadratic polynomials p_1, p_2. Sections of Z can be found by choosing a section $z(\zeta)$, factorising $z - p_1$, $z - p_2$, and assigning two of the roots to x, and the other two to y. Clearly there is a finite indeterminacy, and if we make a "wrong" choice, then we end up not with a positive-definite metric, but with a hyperkähler metric which vanishes on a 3-dimensional submanifold Y (and changes sign there). This is a basic example of Hitchin's *folded hyperkähler structure*.

Sections of Z, which correspond to points of Y, have normal bundle $N \simeq \mathcal{O} \oplus \mathcal{O}(2)$ rather than $\mathcal{O}(1) \oplus \mathcal{O}(1)$. In general (higher dimensions) there are several good reasons to consider a partial compactification $\overline{M}$ of a hyperkähler manifold M by including sections of Z, the normal bundle of which splits into line bundles of *nonnegative* degrees.

Kronecker structures

For one, this is still a smooth manifold, since $h^1(N) = 0$. Secondly, the hypercomplex structure extends to a 2-*Kronecker structure* on $\overline{M}$ (B. – C. Peternell), i.e. a quaternionic vector bundle E on $\overline{M}$ and a homomorphism $\alpha : E \otimes \mathbb{C}^2 \to \mathrm{T}^{\mathbb{C}}\overline{M}$, which is an isomorphism at points of M. There is an appropriate notion of integrability of such a structure.

In particular, the Levi-Civita connection of M extends to a meromorphic connection on $\overline{M}$.

In general (also in 4 dimensions), the geometry of $\overline{M}$ does not have to be folded.

Adjoint orbits

Kronheimer's hyperkähler metrics on adjoint orbits of a complex semisimple Lie group $G^{\mathbb{C}}$ (later generalised by Biquard and Kovalev)

For a regular orbit, they are parametrised by a triple of elements in $\mathfrak{h}$. Such a triple defines a section of $\mathfrak{h}^{\mathbb{C}} \otimes \mathcal{O}_{\mathbb{P}^1}(2)$, and consequently a section S of $\left(\mathfrak{h}^{\mathbb{C}} \otimes \mathcal{O}_{\mathbb{P}^1}(2)\right)/W \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(2d_i)$.

The twistor space of an open dense subset of this hK manifold is

$$X_S = \{x \in \mathfrak{g}^{\mathbb{C}} \otimes \mathcal{O}(2);\, p(x) = S(\pi(x))\},$$

where $p : \mathfrak{g}^{\mathbb{C}} \otimes \mathcal{O}(2) \to \bigoplus \mathcal{O}_{\mathbb{P}^1}(2d_i)$ is the induced projection.

D'Amorim Santa-Cruz (1997): arbitrary $S \implies$ (incomplete) G-invariant (pseudo)-hK metrics on adjoint orbits as Kodaira moduli spaces of section of X_S.

For $G = SU(2)$ these are the Bellinski-Gibbons-Page-Pope metrics.

Proposition [B.-Foscolo] The Kodaira moduli space of sections of X_S with $N \simeq \bigoplus \mathcal{O}(1)$ has at least two components, one where the hK metric is positive definite and one where it is negative definite.

Higher degree curves

Instead of sections of $\pi : Z \to \mathbb{P}^1$ one can consider curves of higher degree/genus in Z.

O. Nash (2006): L^2-metric on the moduli space of charge k $SU(2)$-monopoles as curves in the twistor space of $S^1 \times \mathbb{R}^3$, i.e. the total space of L^2 on $|\mathcal{O}(2)|$ (minus the zero section).

Careful: these curves must lift from $|\mathcal{O}(2)|$. Z has other families of curves!

Generalises to twistor spaces of arbitrary 4-dim'l hK manifolds (B. 2014): the space of σ-invariant degree k curves C, flat over $\mathbb{P}^1$, such that $h^1(N_{C/Z}(-2)) = 0$ has a natural (pseudo)-hK metric.

"Space" means (open subset of) the *Douady space* (or Hilbert scheme) and "curves" can be rather wild (but not very wild in the context of Hilbert schemes).

$Z \to \mathrm{Hilb}^k_{\mathrm{rel}}(Z)$ - fibrewise Hilbert scheme of k points - new twistor space. $\mathrm{Hilb}^k(X)$ parametrises configurations of k points in X.

Fibres (F) are 2-dim'l $\Longrightarrow$ $\mathrm{Hilb}^k(F)$ is smooth and there is an induced complex symplectic form on $\mathrm{Hilb}^k(F)$ (Beauville)

Example: $\mathbb{R}^4$

- Twistor space Z: total space of $\mathcal{O}(1)^{\oplus 2}$, or $\mathbb{P}^3 \backslash \mathbb{P}^1$
- Curves in $Z \iff$ curves in $\mathbb{P}^3$ avoiding a fixed $\mathbb{P}^1$
- Hilbert scheme $H_{k,g}$ of curves in $\mathbb{P}^3$ of degree k and genus g is a classical object
- Describe the *complexified* hyperkähler structure on the appropriate open subset of $H_{k,g}$
- Fibres of Z are just $\mathbb{C}^2$: $\mathrm{Hilb}^k(\mathbb{C}^2)$ - $GL_k(\mathbb{C})$-orbits of pairs of commuting matrices + a cyclic vector (Nakajima)
- B.-C.Peternell $\implies$ curves in Z - commuting matrix polynomials (modulo conjugation)

Theorem (B. – C. Peternell, 2019) The Hilbert scheme of rational curves of degree k, $k \geq 3$, in $\mathbb{P}^3 \backslash \mathbb{P}^1$, without planar components, and satisfying $h^1(N_{C/Z}(-2)) = 0$ is a $\mathbb{C}$-hyperkähler quotient of (an open subset of) a flat vector space of dimension $4k(k-2)$ by a *nonreductive* Lie group.

If we relax to $h^1(N_{C/Z}(-1)) = 0$, we'll get a 2-Kronecker structure.

Transverse Hilbert schemes

Often (e.g. monopoles) one wants to consider a smaller twistor space - an open dense subset of $\mathrm{Hilb}^k_{\mathrm{rel}}(Z)$. This is the case when Z has a holomorphic submersion μ onto $|\mathcal{O}(2r)|$, as for ALE or ALF gravitational instantons.

Then fibres have a submersion $\mu : F \to \mathbb{C}$ and we consider only those $D \in \mathrm{Hilb}^k(F)$ with $\mu(D)$ consisting of k points (with multiplicities).

transverse Hilbert scheme $\mathrm{Hilb}^k_\mu(F)$; Atiyah & Hitchin (1988) for monopoles;

$Z \to \mathrm{Hilb}^k_{\mu,\mathrm{rel}}(Z)$ - get a new hyperkähler manifold $\mathrm{Hilb}^k_\mu(M)$. works well with ALF spaces - get positive definite and complete hyperkähler metrics.

The Hilbert scheme of all curves of degree k in Z can be much larger than $\mathrm{Hilb}^k_\mu(M)$; e.g. in the case of $Z = L^2 \backslash \{0\} \simeq (\mathbb{P}^3 \backslash \mathbb{P}^1)/\mathbb{Z}$ ($M = S^1 \times \mathbb{R}^3$), any curve C of degree k in $\mathbb{P}^3 \backslash \mathbb{P}^1$ such that $C \cap \mathbb{Z}^* C = \emptyset$ descends to Z, but doesn't belong to $\mathrm{Hilb}^k_\mu(M)$.

Higher dimensions?

Can we start with a hk manifold of dimension > 4 and construct a new twistor space by taking fibrewise (transverse or not) Hilbert scheme of points on the original twistor space?

It will not be a twistor space of a hyperkähler or a hypercomplex manifold. If X is symplectic with $\dim X > 2$, then $\mathrm{Hilb}^k(X)$ is never symplectic (even on its smooth locus).

There is an exception...

In the presence of a finite group action, one can define the Γ*-Hilbert scheme* $\mathrm{Hilb}^\Gamma(X)$: a component of $\left(\mathrm{Hilb}^{|\Gamma|}(X)\right)^\Gamma$ defined as the closure of the subset of regular orbits.

Partial resolution of X/Γ and smooth whenever $\mathrm{Hilb}^{|\Gamma|}(X)$ is smooth.

Proposition (B. – Foscolo) Let Γ be a finite Coxeter group acting holomorphically on a complex manifold X, and assume that the subset where the action is not free is a union of codimension 2 submanifolds X^w, each of them being a fixed point set of a reflection $w \in \Gamma$. Then any Γ-invariant symplectic form on X induces a symplectic form on the smooth locus of $\mathrm{Hilb}^\Gamma(X)$.

If X is equipped with a Γ-equivariant submersion μ onto Y, we can also defined the transverse Γ-Hilbert scheme, and:

Proposition (B. – Foscolo) Let X be a complex manifold with a holomorphic action of a finite Coxeter group Γ, and a surjective, holomorphic, and Γ-equivariant map $\mu : X \to \mathbb{C}^n$, with Γ acting on $\mathbb{C}^n$ as a reflection group. Then $\mathrm{Hilb}^\Gamma_\mu(X)$ is smooth.

Hypertoric varieties

These are hyperkähler quotients of a quaternionic vector space by a torus. A $4n$-dimensional hypertoric variety M has a tri-Hamiltonian action of T^n, and is determined by the image of singular orbits under the moment map $\mu_{\mathbb{H}} : M \to \mathbb{R}^{3n}$, i.e. by a collection $\{H_i; i = 1, \dots, d\}$ of codimension 3 flats in $\mathbb{R}^{3n}$.

Consequently, the twistor space of a hypertoric variety comes equipped with a projection $\mu : Z \to \mathcal{O}(2)^{\oplus n}$.

Principle: ALF better than ALE. We consider a *Taub-NUT* modification of M, i.e. the hK quotient of $M \times T^n \times \mathbb{R}^{3n}$ by T^n. We consider only the case where the circumference of each circle is $1/c$. We denote this hK quotient by M_c, its twistor space by Z_c.

The volume growth of M is Euclidean, that of M_c as r^{3n}.

Z_c can be described explicitly. Over $\mathcal{O}(2)^{\oplus n} \setminus \bigcup H_i$ it is certain principal $(\mathbb{C}^*)^n$-bundle determined by the H_i.

QALF metrics from W-invariant hypertoric varieties

Joint project with Lorenzo Foscolo

T - maximal torus of a compact Lie group G, the collection $\{H_i\}$ of flats is W-invariant. We consider the W-equivariant transverse Hilbert scheme $\mathrm{Hilb}_\mu^W(M_c)$, i.e. W-invariant real curves in $\mathfrak{h}\otimes O(2)$ ($\mathfrak{h}\simeq\mathfrak{h}^*$), flat over $\mathbb{P}^1$, which can be lifted to W-invariant real curves in Z_c satisfying $h^1\left(N_{C/Z_c}(-2)^W\right)=0$.

If nonempty, then a (pseudo)-hyperkähler manifold of dimension $4\mathrm{rank}\, G$.

Aim: determine when $\mathrm{Hilb}_\mu^W(M_c)$ is positive-definite and complete; show that they are QALF.

Abelianisation for (co-)Higgs bundles (Faltings, Kanev, Scognamillo, Hurtubise) implies that $\mathrm{Hilb}_\mu^W(M_c)$ corresponds to a moduli space of $\mathfrak{g}$-valued solutions to Nahm's equations on $[0,c]$ with a regular pole at $t=0$.

The condition that a curve C can be lifted from $\mathfrak{h}\otimes O(2)$ to Z_c means that certain W-invariant principal $T^{\mathbb{C}}$-bundle on C is trivial $\implies$ boundary conditions at $t=c$.

The Coulomb branch

The Coulomb branch M_C of a 3-dimensional $N = 4$ SUSY gauge theory is a (possibly singular) hyperkähler manifold with an $SU(2)$-action rotating complex structures, associated to a compact Lie group and its (quaternionic) representation.

Mathematical approach by Nakajima, and Braverman-Finkelberg-Nakajima

Proposal: take a W-invariant configuration of flats, passing through the origin, the normals of which are the weights of the representation. The Coulomb branch is $\mathrm{Hilb}_\mu^W(M_c)$.

Example: no flats ($M \simeq T \times \mathfrak{h}^3$). The condition that a W-invariant curve S in $\mathfrak{h} \otimes O(2)$ can be lifted to $Z_c \iff$ the principal $T^{\mathbb{C}}$-bundle on $\mathfrak{h} \otimes O(2)$ used to define the Taub-NUT deformation is trivial on S $\iff$ the Nahm flow is periodic. $\mathrm{Hilb}_\mu^W(M_c)$ is the moduli space of $\mathfrak{g}$-valued solutions to Nahm's equations on $[-c, 0]$ with regular poles at both ends.

More examples

$G = SU(m)$; the flats H_i are defined by $x_i = \lambda$ for a fixed $\lambda \in \mathbb{R}^3$. The hypertoric variety M is in this case Kronheimer's metric on a minimal adjoint orbit of $SL(m, \mathbb{C})$ (in the case $\lambda = 0$, it is the closure of the minimal nilpotent orbit). (A component of) $\mathrm{Hilb}_\mu^W(M_c)$ is the moduli space of $\mathfrak{su}(m)$-valued solutions to Nahm's equations on $(0, c)$ with a regular pole at $t = 0$, a subregular pole at $t = c$, and matching the constant solution $(\lambda_1, \lambda_2, \lambda_3)$ at $t = c$ in the usual way. The gauge group consists of $PSU(m)$-valued gauge transformations on $[0, c]$ which are equal to identity at $t = 0$ and belong to the centraliser of the subregular triple at $t = c$.

G -simple; the W-orbit of flats generated by a fundamental co-weight w. $\mathrm{Hilb}_\mu^W(M_c)$ - moduli space of $\mathfrak{g}$-valued solutions to Nahm's equations on $(0, c)$ with a regular pole at $t = 0$, a regular pole in the subalgebra of the normaliser of w at $t = c$, and the value of the simple root dual to w equal to the parameter λ.

More examples

$G = SU(m)$ and suppose that we have several W-orbits of flats as above, i.e. the flats are defined by $x_i = \lambda_j$, $i = 1, \dots, m$, $j = 1, \dots k$. $\mathrm{Hilb}_\mu^W(M_c)$ is the Nahm moduli space of $\mathfrak{su}(m)$-solutions on $(0, c)$ with regular pole at $t = 0$ and matching a $\mathfrak{u}(k)$-solution on (c, ∞) corresponding to the adjoint orbit of $(\lambda_1, \dots, \lambda_k)$. Again we have to quotient by an appropriate centraliser.

$G = U(m) \times U(n)$; M defined by flats $x_i = y_j$ in $\mathbb{R}^{3m} \oplus \mathbb{R}^{3n}$, where the $x_i \in \mathbb{R}^3$ are coordinates on $\mathbb{R}^m \otimes \mathbb{R}^3$ and the $y_j \in \mathbb{R}^3$ on $\mathbb{R}^n \otimes \mathbb{R}^3$. (A component of) is $\mathrm{Hilb}_\mu^W(M_c)$ is the moduli space of $SU(3)$-monopoles of charge (m, n) and the Higgs field at infinity having eigenvalues $-c, 0, c$. We can also obtain the moduli space with eigenvalues μ_1, μ_2, μ_3 ($\sum \mu_i = 0$) of the Higgs field at infinity, if we start with a 2-parameter Taub-NUT deformation of the hypertoric variety M. Similarly we can obtain any moduli space of $SU(N)$-monopoles with maximal symmetry breaking, as well as their deformations (the latter defined in the $SU(3)$-case by flats $x_i - y_j = \lambda$ for a fixed $\lambda \in \mathbb{R}^3$).

More examples

The last example suggests an interesting possibility. Let $G = U(m_1) \times U(m_2) \times U(m_3)$ and let M be defined by flats $x_i = y_j$, $y_j = z_k$, $x_i = z_k$ ($i = 1, \ldots, m_1$ etc.). Is $\mathrm{Hilb}_\mu^W(M_c)$ the moduli space of $SO(8)$-monopoles with maximal symmetry breaking and charge (m_1, m_2, m_3)? An analogous question can be asked for other simply laced Dynkin diagrams.