SIMONS COLLABORATION ON SPECIAL HOLONOMY IN
GEOMETRY, ANALYSIS AND PHYSICS

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Hasse Diagrams for Symplectic Singularities via Magnetic Quivers

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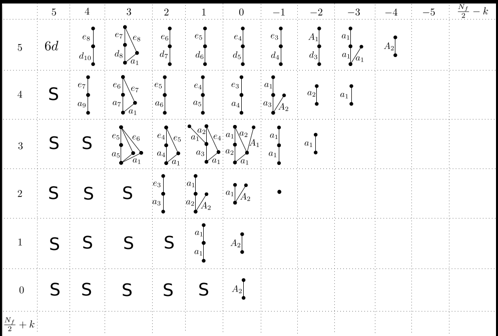


5 4 3 2 1 0 −1 −2 −3 −4 −5
$\frac{N_f}{2} - k$
5 4 3 2 1 0
$\frac{N_f}{2} + k$
6d
S
e_8 d_{10}
e_7 e_8 d_8 a_1
e_6 d_7
e_5 d_6
e_4 d_5
e_3 d_4
A_1 d_3
a_1 a_1 a_1
A_2
e_7 a_9
e_6 e_7 a_7 a_1
e_5 a_6
e_4 a_5
e_3 a_4
a_1 a_3 A_2
a_2
a_1
e_5 e_6 a_5 a_1
e_4 e_5 a_4 a_1
a_1 a_2 e_4 a_3 a_1
a_1 a_2 A_1 a_2 a_1
a_1 a_1
a_1
e_3 a_3
a_1 a_2 A_2
a_1 A_2
a_1 a_1
A_2
A_2

Hanany, Cabrera, Grimminger, Sperling, Zajac, Zhong
Cremonesi, Zaffaroni, Miketa, Kalveks, ...
Beauville, Bellamy, Schedler, Webster, Kaledin, Namikawa.
Nakajima, Braverman, Finkelberg, Weekes

I) Hasse diagrams for symplectic singularities

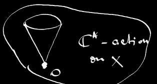
Goal: describe moduli space of vacua of susy QFTs
(Higgs Branch) $\rightsquigarrow$ conic symplectic singularities

Nilpotent cone of simple complex Lie algebra.

ADE du Val / Klein singularities

Def. X - normal affine variety / $\mathbb{C}$. X has symplectic singularities
if (a) $\exists$ symplectic form ω on X_{smooth}
(b) For any resolution of singularities $\rho : Y \longrightarrow X$, $\rho^*\omega$
is a regular 2-form on Y (not necessarily non-degenerate)

$\mathbb{C}[X]$ is a Poisson algebra.



$\mathbb{C}^*$- action on X
0

Def X - normal affine variety. We say that X
is a conic symplectic singularity (CSS) if
(a) X has symplectic singularities
(b) $\mathbb{C}[X] = \bigoplus_{n \geq 0} \mathbb{C}[X]_n$ and $\mathbb{C}[X]_0 = \mathbb{C}$
(c) $\exists\, \ell > 0$, $\forall f \in \mathbb{C}[X]_n$ $g \in \mathbb{C}[X]_m$, $\{f, g\} \in \mathbb{C}[X]_{n+m-\ell}$

Namikawa.

Take $x \in X$. $\lim_{t \to \infty} t \cdot x = 0$ ← origin.

Namikawa Th.

X - CSS. $\{x_i\}$ minimal $\overset{\text{finite}}{\text{set}}$ of $\overset{\text{homo}}{\text{generators}}$ of $\mathbb{C}[X]$

$$\begin{cases} N = \max_i \deg(x_i) \\ d = \dim X \end{cases}$$

- For fixed (N, d), there exists finitely many CSS (up to isom)
- For $N=1$ the degree of $\{.\}$ is 1 or 2
  - if $\ell = 2$ $X = (\mathbb{C}^d, \omega_{st})$
  - if $\underline{\ell = 1}$ $X = (\overline{\mathcal{O}}, \omega_{KKS})$

  $\overline{\mathcal{O}}$ closure of normal nilpotent orbit in semisimple complex Lie algebra

Physically : $\mathbb{C}[X]$ is the chiral ring

degree of generator $\leftrightarrow$ $\begin{cases} \text{R-charge} \\ \text{conformal weight.} \end{cases}$

$f, g \in \boxed{\mathbb{C}[X]_1}$ $\{f, g\} \in \mathbb{C}[X]_1$

$$\boxed{HS_X(t) = \sum_{n \geq 0} t^n \dim \mathbb{C}[X]_n = 1 + t \underset{\uparrow}{\#} + \cdots}$$

$\boxed{\dim \mathbb{C}[X]_1}$

Stratification.

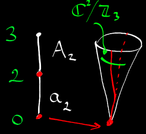
Kaledin

Thm : X — normal affine variety w/ symplectic singularities.

- ∃ finite stratification by $X_i \subseteq X$ with $\mathring{X}_i$, smooth and symplectic, they are the symplectic leaves of X
- The X_i are partially ordered by inclusion ⇝ Hasse diagram
- $\forall x \in \mathring{X}_i$ there is a transverse slice to x in X which is also a CSS.

Example : Nilpotent cone in $\mathfrak{sl}(3, \mathbb{C})$.

$= \{ M \in \mathfrak{sl}(3, \mathbb{C}) \mid M^3 = 0 \}$

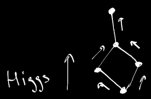

$\mathbb{C}^2/\mathbb{Z}_3$
3
A_2
2
a_2
0

Notation : $\begin{cases} A_n = \mathbb{C}^2 / \mathbb{Z}_{n+1} \\ a_n = \text{closure of minimal nilpotent orbit of } \mathfrak{sl}(n+1, \mathbb{C}) \\ d_n = \text{———————— } \mathfrak{so}(2n, \mathbb{C}) \\ e_n = \text{———————— } E_n \quad (n = 6, 7, 8) \end{cases}$

X	A_n	D_n	E_n	a_n	d_n	e_6	e_7	e_8
$\dim_{\mathbb{H}} X$	1	1	1	n	$2n-3$	11	17	29

Singularity on moduli space of vacua ⟷ appearance of massless states.

↓ Converse of Higgs mech.



(II) Magnetic Quivers.

3d $\mathcal{N}=4$ Coulomb Branch

Consider a renormalizable 3d $\underline{\underline{\mathcal{N}=4}}$ gauge theory

- G complex semisimple Lie group (gauge group)
- N complex finite-dim representation of G (matter content) $(N\oplus N^*)$

Moduli space of vacua of this theory $\longrightarrow$ form an affine alg. variety.

R-symmetry $$so(4) = \underbrace{su(2)_H}_{\substack{\text{acts on}\\ \text{Scalars}\\ \text{in}\\ \text{hyper}\\ \text{multi.}}} \oplus \underbrace{su(2)_C}_{\substack{\text{acts on}\\ \text{Scalars in}\\ \text{vector}\\ \text{multiplets}}}$$

$\swarrow$ Higgs Branch $\quad\downarrow$ Coulomb Branch

$\mathcal{M}_H$ = Higgs Branch
= HyperKähler space whose 3 complex st. are permuted by $su(2)_H$

$\mathcal{M}_C$ = ——— $su(2)_C$

$\mathcal{M}_H$ = hyperKähler quotient $(N\oplus N^*)///G$

$$\mathcal{M}_C^{cl} = \frac{(\mathbb{R}^3\times S^1)^{\text{rank } G}}{\text{Weyl}(G)}$$

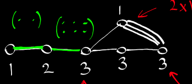
$\mathcal{M}_C$ receives quantum corrections

$\mathcal{M}_C \neq \mathcal{M}_C^{cl}$

$\nearrow$ $\boxed{\text{Moduli space of dressed monopole operators.}}$

Monopole formula and quiver gauge theories

Def. A $\overset{\text{good}}{\underline{\text{quiver}}}$ Q is a $\overset{\text{finite}}{\text{connected}}$ $\underset{\text{by integers}}{\underline{\text{labeled}}}$ graph $N_i>0$ $\underline{\text{without loops}}$ satisfying the "balance condition":
$$2N_i \leqslant \sum_{k \text{ neighb of } i} N_k$$



$2\times 1 \leqslant 3+3+3+3$ $<$

$2\times 3 \underset{=}{\leqslant} 2+3+1$

$2\times 3 \underset{=}{\leqslant} 3+1+1+1$

$\rightsquigarrow$ $\mathcal{M}_C(Q)=X$

$\dim_{\mathbb{H}} X = 12$

$$G = \frac{GL(1)\times GL(2)\times GL(3)^3\times GL(1)}{GL(\underline{\underline{1}})} = \frac{\prod GL(N_i)}{GL(1)}$$

N = Bifundamentals associated to links.

$$\boxed{HS(\mathcal{M}_C(Q),t) = \frac{1}{|W|}\sum_{m\in\mathbb{Z}^r}\ \sum_{\gamma\in W(m)} \frac{t^{\Delta(m)}}{\det(\mathbb{1}_r - t\gamma)}}$$

$W=\text{Weyl}(G)$ $\quad r = \text{Rank } G = \sum N_i$ $\quad W=\prod S_{N_i}$

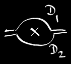
$m\in\mathbb{Z}^r$ $\quad W(m)=\{g\in W \mid g\cdot m = m\}$

Monopole Formula

$$\Delta(m) = -\sum_{\alpha \text{ positive root}} |\alpha(m)| + \frac{1}{2}\sum_{\rho \text{ weights of } N} |\rho(m)|$$

Definition of $\mathcal{M}_C$ by BFN

$$\boxed{\mathcal{M}_C = \mathrm{Spec}\left(H_*^{G_{\mathcal{O}}}(\mathcal{R}) , * \right)}$$

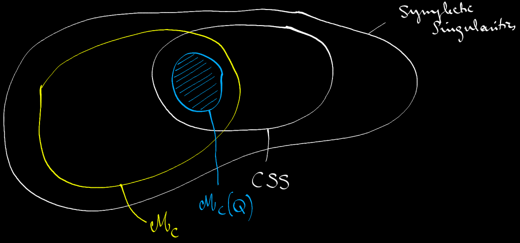


$D_1^* \hookrightarrow D_2^*$

$\mathcal{R}_{G,N}$ = moduli space of triples $(\mathcal{P}, \varphi, s)$ where

- $\mathcal{P}$ = G-bundle over D $\rightsquigarrow \mathcal{P}_N = \mathcal{P} \times_G N$
- φ = trivialization of $\mathcal{P}$ over D^* $\rightsquigarrow \varphi_N : \mathcal{P}_N|_{D^*} \longrightarrow D^* \times N$
- s = section of $\mathcal{P}_N$ such that $\varphi_N(s) \in N_{\mathcal{O}}$ with $\mathcal{O} = \mathbb{C}[z]$

Conjecture : $\mathcal{M}_C$ has symplectic singularities. (cf Weekes)


Symplectic Singularities
CSS
$\mathcal{M}_C(Q)$
$\mathcal{M}_C$

Def. Let X be a CSS, and Q be a good quiver. We say that Q is a magnetic quiver for X if $X = \underline{\underline{\mathcal{M}_C}}(Q)$.

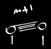
Fact : in many cases, one can compute the Hasse diagram of $X = \mathcal{M}_C(Q)$ by an algorithm applied to Q, quiver subtraction .

Input of algo :

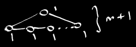
① quiver Q

② List of elementary transverse slices and "their" magnetic quiver.

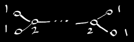
A_n :



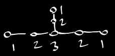
a_n :



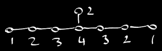
d_n :



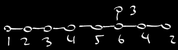
e_6 :

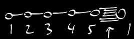


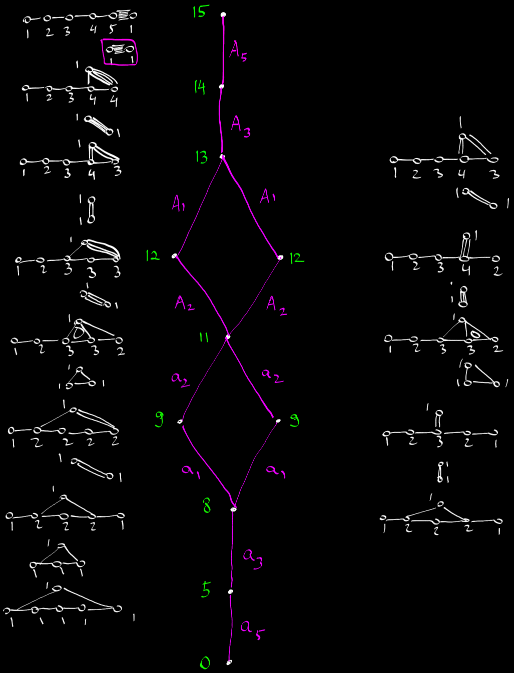
e_7 :



e_8 :



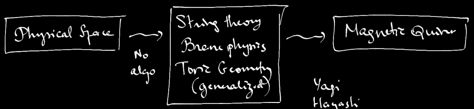
Example : {Nilpotent cone of $\mathfrak{sl}(6,\mathbb{C})$} = $X = \mathcal{M}_C($  $)$


15
A_5
14
A_3
13
A_1
A_1
12
12
A_2
A_2
11
a_2
a_2
9
9
a_1
a_1
8
a_3
5
a_5
0

Ⅲ Applications to physics

Classical Higgs branches $(N \oplus N^*)///G$

Gauge theory with gauge group $\underline{\underline{SU}}(\underline{\underline{N_c}})$ and $\underline{\underline{N_f}}$ fundamentals.


Physical Space

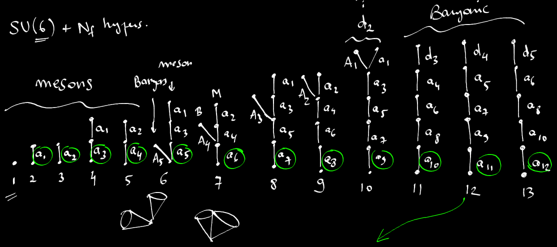
No algo

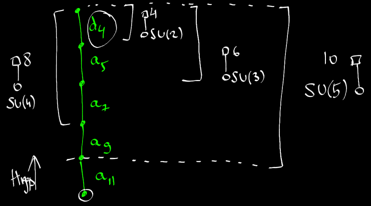
String theory
Brane physics
Toric Geometry (generalized)

Magnetic Quiver

Yagi
Hayashi
Zafrir

SU(6) + N_f hypers.


mesons
meson
Baryons
Baryonic
d_2
d_3
d_4
d_5
A_1
A_3
A_4
A_5
B
M
a_1
a_2
a_3
a_4
a_5
a_6
a_7
a_8
a_9
a_{10}
a_{11}
a_{12}
1
2
3
4
5
6
7
8
9
10
11
12
13


d_4
a_5
a_7
a_9
a_{11}
Higgs
8
SU(4)
4
SU(2)
6
SU(3)
10
SU(5)

$N_f = 12$

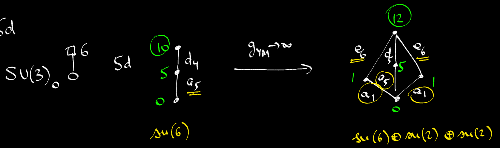
$N_c = 6$

Non-Lagrangian : Higgs branches that are not of the HK form.

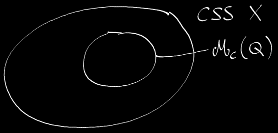
5d gauge th. $g_{YM} \longrightarrow \infty$ Strongly coupled theory. SCFT.

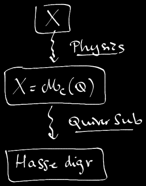
6d

Ex.


$SU(3)_0$
6
5d
10
5
0
d_4
a_5
$su(6)$
$g_{YM} \to \infty$
12
e_6
d_4
5
e_6
1
a_5
1
a_1
a_1
0
$su(6) \oplus su(2) \oplus su(2)$

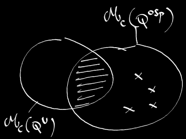
Conclusion.


CSS X
$\mathcal{M}_c(Q)$

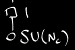

X
Physics
$X = \mathcal{M}_c(Q)$
Quiver Sub
Hasse digr

Questions :

- Details of algorithm in "extreme cases"
- Explore the elementary slices.
- Extension to wider classes of quivers
  - loops?
  - non simply laced $\circ\Leftarrow\circ$
  - orthosymplectic quivers


$\mathcal{M}_c(Q^{osp})$
$\mathcal{M}_c(Q^{u})$

- Nilpotent operators?



$$\mathcal{M}_H = \mathrm{Spec}\left(\frac{\mathbb{C}[x]}{(x^2)}\right)$$

• x 2

- Interplay with toric geometry.

PS : The author I forgot concerning Conformal Fiber Diagrams is Ling Lin, I apologize !