Hodge theory and the topology of hyper-Kähler manifolds: an introduction

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• **Fact.** *Kähler metrics g on Riemannian manifolds M of even real dimension $2n$ are characterized as metrics of holonomy contained in $U(n)$.*

• **Explanation.** Holonomy contained in $U(n)$ says that there is an almost complex structure operator I which is Levi-Civita parallel, hence integrable, and such that $g(u, Iv) := \omega(u, v)$ is a skew-form. Then ω is also parallel hence closed. Hence $X = (M, I)$ is complex and (X, ω) is Kähler.

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• **Holonomy** $\subset SU(n)$**.** This means that there is a $(n, 0)$-form η_X which is parallel for the Levi-Civita connection. Then η_X is holomorphic and nowhere 0. Hence K_X is trivial.

• Conversely: **Yau's theorem.** *If X is compact Kähler and has trivial canonical bundle, for any Kähler class α on X, there is a unique Kähler form ω_α of class α, such that η_X is parallel for the Levi-Civita connection (Ricci flat Kähler-Einstein metric).*

- $n = 2m$. Subgroup $Sp(m) \subset SU(n)$ defined as the group preserving the standard hermitian form on $\mathbb{C}^n$ and symplectic form σ on $\mathbb{C}^{2m}$. σ^m generates $\bigwedge^n(\mathbb{C}^n)$, which gives the inclusion.

Thm. (Beauville) *A simply connected compact Kähler n-fold X admits a metric of holonomy $Sp(m)$ if and only if $H^{2,0}(X) = \mathbb{C}\sigma_X$ for some holomorphic 2-form σ_X which is everywhere nondegenerate.*

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Proof. If X admits a metric of holonomy $Sp(m)$, there exists a unique $(2,0)$-form σ_X on X which is Levi-Civita-parallel and everywhere nondegenerate. σ_X is closed, hence holomorphic. Any holomorphic 2-form on X is parallel by Bochner principle, hence proportional to σ_X.

Conversely. If σ_X exists, then K_X is trivial. Then Yau provides Kähler-Einstein metrics on X. For such a metric, the holomorphic forms are parallel, so in particular the holonomy is contained in $Sp(m)$. If the holonomy is smaller, use Berger classification and conclude that X has parallel $(1,0)$-forms or other parallel $(2,0)$-forms. These forms would be holomorphic, contradicting simple connectedness and/or $H^{2,0}(X) = \mathbb{C}\sigma_X$. **qed**

A global construction: Twistor lines

- X as in the theorem will be called a **hyper-Kähler manifold**.
- Let (X, σ_X, ω) be a hyper-Kähler manifold, where ω is Kähler-Einstein. Then $\mathrm{Re}\,\sigma_X$, $\mathrm{Im}\,\sigma_X$, ω are parallel real 2-forms.
- Write $\mathrm{Re}\,\sigma_X(u, v) = \omega(u, Jv)$ and $\mathrm{Im}\,\sigma_X(u, v) = \omega(u, Kv)$, defining Levi-Civita–parallel operators K, J on T_M.
- Now, use the fact that ω is of type $(1, 1)$ while σ_X is of type $(2, 0)$. Thus $\omega(Iu, Iv) = \omega(u, v)$, $\mathrm{Re}\,\sigma_X(u, Iv) = -\mathrm{Im}\,\sigma_X(u, v)$, $\mathrm{Im}\,\sigma_X(u, Iv) = \mathrm{Re}\,\sigma_X(u, v)$.
- This implies relations $IJ = -JI = K$, $IK = -KI = -J$.
- One also gets that $J^2 = K^2$ is a self-adjoint **parallel** endomorphism of T_M, hence proportional to the identity. After rescalling σ_X, one can arrange $J^2 = K^2 = -Id_{T_M} \rightsquigarrow$ quaternionic structure.
- The operators I, J, K are Levi-Civita parallel hence for any $I_t = \alpha I + \beta J + \gamma K$ in the sphere of pure quaternions of norm 1, the almost complex structure I_t on X is integrable.

• X complex compact. **Deformation functor**:
$(B,0) \mapsto \{\text{isom. classes of families } f : \mathcal{X} \to B, \text{ plus isom.} \mathcal{X}_0 \cong X\}$. Here $(B,0) =$ (germ of) pointed analytic space, $\mathcal{X}$ is complex analytic, f is smooth proper holomorphic,.

• If $H^0(X,T_X)=0$, this functor is representable by a universal family $\mathcal{X}_{univ} \to B_{univ}$.

• The tangent space $T_{B_{univ},0}$ is by definition the set of first order deformations $X_1 \to D_1$ of X, where $D_1 = \operatorname{Spec} \mathbb{C}[t]/t^2$. This set is also isomorphic to $H^1(X,T_X)$ by the Kodaira-Spencer map.

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- **Bogomolov-Tian-Todorov Theorem**. *If X is a compact Kähler manifold with trivial canonical bundle, the deformations of X are unobstructed.*

- This means that B_{univ} is smooth at 0. Equivalently, the first order deformations of X extend to any higher order.

- X compact Kähler $\Rightarrow H^i(X,\mathbb{C}) \cong \oplus_{p+q=i} H^{p,q}(X)$, where $H^{p,q}(X) \subset H^i(X,\mathbb{C})$ is the set of cohomology classes representable by a closed form of type (p,q), and $H^{p,q}(X) \cong H^q(X,\Omega_X^p)$.
- $\Rightarrow b_k(X) = \sum_{p+q=k} h^{p,q}(X) \Rightarrow$ the Frölicher hypercohomology spectral sequence of $H^i(X,\mathbb{C}) \cong \mathbb{H}^i(X,\Omega_X^{\cdot})$ degenerates at E_1.

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• $\Rightarrow$ $b_k(X) = \sum_{p+q=k} h^{p,q}(X)$ $\Rightarrow$ the Frölicher hypercohomology spectral sequence of $H^i(X,\mathbb{C}) \cong \mathbb{H}^i(X,\Omega^{\cdot}_X)$ degenerates at E_1.

• $\Rightarrow$ Hodge numbers $h^{p,q}(X_t)$ remain constant under a small **or infinitesimal** deformation and the Frölicher spectral sequence of a small deformation also degenerates at E_1.

• **Proof.** $b_k(X_t) \leq \sum_{p+q=k} h^{p,q}(X_t)$ and equality is equivalent to degeneracy at E_1. Under a small deformation, X_t remains homeomorphic to X so $b_k(X_t) = b_k(X)$. But also (upper-semicontinuity) $h^{p,q}(X_t) \leq h^{p,q}(X)$.

• The schematic version of this argument, due to Deligne, gives an algebraic proof of BTT in the form:

Thm. *Let X be complex compact, with trivial canonical bundle and Frölicher spectral sequence degenerating at E_1. Then the deformations of X are unobstructed.*

• When (the complex structure of) X deforms, say along a 1-parameter family $(X_t)_{t\in\Delta}$, $H^{p,q}(X_t) \subset H^i(X_t,\mathbb{C}) = H^i(X,\mathbb{C})$ varies in a C^∞ way. It does not vary holomorphically but

$$F^pH^i(X_t) := \oplus_{r\geq p} H^{r,i-r}(X_t) \subset H^i(X_t,\mathbb{C}) = H^i(X,\mathbb{C})$$

does (Griffiths).

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It follows that the first order variation of $F^pH^i(X_t)\subset H^i(X,\mathbb{C})$ is described by a linear map

$$\begin{array}{cccc} \phi_p: & H^{p,i-p}(X) & \to & H^{p-1,i-p+1}(X) \\ & \| & & \| \\ & H^{i-p}(X,\Omega_X^p) & \to & H^{i-p+1}(X,\Omega_X^{p-1}) \end{array}$$

Thm. (Griffiths) ϕ_p *is given by interior product/cup-product with* $u\in H^1(X,T_X)$, *where* u *is the Kodaira-Spencer class of the first order deformation* $(X_t)_{t\in\Delta}$.

Corollary. *Let X be a hyper-Kähler manifold. Then the local period map: $\mathcal{P} : B_{univ} \to \mathbb{P}(H^2(X, \mathbb{C}))$, which to t associates the line $H^{2,0}(X_t) = \mathbb{C}\sigma_{X_t} \subset H^2(X, \mathbb{C})$, is an immersion whose image is a smooth (germ of) hypersurface.*

Proof. B_{univ} is smooth with tangent space $H^1(X, T_X)$. By Griffiths, $d\mathcal{P}_0$ is the composite $H^1(X, T_X) \to H^1(X, \Omega_X) \hookrightarrow H^2(X, \mathbb{C})/H^{2,0}(X)$, where the first map is the isomorphism given by $\sigma_X \lrcorner$, the second map is the inclusion of a hyperplane. **qed**.

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Corollary. (Beauville-Bogomolov-Fujiki) *Let X be a hyper-Kähler n-fold with $n = 2m$. There exists a quadratic form q on $H^2(X, \mathbb{Q})$ and a coefficient $\lambda \in \mathbb{Q}$ such that $\forall \alpha \in H^2(X, \mathbb{Q})$, $(*)$ $\int_X \alpha^n = \lambda q(\alpha)^m$.*

Proof. The left hand side of (*) defines a nonzero degree n homogeneous function on $H^2(X, \mathbb{C})$. As $\sigma_t^{m+1} = 0$ in $H^{2m+2}(X, \mathbb{C})$ (for type reasons) for $\sigma_t \in H^{2,0}(X_t)$, this function vanishes to order $\geq m$ along the (germ of) hypersurface $\operatorname{Im} \mathcal{P}$. So either $\operatorname{Im} \mathcal{P}$ is open, hence Zariski dense, in a quadric Q defined by a quadratic form q satisfying (*), or $\operatorname{Im} \mathcal{P}$ is an open set of a hyperplane, which one easily excludes. **qed**

- The form q can be normalized so that q is integral, $\lambda > 0$.

Thm. *The signature of q is $(3, b_2 - 3)$.*

Proof. Let $h \in H^2(X, \mathbb{Q})$ be a Kähler class. Then by differentiating (*)
$\int_X nh^{n-1}\alpha = 2mq(h)^{m-1}q(\alpha, h)$, so α is h-primitive iff $q(\alpha, h) = 0$.

Differentiating again,
(**) $\int_X n(n-1)h^{n-2}\alpha\beta = 2mq(h)^{m-1}q(\alpha, \beta)$
if α, β are primitive. Conclude by Hodge index theorem. **qed**

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- **More properties of q.** Let $Q = \{q = 0\} \subset \mathbb{P}(H^2(X, \mathbb{C}))$. By construction an open set of Q is made of $\sigma_{X_t} =: \mathcal{P}(t)$ for some small deformations X_t of X.

Thm. *One has $q(\sigma_{X_t}, \overline{\sigma_{X_t}}) > 0$ and $q(\sigma_{X_t}, \alpha) = 0$ iff $\alpha \in F^1H^2(X_t)$.*

Proof. Let h be a Kähler class on X_t. Since σ_{X_t} and $\overline{\sigma_{X_t}}$ are primitive, we can apply (**).
The first statement thus follows from Hodge index thm.
For the second statement: $F^1H^2(X_t)$ is by Griffiths the image of $d\mathcal{P}$ but it is also the tangent space to Q at σ_{X_t}. **qed**

Remark. The deformations along twistor lines provide conics in Q. The projective plane of the conic has to be real positive.

- **Thm.** (Huybrechts) *Any $\sigma \in Q$ with $q(\sigma, \overline{\sigma}) > 0$ is $\mathcal{P}(X_t)$ for some hyper-Kähler deformation X_t of X.*

Proof. Use the (iterated) twistor lines to get **large** deformations and all period points. **qed**

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- **Cohomology ring**. $\mu : \operatorname{Sym}^* H^2(X, \mathbb{C}) \to H^{2*}(X, \mathbb{C})$.

Thm. (Verbitsky) *The kernel of μ is generated by the relations (*) $\alpha^{m+1} = 0$ when $q(\alpha) = 0$. In particular μ is injective in degree $2* \leq 2m$.*

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Proof. **The relations (*) are satisfied**, because a Zariski dense subset of $Q \subset H^2(X, \mathbb{C})$ consists of forms σ_{X_t} of type $(2,0)$ on a deformation of X.
These are all the relations: $I_m :=$ ideal generated by (*), $T^* := \mathrm{Sym}^* H^2(X, \mathbb{C})/I_m$:

Fact. *$T^k = 0$ for $k > n$, $\dim T^n = 1$ and the pairing $T^k \otimes T^{n-k} \to T^n$ is perfect.*

If $k \leq n$ and $\exists 0 \neq \beta \in \mathrm{Ker}\,(\mu : T^k \to H^{2k}(X, \mathbb{C}))$, there is an $\alpha \in T^{n-k}$ such that $\alpha\beta \neq 0$ in T^n and then $\mu(T^n) = 0$. Absurd because $\mu(h^n) \neq 0$. **qed**

Thm. (S. Salamon) *Let X be a HK manifold of dimension $n = 2m$. Then* $mb_{2m}(X) = 2\sum_{j=1}^{2m}(-1)^j(3j^2 - m)b_{2m-j}(X)$.

Sketch of proof. Riemann-Roch applied to the vector bundles Ω_X^i gives $\int_X c_1(X)c_{n-1}(X) = \sum_{p=0}^{n}(-1)^p(6p^2 - \frac{1}{2}n(3n+1))\chi_p$, where $\chi_p = \chi(X, \Omega_X^p) = \sum_j(-1)^j h^{p,j}(X)$. So if K_X trivial, $\sum_{p=0}^{n}(-1)^p(6p^2 - \frac{1}{2}n(3n+1))\chi_p = 0$.

When X is Kähler, uses the Hodge symmetry $h^{p,q}(X) = h^{q,p}(X)$. When X is hyper-Kähler, use the extra symmetry $h^{n-p,q}(X) = h^{p,q}(X)$ given by isomorphism $\sigma_X^{m-p} : \Omega_X^p \cong \Omega_X^{n-p}$. Regroup... **qed**

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Thm. (Guan) *Let X be HK of dim 4. Then $b_2(X) = 23$ or $b_2(X) \leq 8$.*

Proof (of $b_2 \leq 23$, also due to Beauville). **Salamon** gives the equality $2b_4(X) = -2b_3(X) + 20b_2(X) + 92$, hence $b_4(X) \leq 10b_2(X) + 46$.

Verbitsky gives $b_4(X) \geq \frac{b_2(X)(b_2(X)+1)}{2}$.

Hence $\frac{b_2(X)(b_2(X)+1)}{2} \leq 10b_2(X) + 46$ and $b_2(X) \leq 23$ (equality only if $b_3(X) = 0$). **qed**

- (Conjectural) results by Kurnuzov, Sawon, Laza et col. Bound on b_2 ?

The quadratic form q appears in

Thm (Fujiki) *Let X be HK $2m$-fold. Then there exists a degree m polynomial P with rational coefficients, such that for any holomorphic line bundle L on X, $\chi(X, L) = P(q(c_1(L)))$.*

Proof. Apply Riemann-Roch. $\chi(X, L) = \sum_{i=0}^{n} \int_X Q_i(c_l(X))c_1(L)^{n-i}$, for some polynomials Q_i in the Chern classes of X.
Then the result follows from $c_l(X) = 0$ for l odd and **Thm**' applied to $\alpha = c_1(L)$.

Thm'. *For any $j \leq m$, any polynomial Q of weighted degree $2j$ in the Chern classes $c_l(X)$, there exists a rational number λ_j such that $\int_X Q(c_l(X))\alpha^{n-2j} = \lambda_j q(\alpha)^{m-j}$ for any $\alpha \in H^2(X, \mathbb{Q})$.*

• The last statement is proved as the absolute Fujiki relations, using the fact that the class $Q(c_l(X)) = Q(c_l(X_t))$ is of type $(2j, 2j)$ on any deformation X_t of X, hence

$$\sigma_{X_t}^{m-j+1} Q(c_l(X_t)) = 0 \text{ in } H^{2n}(X_t, \mathbb{C}).$$ **qed**

Thm. *Let X be a hyper-Kähler manifold with universal deformation $\mathcal{X} \to B_{univ}$. Then the set of points $t \in B_{univ}$ such that $\mathcal{X}_t$ is projective is dense in B_{univ}.*

Proof. $H^{1,1}(X)$ is the orthogonal complement of $\langle \sigma_X, \overline{\sigma_X} \rangle$ wrt q. For $\lambda \in H^2(X, \mathbb{Q})$, $\lambda \in H^{1,1}(X)$ iff $q(\lambda, \sigma_X) = 0$. Let ω be a Kähler class on X and let $\lambda_n \in H^2(X, \mathbb{Q})$ with $\lim_{n\to\infty} \lambda_n = \lambda$. Then, as $q(\omega, \sigma_X) = 0$ and $\mathcal{P}$ is submersive onto an open set of Q, $B_{\lambda_n} = \{t \in B_{univ},\ q(\lambda_n, \sigma_{X_t}) = 0\}$ has points t_n tending to 0 with n. For n large enough, λ_n is a Kähler class on X_{t_n} by openness of the Kähler condition. Then X_{t_n} is projective by Kodaira. **qed**

Thm. (Huybrechts) *Let X be a hyper-Kähler manifold. Then X is projective if and only X has a holomorphic line bundle L with $q(c_1(L)) > 0$.*

The proof uses Demailly-Paun theorem describing the Kähler cone of a compact Kähler manifold.