Global aspects of Calabi-Yau moduli space

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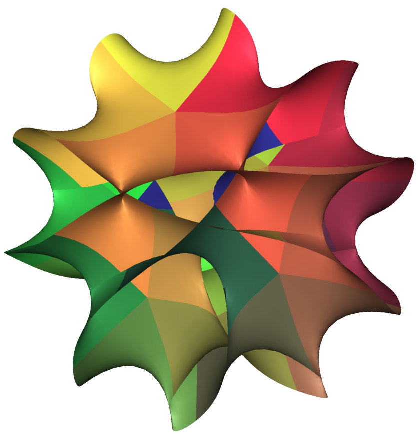


$SU(n)$ holonomy

For (X, g) a compact complex manifold of dimension n equipped with a Kähler metric, the holonomy of the metric is contained in $U(n)$.

Fact For (X, g) compact Kähler, the following are equivalent.

(i) The holonomy of (X, g) is contained in $SU(n)$.

(ii) There exists a nowhere vanishing holomorphic n-form on X.

(iii) The canonical bundle $K_X = \bigwedge^{\dim X} \Omega_X$ is trivial.

One special case is when the holonomy of (X, g) is **exactly** $SU(n)$. This is the case we will look at.

Fact For (X, g) compact Kähler with holonomy $SU(n)$, we have

(i) $H^0(X, \Omega_X^p) = 0$ for all $0 < p < n$;

(ii) $\pi_1(X)$ is finite.

$SU(n)$ holonomy

Yau's theorem (Calabi conjecture) Given X a compact complex manifold with K_X trivial, and a Kähler class $\alpha \in H^2(X, \mathbb{R})$ (cohomology class of a Kähler form corresponding to a Kähler metric on X), there is a unique Kähler form of class α on X corresponding to a **Ricci flat** Kähler metric.

The first interesting case is $n = 2$: the case of K3 surfaces. This case is of different flavour; also covered in other talks at this meeting.

For the rest of the talk, X is a compact complex manifold, admitting a (Ricci flat) metric with holonomy $SU(n)$, of dimension $n > 2$: **Calabi–Yau n-fold**.

Proposition A Calabi–Yau n-fold is automatically projective for $n > 2$.

Proof We have $H^2(X, \mathbb{C}) \cong H^{1,1}(X)$. So near a Kähler form $\alpha \in H^2(X, \mathbb{R})$, there is a **rational** Kähler form $\alpha' \in H^2(X, \mathbb{Q})$. An integral multiple of such a form must come from a projective embedding $X \subset \mathbb{P}^N$ by Kodaira's Embedding Theorem.

Deformation theory

A **deformation** of X is a proper holomorphic submersion $f\colon \mathcal{X} \to B$ between complex manifolds such that for $0 \in B$, $X_0 := f^{-1}(0) \cong X$.

Bogomolov–Tian–Todorov theorem A Calabi–Yau n-fold $X = X_0$ has **unobstructed deformations**: there exists a universal deformation (germ) $\tilde{f}\colon \tilde{\mathcal{X}} \to \tilde{B}$, with $0 \in \tilde{B} \subset H^1(X, T_X)$ a (germ of a) polydisk.

Existence of universal deformation follows from

$$H^0(X, T_X) \cong H^0(X, \Omega_X^{\dim X - 1}) = 0.$$

Now Kodaira–Spencer theory gives a universal deformation

$$\tilde{f}\colon \tilde{\mathcal{X}} \to \tilde{B} \subset H^1(X, T_X).$$

In general, we would expect this to be **obstructed**, as $H^2(X, T_X) \neq 0$. But here, all obstructions vanish. Original proofs complex analytic; there are several algebraic proofs known.

[Goto 2004] proves this uniformly for compact manifolds with special holonomy.

Topology and Hodge decomposition

Let $H^*(X, \mathbb{Z})$ denote integral cohomology modulo torsion. Torsion phenomena not without interest.

Hodge decomposition:

$$H^k(X, \mathbb{Z}) \otimes \mathbb{C} \cong \bigoplus_{p=0}^{k} H^{p,k-p}(X) \quad \text{with} \quad H^{p,q} \cong H^q(X, \Omega_X^p).$$

We will assume X connected, so $H^0(X, \mathbb{Z}) \cong \mathbb{Z}$. We also have $H^1(X, \mathbb{Z}) = 0$, and trivial Hodge decomposition in degree 2:

$$H^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^{1,1}(X) \cong \mathrm{Pic}(X) \otimes \mathbb{C}.$$

Most interesting Hodge decomposition on **middle cohomology**

$$H^n(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^0(X, K_X) \oplus H^1(X, \Omega_X^{n-1}) \oplus \ldots \oplus H^n(X, \mathcal{O}_X).$$

This is **polarized** by the intersection form $Q : H^n(X, \mathbb{Z}) \times H^n(X, \mathbb{Z}) \to \mathbb{Z}$.

Hodge decomposition in families

Assume $f\colon \mathcal{X} \to B \ni 0$ is a deformation of $X = X_0$ over a contractible base B and fibres X_t for $t \in B$. Get identification $H^k(X_t, \mathbb{Z}) \cong H^k(X, \mathbb{Z})$.

Thus the Betti numbers $b_k(X_t)$ and also the Hodge numbers $h^{p,q}(X_t)$ are constant. But the vector subspaces

$$H^{p,k-p}(X_t) \subset H^k(X_t, \mathbb{C}) \cong H^k(X, \mathbb{C})$$

vary, and in fact vary **non-holomorphically**.

Griffiths The **Hodge filtration** $F^\bullet$ on $H^k(X, \mathbb{C})$ given by

$$F^p H^k(X_t, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,k-r}(X_t)$$

varies **holomorphically** with $t \in B$.

Griffiths transversality This variation satisfies

$$\frac{d}{dt}\left(F^p H^k(X_t, \mathbb{C})\right)\Big|_{t=0} \subseteq F^{p-1} H^k(X_0, \mathbb{C}).$$

The period domain

Consider middle cohomology $(H^n(X,\mathbb{Z}), Q)$. Define the **period domain** $\mathcal{D}$ to be the space parametrising all flags $F^\bullet$ in $H^n(X,\mathbb{C})$ satisfying the following conditions:

(1) $\dim F^k = \dim F^k(X)$;

(2) $Q(F^k, F^{n-k+1}) = 0$;

(3) certain positivity properties with respect to the product Q.

The period domain $\mathcal{D}$ is an analytic open subset of its **compact dual** $\overline{\mathcal{D}}$, the projective variety of flags in $H^n(X,\mathbb{C})$ given by conditions (1)-(2).

Local period map and infinitesimal Torelli

Consider a deformation $f\colon \mathcal{X} \to B \ni 0$ of $X = X_0$ over a contractible base. Fix isomorphisms $(H^n(X_t,\mathbb{Z}), Q_t) \cong (H^n(X,\mathbb{Z}), Q)$.

The variation of the Hodge filtration in middle cohomology gives rise to the **local period map**

$$\begin{array}{rcccl} \varphi_B & : & B & \to & \mathcal{D} \\ & & t & \mapsto & (F^\bullet H^n(X_t,\mathbb{C})) \end{array}$$

Infinitesimal Torelli theorem For the universal deformation $\tilde{f}\colon \tilde{\mathcal{X}} \to \tilde{B}$, the local period map $\varphi_{\tilde{B}}$ is a complex analytic embedding.

There is a simple proof in Claire's online lecture.
[Goto 2004]: uniform proof for compact manifolds with special holonomy.

Theorem [Bryant–Griffiths 1983] The image of any period map φ_B is contained in a so-called **horizontal** submanifold of $\mathcal{D}$, an integral manifold for a certain differential system corresponding to Griffiths transversality.

Teichmüller space

To define a version of the period map globally, we need to construct a global moduli space of Calabi–Yau n-folds. Fix a lattice (Λ, Q_Λ) with the correct symmetry and signature properties. A **marked Calabi–Yau n-fold** is a pair (X, γ) with X a Calabi–Yau n-fold, and an isomorphism

$$\gamma\colon (H^n(X, \mathbb{Z}), Q) \cong (\Lambda, Q_\Lambda).$$

Consider triples (X, L, γ), where (X, γ) is a marked Calabi–Yau n-fold and L (the Kähler class of) an ample line bundle on X. The space of all such triples up to isomorphism has a natural topology.

Fact Every connected component $\mathcal{T}$ of the space of triples (X, L, γ) has the structure of a (connected, Hausdorff) complex manifold.

$\mathcal{T}$ is a **Teichmüller space** of marked, polarized Calabi–Yau n-folds.

For any $t \in \mathcal{T}$ corresponding to a triple (X_t, L_t, γ_t), the local germ of $t \in \mathcal{T}$ can be identified with the universal deformation germ of X_t. In particular, there is a global family of Calabi–Yau manifolds $f_{\mathcal{T}}\colon \mathcal{X}_{\mathcal{T}} \to \mathcal{T}$.

The global period map on Teichmüller space

Fix a base point $0 \in \mathcal{T}$ corresponding to a triple (X, L, γ). Using the markings, we have consistent identifications

$$(H^n(X_t, \mathbb{Z}), Q_t) \cong (\Lambda, Q_\Lambda) \cong (H^n(X, \mathbb{Z}), Q)$$

for $t \in \mathcal{T}$ and fibres X_t of the family $f_{\mathcal{T}} \colon \mathcal{X}_{\mathcal{T}} \to \mathcal{T}$.

We can thus define the period map **globally** by

$$\varphi \colon \mathcal{T} \to \mathcal{D}$$

to the corresponding period domain, using the Hodge filtration on middle cohomology.

For Calabi–Yau n-folds, this holomorphic map is **locally injective** by the infinitesimal Torelli theorem. Its image lies in a horizontal submanifold of $\mathcal{D}$.

Metric aspects of Teichmüller space

Teichmüller space $\mathcal{T}$ carries a natural metric, the Weil–Petersson metric. For the Teichmüller space of elliptic curves, K3 surfaces and abelian varieties, this is a complete metric which tends to be negatively curved.

- W–P metric on $\mathcal{T}$ is incomplete; there are finite distance singularities [Candelas–Green–Hubsch 1990].
- A certain 1-dimensional $\mathcal{T}$ has W–P curvature tending to $+\infty$ near a point [Candelas, de la Ossa et al 1991].
- Whenever $\mathcal{T}$ is 1-dimensional, the W–P metric is asymptotic to the Poincaré metric near certain boundary points [Wang 2003].
- Ooguri–Vafa conjectured that the scalar curvature of the W–P metric should be non-positive near the boundary; disproved by [Trenner-Wilson 2011].

Hodge metric on $\mathcal{T}$: natural metric pulled back from $\mathcal{D}$ via φ. This has better curvature properties [Lu–Sun 2004].

Moduli space

We can also consider polarized Calabi–Yau manifolds (X, L) up to isomorphism, without marking. This also has a natural topology. Let $\mathcal{M}$ be a connected component, a **moduli space of polarized Calabi–Yau n-folds**.

Theorem (Viehweg) The space $\mathcal{M}$ carries the structure of a quasiprojective algebraic variety (orbifold).

For $t \in \mathcal{M}$, representing an isomorphism class $[(X_t, L_t)]$ of polarized Calabi–Yau n-folds, the topological invariants $\{b_i(X_t) \mid 2 \leq i \leq n\}$ are independent of t.

$\mathcal{M}$ is what is known as a **coarse** moduli space: there is no family over it. There exists a family over a finite cover of $\mathcal{M}$.

Key questions about the moduli space

Key questions:

(i) In a fixed dimension n, how many (substantially) different moduli spaces $\mathcal{M}_k$ exist? Is this number finite?

(ii) Fixing topological invariants $\{b_i \mid 2 \leq i \leq n\}$, how many different moduli spaces $\mathcal{M}_k$ exist? Is this number finite?

(iii) "Reid's fantasy" [Reid 1983]: are all Calabi-Yau-n moduli spaces connected by geometric transitions?

Known:

(i) Already for $n = 3$, at least tens of thousands of different $\mathcal{M}_i$

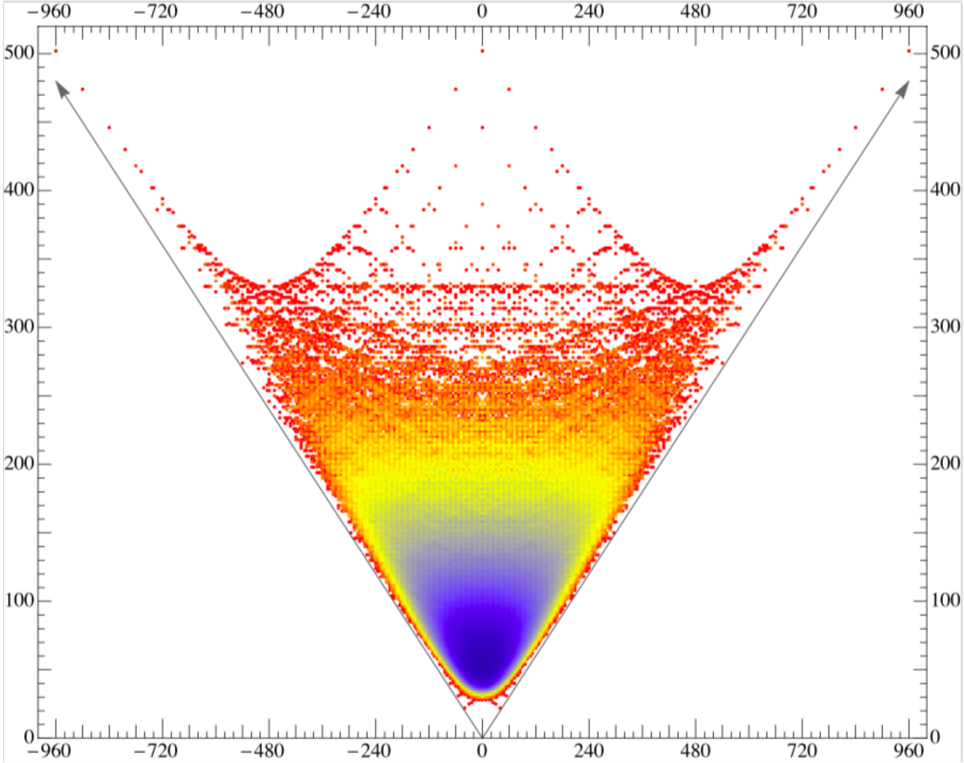
(ii) Examples known of different families with the same data $\{b_i \mid 2 \leq i \leq n\}$

(iii) Bounds for b_i known if we assume extra structure (e.g. elliptic fibrations)

(iv) Many moduli spaces can be connected by geometric transitions [Gross 1997].

Geography of Calabi–Yau threefolds

For Calabi–Yau 3-folds, have (b_2, b_3). Plot of Kreuzer–Braun–Candelas of different possible topologies, in coordinates $(\chi(X), h^{1,1}(X) + h^{2,1}(X))$:



The global period map on moduli space

To define the period map on the moduli space, we need to “forget the marking”. Let

$$\Gamma = \mathrm{Aut}(\Lambda, Q_\Lambda).$$

This is an arithmetic group that is known to act properly and discontinuously on $\mathcal{D}$.

We get a commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\varphi} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\bar{\varphi}} & \mathcal{D}/\Gamma. \end{array}$$

This defines the period map $\bar{\varphi}$ on moduli space.

The global Torelli problem

The **global Torelli problem** is the question whether the period map

$$\bar{\varphi}\colon \mathcal{M} \to \mathcal{D}/\Gamma$$

is **injective** for a connected moduli space $\mathcal{M}$ of polarized Calabi–Yau n-folds.

This is equivalent to the following concrete formulation: given polarised Calabi–Yau n-folds (X_1, L_1), (X_2, L_2) which live in the same deformation family, does the existence of an isomorphism

$$(H^n(X_1, \mathbb{Z}), Q_{X_1}) \cong (H^n(X_2, \mathbb{Z}), Q_{X_2})$$

respecting the Hodge filtrations imply that X_1, X_2 are **isomorphic**?

Variants of the global Torelli problem:

- We could ask for **generic** injectivity of $\bar{\varphi}$: **weak global Torelli**.
- We could change the group Γ to a different group Γ' acting discontinuously on $\mathcal{D}$, perhaps arising from monodromy considerations.

Example: quintic threefolds

Consider the family of smooth quintic threefolds

$$X = \{f_5(x_i) = 0\} \subset \mathbb{P}^4.$$

These have K_X trivial by the Adjunction Formula; $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ by Lefschetz; also $H^3(X, \mathbb{Z}) \cong \mathbb{Z}^{204}$ with Hodge numbers $(1, 101, 101, 1)$.

Connected moduli space

$$\mathcal{M}_X \cong U/\mathrm{PGL}(5, \mathbb{C})$$

of dimension $101 = 125 - 24$, where

$$U \subset \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)))$$

is the open set of non-singular quintic polynomials.

The period domain $\mathcal{D}$ parametrises flags $\mathbb{C} \subset \mathbb{C}^{102} \subset \mathbb{C}^{203} \subset \mathbb{C}^{204} \cong H^3(X, \mathbb{C})$. A lot of information in the period point!

Weak global Torelli for quintic threefolds

Theorem (Voisin) Quintic threefolds satisfy weak global Torelli: a general quintic threefold can be recovered from its period point.

Discussion of proof [Donagi 1983]: for projective hypersurfaces, the knowledge of the Hodge structure is equivalent to the knowledge of a certain algebraic structure on the Jacobian ring of the hypersurface. Using a trick called the **Symmetrizer Lemma**, this information is often enough to recover the equation of the hypersurface. Not in Calabi–Yau cases!

[Voisin 1999]: a (large!) bag of special tricks for the quintic threefold, invented while being prevented from doing more exciting mathematics by other concerns.

[Voisin 2020]: a systematic algebraic treatment of weak global Torelli for classes of hypersurfaces where the Symmetrizer Lemma does not apply.

Failure of weak global Torelli for Calabi–Yau threefolds

Next, let Y be a resolution of singularities of a hypersurface

$$\bar{Y} = \{f_8(x_i, y_j) = 0\} \subset \mathbb{P}^4[1, 1, 2, 2, 2].$$

This family of Calabi–Yau threefolds has $b_2(Y) = 2$ and middle cohomology of dimension $b_3(Y) = 174$. Thus the moduli space $\mathcal{M}_Y$ has dimension 86.

Theorem [Szendrői 2000] The period map

$$\bar{\varphi}\colon \mathcal{M}_Y \to \mathcal{D}/\Gamma$$

is of degree at least 2.

These are weak counterexamples: 3-folds in the family with the same period point are **birational** but not **isomorphic**.

Examples of small moduli spaces

At the other extreme are Calabi–Yau threefolds with small moduli spaces. $\dim \mathcal{M} = 0$, the case of rigid Calabi–Yau threefolds, is not without interest. But here focus on some cases when $\dim \mathcal{M} = 1$.

The best-known example is the **mirror quintic** $X^\vee$, the resolution of a finite quotient of the Fermat quintic threefold:

$$X^\vee \to \bar{X}^\vee = \left\{ \sum_{i=0}^{4} x_i^5 = 0 \right\} \Big/ G \subset \mathbb{P}^4/G$$

with $G \cong (\mathbb{Z}/5\mathbb{Z})^3 \subset \mathrm{PGL}(4, \mathbb{C})$ a certain finite subgroup. This has a one-dimensional moduli space

$$\mathcal{M}_{X^\vee} \cong \mathbb{P}^1 \setminus \{1, \infty\}, \text{ with } 0 \in \mathcal{M}_{X^\vee} \text{ an orbifold point of order } 5.$$

Theorem [Usui 2008] The quintic mirror family satisfies weak global Torelli: its period map $\bar{\varphi}\colon \mathcal{M}_{X^\vee} \to \mathcal{D}/\Gamma$ is generically injective.

Examples of small moduli spaces and failure of global Torelli

There is a related example of a family of Calabi–Yau threefolds

$$Z \to \bar{Z} = \left\{ \sum_{i=0}^{4} x_i^5 = 0 \right\} \Big/ H \subset \mathbb{P}^4/H$$

with $H \cong \mathbb{Z}/5\mathbb{Z} \ltimes (\mathbb{Z}/5\mathbb{Z})^2 \subset \mathrm{PGL}(4, \mathbb{C})$ another specific finite subgroup, studied by [Aspinwall–Morrison 1994].

These Calabi–Yau threefolds have fundamental group $\pi_1(Z) \cong \mathbb{Z}/5\mathbb{Z}$, and a one-dimensional moduli space $\mathcal{M}_Z$.

Theorem [Szendrői 2004] The period map $\bar{\varphi}\colon \mathcal{M}_Z \to \mathcal{D}/\Gamma'$ is of degree at least 5, for a certain group Γ' acting properly discontinuously on the relevant period domain $\mathcal{D}$.

I conjecture that the Calabi–Yau threefolds in this family with the same period point are not biratonal (and have non-equivalent derived categories of coherent sheaves). There may be interesting phenomena lurking here.

Global considerations

Let us return to the diagram

$$\begin{array}{ccccc} \mathcal{T} & \stackrel{\varphi}{\longrightarrow} & \mathcal{D} & \hookrightarrow & \overline{\mathcal{D}} \\ \downarrow & & \downarrow & & \\ \mathcal{M} & \stackrel{\bar{\varphi}}{\longrightarrow} & \mathcal{D}/\Gamma & & \end{array}$$

involving Teichmüller and moduli spaces of some Calabi–Yau n-fold. We know

(i) $\overline{\mathcal{D}}$ is a projective variety containing as an analytic open subset the period domain $\mathcal{D}$;

(ii) $\mathcal{M}$ is a quasiprojective orbifold;

(iii) φ is locally injective;

(iv) the image of φ is constrained to lie in a horizontal submanifold of $\mathcal{D}$, by Griffiths transversality.

It appears that in most cases, the images of the maps $(\varphi, \bar{\varphi})$ are too transcendental to describe in terms that would help us understand the moduli space.

Semi-algebraic subvarieties of the period domain

Definition A closed horizontal submanifold $Z \subset \mathcal{D}$ is called **semi-algebraic**, if Z is a connected component of an intersection $\overline{Z} \cap \mathcal{D}$ for a Zariski closed subvariety $\overline{Z} \subset \overline{\mathcal{D}}$.

Theorem [Friedmann–Laza 2013] Suppose $Z \subset \mathcal{D}$ is a closed horizontal subvariety, with stabilizer group $\Gamma_Z \subset \Gamma$. Assume that

(i) $Z \subset \mathcal{D}$ is semi-algebraic, and

(ii) Z/Γ_Z is quasi-projective.

Then Z itself is a Hermitian symmetric domain, whose embedding into $\mathcal{D}$ is an equivariant, holomorphic, horizontal embedding.

$$\begin{array}{ccccc} Z & \hookrightarrow & \mathcal{D} & \hookrightarrow & \overline{\mathcal{D}} \\ \downarrow & & \downarrow & & \\ Z/\Gamma_Z & \hookrightarrow & \mathcal{D}/\Gamma & & \end{array}$$

Semi-algebraic subvarieties of the period domain

We have a diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & \mathcal{D} & \hookrightarrow & \overline{\mathcal{D}} \\ \downarrow & & \downarrow & & \\ Z/\Gamma_Z & \hookrightarrow & \mathcal{D}/\Gamma & & \end{array}$$

Message: the image of the period map could sometimes be described explicitly in geometric terms, and interesting conclusions drawn.

Caveat: most Calabi–Yau n-folds will not have period maps with semi-algebraic image.

For Calabi–Yau threefolds, semi-algebraicity of the image of the period map is equivalent to the fact that the period map involves no "quantum corrections" [Liu–Yin 2014].

Semi-algebraic subvarieties of the period domain: an example

Let $D = \cup_{i=1}^{6} H_i \subset \mathbb{P}^3$ be a union of general hyperplanes, $\bar{X} \to \mathbb{P}^3$ the triple cover of $\mathbb{P}^3$ branched along D, and $X \to \bar{X}$ a small resolution.

Get a Calabi–Yau 3-fold X with $b_3(X) = 8$, and 3-dimensional moduli space $\mathcal{M}_X$.

Let (Λ, h) be a $\mathbb{Z}$-lattice of signature $(3, 1)$. Let

$$\mathbb{B}_3 = \{v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid h(v, v) < 0\},$$

a complex unit 3-ball. Let $G = \mathrm{Aut}(\Lambda, h)$.

Theorem [Sheng–Xu 2019] The period map for the family $\mathcal{M}_X$ factors

$$\mathcal{M}_X \hookrightarrow \mathbb{B}_3/G \hookrightarrow \mathcal{D}/\Gamma$$

where the first map is an open embedding, and the second a semi-algebraic closed embedding.

In particular, for this family, global Torelli holds.

Note that $H^3(X, \mathbb{C})$ is built from one-dimensional Hodge structures.

Conclusions

Global moduli theory of Calabi–Yau n-folds

(i) Local theory very pleasant, and well described by periods

(ii) There is a reasonable global algebraic theory of the moduli space

- Mirror symmetry predictions from precise form of periods plus monodromy data

(iii) Image of period map is a horizontal submanifold of the period domain; usually a transcendental condition

- Similar "mirror" problem: "stringy Kähler moduli space" inside space of Bridgeland stability conditions

(iv) Period map can sometimes be of use to describe the global moduli space, but these are "very algebraic" examples

(v) Finitely many or infinitely many families???

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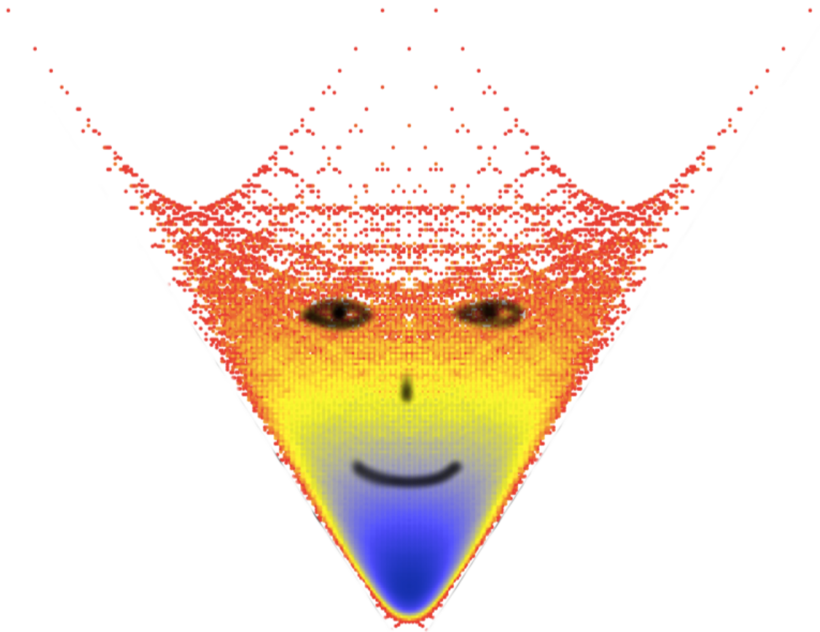

Thank you for your attention!