Constructing shifted symplectic structures part 2

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Integrability, Geometry and Moduli
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Outline

- joint with Dima Arinkin and Bertrand Toën
- the pushforward theorem;
- symplectic geometry of the moduli of tame or wild local systems on a smooth variety over $\mathbb{C}$:
  - shifted Poisson structures;
  - description of symplectic leaves.

Relative symplectic structures (i)

Recall from last lecture that given

$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of stacks;

$\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F}, n)$ - a relative closed n-shifted 2-form on $\mathscr{F}$;

we get, for every stratum X_α in X natural data:

- a derived stack $\mathscr{F}_\alpha$ - the fiber of $\mathscr{F}$ at X_α;
- a derived stack $\overset{\circ}{\mathscr{F}}_\alpha$ - the nearby cycles of $\mathscr{F}$ for X_α;
- a cospecialization morphism of derived stacks $\mathrm{cosp}^{\mathscr{F}}_\alpha : \mathscr{F}_\alpha \to \overset{\circ}{\mathscr{F}}_\alpha$;
- a closed $(n+1+\dim X_\alpha)$-shifted 2-form $\overline{\omega}_\alpha$ on $\overset{\circ}{\mathscr{F}}_\alpha$;
- a homotopy h_α between $\mathrm{cosp}^{\mathscr{F}}_\alpha{}^* \overline{\omega}_\alpha$ and 0 in the complex $\mathbb{A}^{2,cl}_X(\mathscr{F}_\alpha, n+1+\dim X_\alpha)$.

Relative symplectic structures (ii)

It turns out that the non-degenracy of the relative form ω is equivalent to a set of conditions on these data.

Theorem: **[Arinkin-P-Toën]** Suppose

$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of derived Artin stacks, locally of f.p.;

$\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F}, n)$ - a relative closed n-shifted 2-form on $\mathscr{F}$.

Then ω is symplectic if and only if for every $\alpha \in I$ the following two conditions hold:

- **(a)** $\overline{\omega}_\alpha$ is symplectic.
- **(b)** $\mathrm{cosp}^{\mathscr{F}}_\alpha : (\mathscr{F}_\alpha, h_\alpha) \to (\overset{\circ}{\mathscr{F}}_\alpha, \overline{\omega}_\alpha)$ is Lagrangian.

Push-forwards (i)

Pushing forward also makes sense for relative symplectic forms. Suppose

- $f : X \to Y$ is a stratified map of nicely stratified spaces;
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$;
- $E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$;

We have push-forwards

$$f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{dSt}_{\mathbb{C}}) \quad \text{and} \quad f_*E \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{Vect}_{\mathbb{C}})$$

computed via the right Kan extensions along the functor between exit path categories induced from f.
The pushforward of $\omega \in \mathbf{A}^{p,cl}_X(\mathscr{F})_E$ then is a closed relative form:

$$\boxed{f_*\omega \in \mathbf{A}^{p,cl}_Y(f_*\mathscr{F})_{f_*E}.}$$

Push-forwards (ii)

Theorem: **[Arinkin-P-Toën]** Suppose that $f : X \to Y$ is proper and that $\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X[n]}$ is symplectic. Then the pushforward

$$\mathrm{tr}\, f_*\omega \in \mathbf{A}^{2,cl}_Y(f_*\mathscr{F})_{K_Y[n]}$$

is symplectic as well.

Remark:

- Here $\mathrm{tr} : f_*K_X \to K_Y$ denotes the canonical trace map.
- Most of the standard constructions of shifted symplectic structures arise as special cases of the above theorem.

Recall: unramified character varieties

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- X - smooth projective curve/$\mathbb{C}$,
- G - a complex reductive group,
- $M_G(X)$ - the coarse moduli space of representations $\rho : \pi_1(X, x) \to G$.

Classical story:

- The smooth part $M_G^{\mathrm{sm}}(X)$ of $M_G(X)$ admits an algebraic symplectic structure;
- There are explicit descriptions:
  - **cohomological** construction in deformation theory Goldman, Karshon, Weinstein, . . . ;

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  - **quasi-Hamiltonian reduction** construction Alekseev-Malkin-Meinrenken;

Recall: tame character varieties

- X - a smooth quasi-projective curve/$\mathbb{C}$.

Classical story: Fock-Rosly, Goldman, Guruprasad-Rajan, Guruprasad-Huebschmann-Jeffrey-Weinstein, ...

- $M_G^{\mathrm{sm}}(X)$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G^{\mathrm{sm}}(X)$ are the moduli spaces of ρ with fixed monodromy at infinity.
- There are **cohomological** and **quasi-Hamiltonian** descriptions of symplectic leaves.

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non-degeneracy = Lefschetz duality

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Note: Symplectic leaves = **tame character varieties**.

Recall: wild character varieties

- $\mathfrak{X}$ is a smooth projective curve/$\mathbb{C}$, $X = \mathfrak{X} - \{x_1, \ldots, x_k\}$;
- $\mathscr{I} = \{\mathscr{I}_1, \ldots, \mathscr{I}_k\}$, $\mathscr{I}_i$ irregular type at x_i;
- $M_G(X, \mathscr{I})$ - the moduli of representations of $\pi_1(X)$ equipped with Stokes data of type $\mathscr{I}$.

Classical story: Boalch, Boalch-Yamakawa

- $M_G^{\mathrm{sm}}(X, \mathscr{I})$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G^{\mathrm{sm}}(X, \mathscr{I})$ are the moduli spaces of ρ with fixed formal monodromy at infinity.
- There is a **quasi-Hamiltonian** descriptions of symplectic leaves.

Goal

Remark:

- The ramified setting is often better behaved. Generic choices of local monodromies (in the tame case) or of irregular types and formal monodromies (in the wild case) ensure that $M_G^{\mathrm{sm}} = M_G$
- The **cohomological** description and the linear reason for non-degeneracy of the symplectic structure on the wild character varieties is not immediately clear.

Goal

- Construct Poisson structures everywhere on $M_G(X)$ and $M_G(X, \mathscr{I})$ including the singular points.
- Describe their symplectic leaves.
- Find **cohomological** and **quasi-Hamiltonian** descriptions of the symplectic form at the singular points.
- Extend the whole story to higher dimensional smooth varieties X.

Setup

Natural approach: Resolve the singularities of M_G in a minimal way so that the Poisson and symplectic structures extend to the resolution.

Lucky break: $M_G(X)$ and $M_G(X, \mathscr{I})$ admit natural resolutions which are again moduli spaces.

Note: These resolutions/refinements of M_G are not schemes but rather are derived algebraic stacks whicih are locally finitely presentable and in particular have perfect tangent complexes.

Refinements (i)

Unramified+tame cases

The moduli $M_G(X)$ can be refined to the derived stack

$$Loc_G(X) = \mathrm{Map}_{\mathrm{dSt}}(X, BG)$$

parametrizing G-local systems on X.

Key point:

- Any non-degenerate $\kappa \in (\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$ corresponds to a 2-shifted symplectic structure ω_κ on the Artin stack BG
- ω_κ induces 0-shifted symplectic or Poisson structures on $Loc_G(X)$ in the unramified or tame case respectively.

Refinements (ii)

Wild case:

The moduli $M_G(X, \mathscr{I})$ can be refined to the derived stack

$$Loc_G(X, \mathscr{I}) = \Gamma\left(\widehat{\mathfrak{X}}, \mathrm{DMS}_{G,\mathscr{I}}\right)$$

parametrizing Stokes filtered G-local systems of irregular type $\mathscr{I}$.

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Note:

- $\mathrm{DMS}_{G,\mathscr{I}}$ denotes the Deligne-Malgrange-Stokes sheaf of Artin stacks on X classifying Stokes data of type $\mathscr{I}$.
- $\widehat{\mathfrak{X}}$ denotes the real oriented blow up of $\mathfrak{X}$ in the x_i.

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parametrizing Stokes filtered G-local systems of irregular type $\mathscr{I}$.

Key point: $\mathrm{DMS}_{G,\mathscr{I}}$ is equipped with a natural 0-shifted relative symplectic structure which induces a Poisson structure on $Loc_G(X, \mathscr{I})$.

Deligne-Malgrange-Stokes sheaves (i)

Suppose X is a smooth surface underlying a quasi-projective complex algebraic curve and stratified with the following strata:

- a connected open stratum X_{in};
- a (not necessarily connected) open stratum X_{out};
- arcs X_e, $e \in E$;
- endpoints X_v, $v \in V$.

We require that the strata satisfy the following conditions:

(1) exactly two arcs meet at each endpoint;

(2) each arc separates X_{in} and X_{out};

(3) X_{out} retracts onto its boundary.

Deligne-Malgrange-Stokes sheaves (ii)

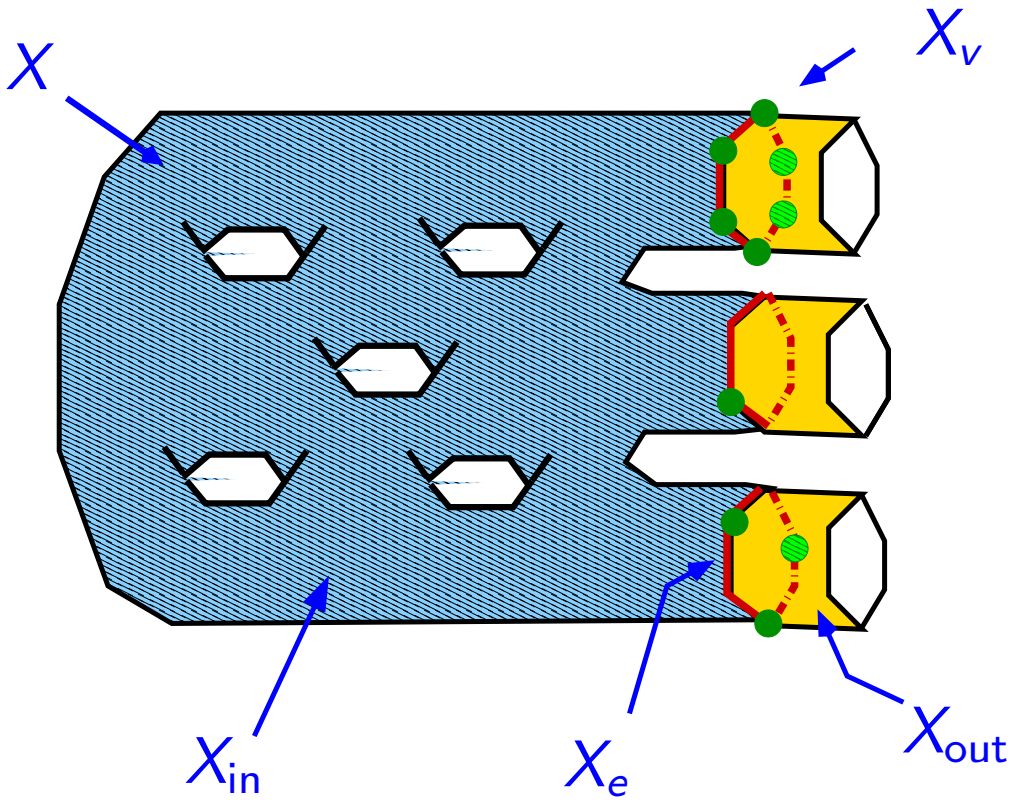

Figure: Stratified surface

Deligne-Malgrange-Stokes sheaves (iii)

Setup: Consider $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of stacks satisfying:

$\boxed{\text{codim} = 0}$ Locally on X_{in} and on X_{out}, $\mathscr{F}$ is isomorphic to BG for some reductive G.

Deligne-Malgrange-Stokes sheaves (iii)

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$\boxed{\text{codim} = 0}$ Locally on X_{in} and on X_{out}, $\mathscr{F}$ is isomorphic to BG for some reductive G.

typically the groups will be different on different connected components of strata

Deligne-Malgrange-Stokes sheaves (iv)

$\boxed{\text{codim} = 1}$ For each point $x \in X_e$, $e \in E$ the specialization of strata

$$X_{\text{in}} \rightsquigarrow X_e \leftsquigarrow X_{\text{out}}$$

leads to a cospecialization diagram of stacks:

$$\mathscr{F}_{\text{in}} \longleftarrow \mathscr{F}_e \longrightarrow \mathscr{F}_{\text{out}} \ .$$

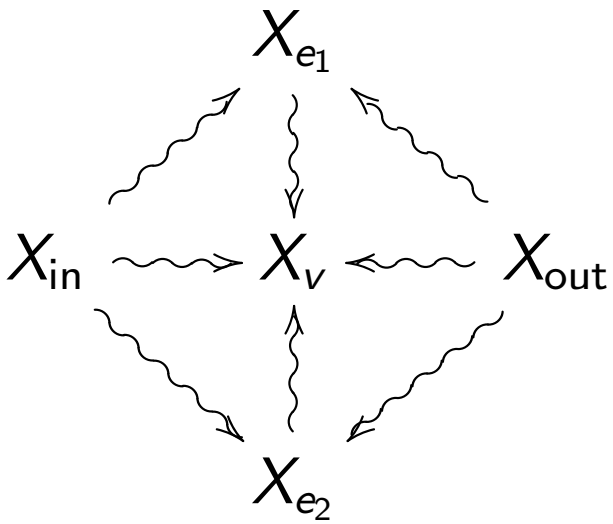
We require that this diagram must be isomorphic to

$$BG \longleftarrow BP \longrightarrow BL \ ,$$

where G is a reductive group, $P \subset G$ is a parabolic subgroup, and $P \to L$ is the Levi quotient.

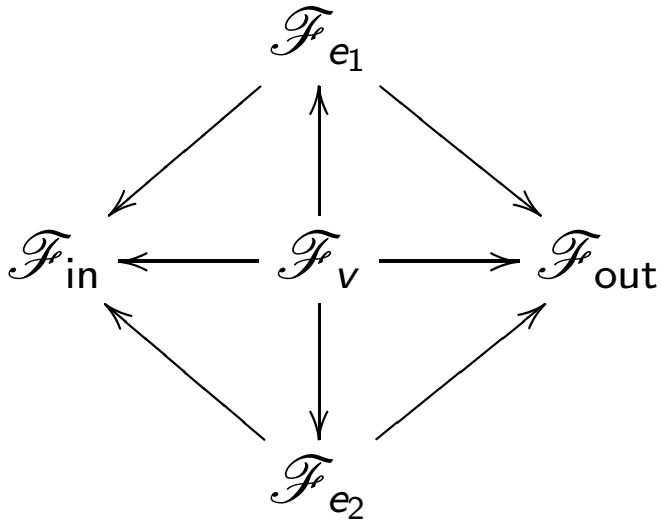
Deligne-Malgrange-Stokes sheaves (iv)

$\boxed{\text{codim} = 2}$ In a neighborhood of a point $\{b\} = X_v$, $v \in V$, the specialization of strata



Deligne-Malgrange-Stokes sheaves (iv)

codim $= 2$ In a neighborhood of a point $\{b\} = X_v$, $v \in V$, the specialization of strata leads to a diagram of stacks



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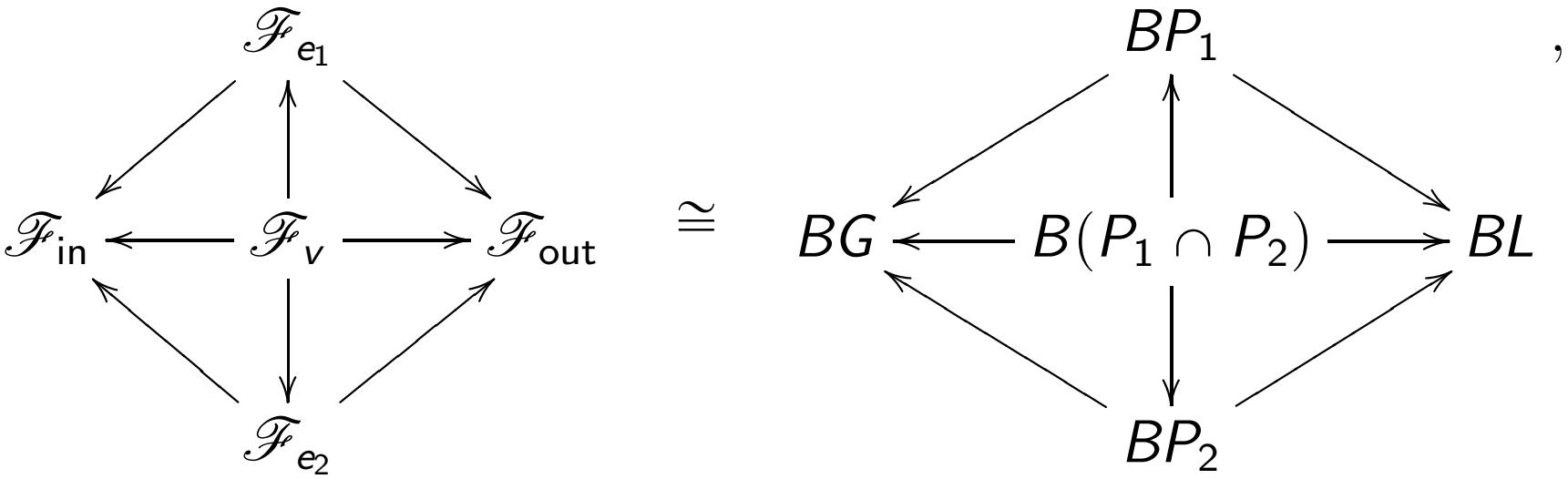
Deligne-Malgrange-Stokes sheaves (v)

codim $= 2$ We require that



where G is a reductive group, $P_1, P_2 \subset G$ are parabolic subgroups that admit a common Levi, and $P_{1,2} \to L$ are the Levi quotients.

Deligne-Malgrange-Stokes sheaves (vi)

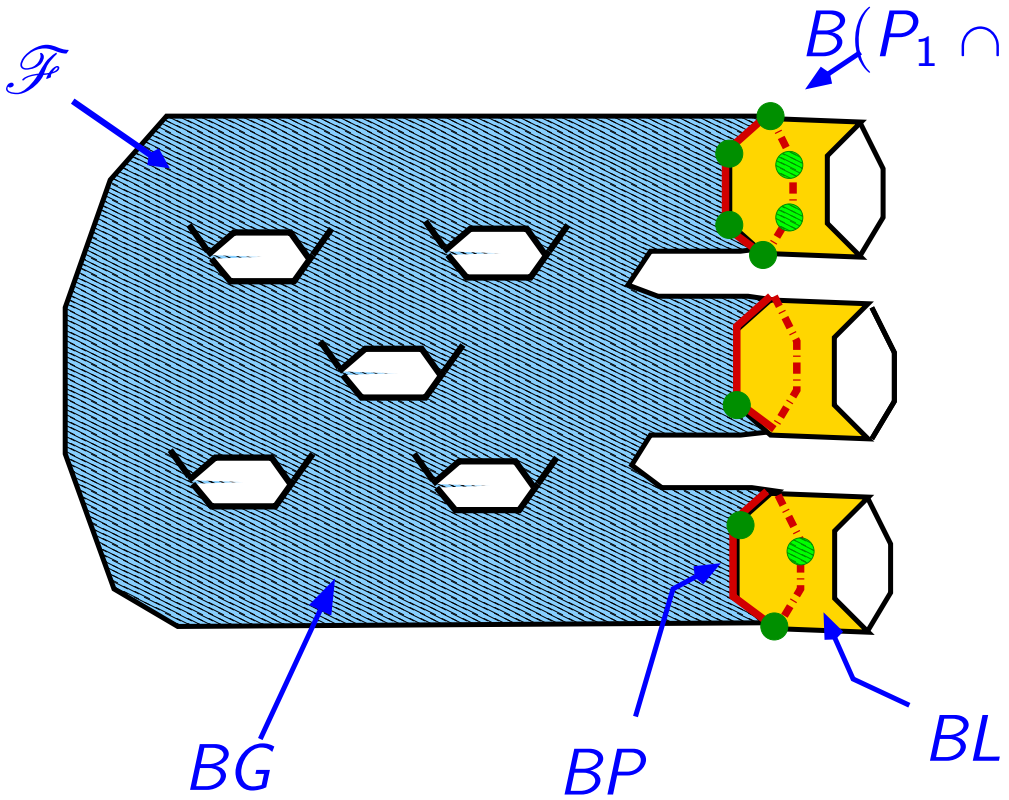


Figure: DMS sheaf

Deligne-Malgrange-Stokes sheaves (vii)

Theorem: **[Arinkin-P-Toën]** Let $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ be a DMS sheaf. Then

(1) The restriction map $\mathbf{A}^{2,cl}(\mathscr{F})_{K_X} \to \mathbf{A}^{2,cl}(\mathscr{F}_{\mathrm{in}})_{K_{X_{\mathrm{in}}}}$, is a homotopy equivalence.

(2) The extension ω of a form ω_{in} is non-degenerate if and only if ω_{in} is non-degenerate.

Deligne-Malgrange-Stokes sheaves (ix)

Remarks:

- $\mathscr{F}_{\text{in}}$ is a local system of stacks with fiber BG. Hence $\mathbb{T}_{\mathscr{F}_{\text{in}}}$ is a local system of complexes on X_{in} with fiber $\mathfrak{g}[1]$.
- In particular
$$\begin{pmatrix} \text{a} \quad \Pi_1(X_{\text{in}}) \quad \times \quad G\text{-invariant} \\ \text{symmetric pairing } \kappa \text{ on } \mathfrak{g} \end{pmatrix}$$
$$\Updownarrow$$
$$\begin{pmatrix} \text{a} \quad K_{X_{\text{in}}}\text{-valued} \quad \text{relative} \quad \text{symplectic} \\ \text{form } \omega_{\text{in}} = \omega_\kappa \text{ on } \mathscr{F}_{\text{in}} \end{pmatrix}$$
$$\Updownarrow$$
$$\begin{pmatrix} \text{a} \quad K_X\text{-valued} \quad \text{relative} \quad \text{symplectic} \\ \text{form } \omega \text{ on } \mathscr{F}. \end{pmatrix}$$
- When $\mathscr{F}_{\text{in}}$ is **constant**, the form κ exists automatically since G is assumed to be reductive.

Deligne-Malgrange-Stokes sheaves (ix)

Suppose

- $\mathfrak{X}$ is a smooth projective curve/$\mathbb{C}$;
- $X = \mathfrak{X} - \{x_1, \ldots, x_k\}$;
- $\mathscr{I} = \{\mathscr{I}_1, \ldots, \mathscr{I}_k\}$, $\mathscr{I}_i$ an irregular type at x_i.

Then:

• $\mathscr{I}$ can be recorded equivalently in $\mathsf{DMS}_{G,\mathscr{I}} \in \mathsf{Sh}^{\mathsf{str}}(X, \mathsf{dSt}_{\mathbb{C}})$;

• $\mathsf{DMS}_{G,\mathscr{I}}$ classifies Stokes data on $\mathfrak{X}$ of irregular type $\mathscr{I}$, in the sense that

$$Loc_G(X, \mathscr{I}) = \Gamma\left(\hat{\mathfrak{X}}, \mathsf{DMS}_{G,\mathscr{I}|\hat{\mathfrak{X}}}\right).$$

Here $\hat{\mathfrak{X}} \subset X$ denotes the real oriented blow-up of $\mathfrak{X}$ at the points x_i.

Deligne-Malgrange-Stokes sheaves (x)

Note:

- The sheaf $\mathrm{DMS}_{G,\mathscr{J}}$ tautologically satisfies the $\mathrm{codim} = 0, 1, 2$ properties.
- Since the underlying G-local systems are untwisted $\mathrm{DMS}_{G,\mathscr{J}}$ comes with a canonical relative symplectic form which depends only on a choice of a non-degenerate $\kappa \in (\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$.

Because of this we introduce the following:

Definition: A **DMS sheaf** on X is a sheaf $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ that satisfies the $\mathrm{codim} = 0, 1, 2$ properties and admits a relative K_X-valued symplectic structure.

Stokes data in dimension one (i)

Suppose

$\mathscr{F}$ is a DMS sheaf of stacks equipped with a K_X-valued relative symplectic form ω.

$f : X \to (0,1]$ is the stratified map which collapses $X - X_{\text{out}}$ to 1 and projects each cylinder component of X_{out} onto its ruling $(0,1)$.

Then f is a proper stratified map and

Pushforward theorem $\Longrightarrow$ $\operatorname{tr} f_*\omega$ is a relative $K_{(0,1]}$-valued symplectic structure on $f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$.

Stokes data in dimension one (ii)

Hence

- $\mathrm{tr}\, f_*\omega$ defines a 1-shifted symplectic structure on $\Gamma(X_{\mathrm{out}}, \mathscr{F}_{\mathrm{out}})$.
- $\mathrm{tr}\, f_*\omega$ defines a 0-shifted Lagrangian structure on the cospecialization map

$$\Gamma(X - X_{\mathrm{out}}, \mathscr{F}) \to \Gamma(X_{\mathrm{out}}, \mathscr{F}_{\mathrm{out}}).$$

Stokes data in dimension one (iii)

In the special case when $\mathscr{F} = \mathrm{DMS}_{G,\mathscr{I}}$ we get

$$Loc_G(X, \mathscr{I}) = \Gamma(X - X_{\mathrm{out}}, \mathscr{F})$$
$$Loc_{\widetilde{L}}(\partial X) = \Gamma(X_{\mathrm{out}}, \mathscr{F}_{\mathrm{out}})$$

where $\widetilde{L}$ is the local system of Levi subgroups on X_{out} for which $\mathscr{F}_{\mathrm{out}} = B\widetilde{L}$.

Moreover in this setting the map

$$r_{\mathscr{I}} : Loc_G(X, \mathscr{I}) \to Loc_{\widetilde{L}}(\partial X)$$

assigns to each Stokes filtered local system its formal monodromy at ∞.

Stokes data in dimension one (iv)

Since $\widetilde{L}$ is a locally constant sheaf we again have that fixing a flat section $\lambda \in \Gamma(\partial X, \widetilde{L})$ gives a Lagrangian map

$$\prod_{i=1}^{k} BG_{\lambda_i} \to \prod_{i=1}^{k} [G/G] = Loc_{\widetilde{L}}(\partial X).$$

The intersection

$$Loc_G(X, \mathscr{I}; \lambda)$$

of this Lagrangian with the Lagrangian map $r_{\mathscr{I}}$ is the moduli of Stokes data of type $\mathscr{I}$ having a fixed formal monodromy at ∞ and is therefore symplectic.

Extraction (i)

M - topological space, $N \subset M$ - a closed subspace.
Half open cylinder of the inclusion $N \hookrightarrow M$:

$$M^+ := (N \times (0,1]) \bigsqcup_N M,$$

where

- $N \to (0,1] \times N$ identifies N with the base $N \times \{1\}$;
- $M \to M^+$ identifies M with a closed subset of M^+, with complement $M - M^+ = N \times (0,1)$.

Note: For us M will be a topological manifold with boundary and $N = \partial M$. In this case M^+ is a topological manifold as well.

Extraction (ii)

Given $F \in \mathrm{Sh}^{\mathrm{str}}(M, C)$, $F^e \in \mathrm{Sh}^{\mathrm{str}}(N, C)$, $F_{|N} \to F^e$, define the **extract** $F^+ \in \mathrm{Sh}^{\mathrm{str}}(M^+, C)$ of (F, F^e) as the unique sheaf s.t.:

- The restriction $F^+|_M$ equals F;
- The restriction $F^+|_{N\times(0,1)}$ equals $\pi^* F^e$;
- The cospecialization map $F \to \imath^* \jmath_* (\pi^* F^e)$ corresponds to

$$F \to \iota_*(F|_N) \to \iota_*(F^e)$$

under the identification $\imath^* \jmath_* (\pi^* F^e) = \iota_* F^e$.

Here $\imath : M \hookrightarrow M^+$, $\jmath : N \times (0,1) \hookrightarrow M^+$, $\iota : N \hookrightarrow M$ are the natural inclusions, while $\pi : N \times (0,1) \twoheadrightarrow N$ is the projection.

Local model (i)

Group theoretic data:

- G - a complex reductive group, $T \subset G$ - a maximal torus.
- $\Lambda = \mathrm{cochar}(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Local model (i)

Constructions:

- $\xi \in \Lambda$ defines a parabolic $P(\xi) \subset G$ with $T \subset P(\xi)$.
- a Levi $L \supset T$ defines a subspace

$$\Lambda_{\mathbb{R}}^{L} = \{\xi \in \Lambda_{\mathbb{R}} \,|\, P(\xi) \supset L\}\,.$$

- a continuous map $\xi : Z \to \Lambda_{\mathbb{R}}^{L}$ defines a subsheaf of groups $P_\xi \subset G_Z$ via

$$\Gamma(U, P_\xi) := \left\{ g : U \to G \;\middle|\; \begin{array}{l} g \text{ is locally constant and} \\ g(z) \in P(\xi(z)) \text{ for all } z \in U \end{array} \right\}.$$

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Remark: the stalk of P_ξ at $z \in Z$ is not equal to $P_{\xi(z)}$: $P_{\xi(z)}$ depends upper-semicontinuously on z, while the stalk is lower-semicontinuous and in general not parabolic.

Local model (ii)

Setup:

- Y - a manifold with corners, $\partial Y \subset Y$ - its boundary.
- $\pi : Z \to Y$ a smooth fibration.
- $\xi : \partial Z \to \Lambda_{\mathbb{R}}$ an analytic function such that for any root $\alpha : \Lambda_{\mathbb{R}} \to \mathbb{R}$, if $\alpha(\xi(z)) = 0$, then either $\alpha \circ \xi = 0$ identically, or the derivative of $\alpha \circ \xi$ is non-zero on the vertical tangent space $T_{\pi,z}$

These data give rise a constructible subsheaf $P_\xi \subset G_Z$ defined by essentially the same formula:

$$\Gamma(U, P_\xi) := \left\{ g : U \to G \;\middle|\; \begin{array}{l} g \text{ is locally constant and} \\ g(z) \in P(\xi(z)) \text{ for all} \\ z \in U \cap \partial Z \end{array} \right\}.$$

Local model (iii)

Consider $L \subset G$ - the Levi whose root system consists of all roots α such that $\alpha \circ \xi = 0$ identically. Then for all $z \in \partial Z$ we have $L \subset P(\xi(z))$ and hence we get an embedding of sheaves

$$L_{\partial Z} \subset P_\xi|_{\partial Z}$$

which has a natural splitting:

Proposition [Arinkin-P-Töen]: There is a retract

$$r : P_\xi|_{\partial Z} \to L_{\partial Z}$$

that is a stalk-wise Levi quotient: the germ of r at every point of z identifies L with the maximal reductive quotient of $(P_\xi)_z$.

Local model (iv)

Apply extraction to $\partial Z \subset Z$ and the map of sheaves $r : P_\xi|_{\partial Z} \to L_{\partial Z}$. Then:

- we get a constructible sheaf of groups P^+_ξ on the topological manifold Z^+.
- Z^+ contains $\overset{\circ}{Z} = Z - \partial Z$ as an open and $BP^+_\xi|_{\overset{\circ}{Z}} = BG_{\overset{\circ}{Z}}$.

Local model (iv)

With this notation we then have:

Theorem [Arinkin-P-Töen]:

(1) Any $K_Z^\circ[2-\dim(Z)]$-valued closed 2-form $\overset{\circ}{\omega}$ on BG_Z° extends to a form ω on $B(P_\xi^+)$, and the space of such extensions is contractible, i.e. the map

$$\mathbf{A}_Z^{2,cl}(B(P_\xi^+))_{K_Z[2-\dim(Z)]} \to \mathbf{A}_{\overset{\circ}{Z}}^{2,cl}(BG_Z^\circ)_{K_Z^\circ[2-\dim(Z)]}$$

is a homotopy equivalence.

(2) The extension ω is non-degenerate iff the form $\overset{\circ}{\omega}$ is non-degenerate.

Deligne-Malgrange-Stokes gerbes (i)

Setup:

- $\mathfrak{X}$ - a compact complex manifold, $\dim_{\mathbb{C}} \mathfrak{X} = d$.
- $D \subset \mathfrak{X}$ - a simple normal crossings divisor.
- $\sigma : \widehat{\mathfrak{X}} \to \mathfrak{X}$ - the real oriented blow-up of $\mathfrak{X}$ along D.

Notation:

$$\begin{aligned} \overset{\circ}{\widehat{\mathfrak{X}}} &:= \mathfrak{X} - D = \widehat{\mathfrak{X}} - \sigma^{-1}(D); \\ \partial\widehat{\mathfrak{X}} &:= \sigma^{-1}(D). \\ X &:= \text{the cylinder } \widehat{\mathfrak{X}}^{+} \text{ of } \widehat{\mathfrak{X}} \hookrightarrow \widehat{\mathfrak{X}}. \end{aligned}$$

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Setup:

- $\mathfrak{X}$ - a compact complex manifold, $\dim_{\mathbb{C}} \mathfrak{X} = d$.
- $D \subset \mathfrak{X}$ - a simple normal crossings divisor.
- $\sigma : \widehat{\mathfrak{X}} \to \mathfrak{X}$ - the real oriented blow-up of $\mathfrak{X}$ along D.

Plan: Introduce a special class of sheaves of stacks on $\widehat{\mathfrak{X}}$ - **DMS gerbes** - which support tautologically relative shifted symplectic structures.

Note: • These sheaves are constructible with respect to some stratification, which will be considered a part of the data.

• The class of DMS gerbes includes the classifying stacks for Stokes data of given irregular type (with eliminated turning points).

Deligne-Malgrange-Stokes gerbes (ii)

Definition: A **DMS gerbe** on $\widehat{\mathfrak{X}}$ is a sheaf $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(\widehat{\mathfrak{X}}, \mathrm{dSt}_{\mathbb{C}})$ such that

- $\mathscr{F}$ is locally constant on $\overset{\circ}{\widehat{\mathfrak{X}}}$ with fiber BG for some reductive G.
- For each point $p \in \partial\widehat{\mathfrak{X}}$ there exists a reductive group G, an open neighborhood $p \in U \subset \widehat{\mathfrak{X}}$, and a real analytic function $\xi : \partial\widehat{\mathfrak{X}} \cap U \to \Lambda_{\mathbb{R}}$ so that $\mathscr{F}|_U \cong BP_\xi$.
- If $\alpha : \Lambda_{\mathbb{R}} \to \mathbb{R}$ is a root such that $\alpha \circ \xi(p) = 0$, then either $\alpha \circ \xi = 0$ identically, or the derivative of $\alpha \circ \xi|_{\widehat{\mathfrak{X}}_p}$ is non-zero at p

Note: The fiber $\widehat{\mathfrak{X}}_p = \sigma^{-1}(\sigma(p))$ is a k-dimensional torus, $k =$ the number of components of D passing through p.

Deligne-Malgrange-Stokes gerbes (iii)

Proposition [Arinkin-P-Töen]: Suppose $\mathscr{F}$ is a DMS gerbe on $\hat{\mathfrak{X}}$. Then:

- There is a locally constant sheaf of stacks $\mathcal{L}$ over $\partial\hat{\mathfrak{X}}$ equipped with a morphism

$$r : \mathscr{F}|_{\partial\hat{\mathfrak{X}}} \to \mathcal{L}.$$

- The pair $(\mathcal{L}, r)$ is uniquely determined (up to a contractible space of choices) by the condition that the germ of r at all points $z \in \partial\hat{\mathfrak{X}}$ is isomorphic to a map $BH \to BL_H$, where H is a connected linear group and L_H is its maximal reductive quotient.

Deligne-Malgrange-Stokes gerbes (iv)

The extraction construction applied to $\mathscr{F}$ and the map

$$r : \mathscr{F}|_{\partial\hat{\mathfrak{X}}} \to \mathcal{L},$$

yields a constructible sheaf of stacks $\mathscr{F}^+$ on $X = \hat{\mathfrak{X}}^+$.
Note also that

- $\overset{\circ}{\mathfrak{X}} \subset X$ is open;
- X is an oriented topological manifold of dimension $2d$;
- $K_X = \mathbb{C}_X[2d]$ or equivalently $K_X[2-2d] = \mathbb{C}_X[2]$.

We can now formulate our general existence theorem

Deligne-Malgrange-Stokes gerbes (v)

Theorem [Arinkin-P-Töen]: Suppose $\mathscr{F}$ is a DMS gerbe on $\hat{\mathfrak{X}}$. Then:

(1) Any $\mathbb{C}^{\circ}_{\mathfrak{X}}[2]$-valued closed 2-form $\overset{\circ}{\omega}$ on $\mathscr{F}^{+}|^{\circ}_{\mathfrak{X}} = \mathscr{F}|^{\circ}_{\mathfrak{X}}$ extends to a $\mathbb{C}_X[2]$-valued 2-form ω on X, and the space of such extensions is contractible. Equivalently, the restriction map

$$\mathbf{A}^{2,cl}_X(\mathscr{F}^{+})_{K_X[2-2d]} \to \mathbf{A}^{2,cl}(\mathscr{F})_{K^{\circ}_{\mathfrak{X}}[2-2d]}$$

is a homotopy equivalence.

(2) The extension ω is non-degenerate iff $\overset{\circ}{\omega}$ is non-degenerate.

Stokes data in higher dimension (i)

Suppose

$\mathscr{F}$ is a DMS gerbe equipped with a $K^{\circ}_{\mathfrak{X}[2-2d]}$-valued relative symplectic form ω.

$f : X \to (0,1]$ is the stratified map which collapses $X - \mathring{\mathfrak{X}}$ to 1 and projects each cylinder component of X onto its ruling $(0,1)$.

Then f is a proper stratified map and

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\, f_*\omega$ is a relative $K_{(0,1]}[2-2d]$-valued symplectic structure on $f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$.

Stokes data in higher dimension (ii)

Hence

- tr $f_*\omega$ defines a $3-2d$-shifted symplectic structure on $\Gamma(\partial\widehat{\mathfrak{X}}, \mathscr{F})$.
- tr $f_*\omega$ defines a $2-2d$-shifted Lagrangian structure on the cospecialization map

$$\Gamma(X, \mathscr{F}^+) \to \Gamma(\partial\widehat{\mathfrak{X}}, \mathscr{F}).$$

Stokes data in higher dimension (iii)

In the special case when $\mathscr{F} = \mathrm{DMS}_{G,\mathscr{I}}$ we get

$$Loc_G(\mathring{\mathfrak{X}}, \mathscr{I}) = \Gamma(\mathring{\mathfrak{X}}, \mathscr{F}),$$
$$Loc_{\mathcal{L}}(\partial\hat{\mathfrak{X}}) = \Gamma(X - \mathring{\mathfrak{X}}, \mathscr{F}).$$

Moreover in this setting the map

$$r_{\mathscr{I}} : Loc_G(X, \mathscr{I}) \to Loc_{\mathcal{L}}(\partial X)$$

assigns to each Stokes filtered local system its formal monodromy at ∞.

Constant sheaves (i)

Setup:

- X - a topological space = a stratified space with a single stratum.
- $F \in \mathrm{dSt}_{\mathbb{C}}$ - a derived Artin stack, locally of finite presentation.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - the constant sheaf on X with fiber F.
- $\omega \in \mathcal{A}^{2,cl}(F, n)$ - an (absolute) n-shifted symplectic form on F.

Constant sheaves (ii)

Note:

- ω corresponds to a relative $\mathbb{C}_X[n]$-valued closed 2-form $\omega_X \in \mathbf{A}_X^{2,cl}(\mathscr{F})_{\mathbb{C}[n]}$ on $\mathscr{F}$.
- If X is an oriented manifold of pure dimension d, then $K_X = \mathbb{C}_X[d]$ and so $\omega_X \in \mathbf{A}_X^{2,cl}(\mathscr{F})_{K_X[n-d]}$.

$$\begin{pmatrix}\omega \text{ is non-degenerate as}\\ \text{an } n\text{-shifted absolute}\\ \text{form on } F\end{pmatrix} \iff \begin{pmatrix}\omega_X \text{ is non-degenerate}\\ \text{as a } K_X[n-d]\text{-valued}\\ \text{relative form on } \mathscr{F}.\end{pmatrix}$$

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega_X))$ is an $(n-d)$-shifted symplectic structure on $\Gamma(X,\mathscr{F}) = Map_{\mathrm{dSt}_{\mathbb{C}}}(X,F)$.

Constant sheaves (ii)

Note:

- ω corresponds to a relative $\mathbb{C}_X[n]$-valued closed 2-form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{\mathbb{C}[n]}$ on $\mathscr{F}$.
- If X is an oriented manifold of pure dimension d, then $K_X = \mathbb{C}_X[d]$ and so $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X[n-d]}$.

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Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega_X))$ is an $(n-d)$-shifted symplectic structure on $\Gamma(X,\mathscr{F}) = Map_{\mathrm{dSt}_{\mathbb{C}}}(X,F)$.
This is the mapping stack theorem from **[PTVV]**.

Lagrangian maps

Setup:

- $X = (0,1]$ is the half-open interval stratified by the strata $X_{\mathsf{in}} = (0,1)$ and $X_1 = \{1\}$, labeled by $I = \{\mathsf{in}, 1\}$ with $1 < \mathsf{in}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_1, \mathscr{F}_{\mathsf{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and one cospecialization map $\mathscr{F}_1 \to \mathscr{F}_{\mathsf{in}}$.

Note: $K_X = (\text{extension by zero of } \mathbb{C}_{X_{\mathsf{in}}}[1] \text{ from } X_{\mathsf{in}} \text{ to } X)$.
Thus a relative n-shifted symplectic structure on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\mathsf{in}}$;
- an n-shifted Lagrangian structure on $\mathscr{F}_1 \to \mathscr{F}_{\mathsf{in}}$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, and $X_1 = \{1\}$, where $I = \{0, 1, \text{in}\}$ with $0 < \text{in}$, $1 < \text{in}$, while 0 and 1 are incomparable.
- $\mathscr{F} \in \text{Sh}^{\text{str}}(X, \text{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\text{in}} \in \text{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

Note: $K_X = (\text{extension by zero of } \mathbb{C}_{X_{\text{in}}}[1] \text{ from } X_{\text{in}} \text{ to } X)$. A relative n-shifted symplectic structure on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\text{in}}$;
- n-shifted Lagrangian structures on $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\text{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

A relative n-shifted symplectic structure ω on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\text{in}}$;
- n-shifted Lagrangian structures on $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\left(\Gamma(\omega)\right)$ is an n-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = \mathscr{F}_0 \times^h_{\mathscr{F}_{\text{in}}} \mathscr{F}_1$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\text{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

A relative n-shifted symplectic structure ω on $\mathscr{F}$ consists of:

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This is the Lagrangian intersection theorem from **[PTVV]**.

Hamiltonian reduction (iii)

Setup:

- $X = [0,1]$ with strata $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- G - a linear algebraic group/$\mathbb{C}$, and $\mathbb{O} \subset \mathfrak{g}^\vee$ - a coadjoint orbit.
- (M, ω) - an algebraic symplectic manifold equipped with a Hamiltonian G-action.
- $\mu : M \to \mathfrak{g}^\vee$ - a G-equivariant moment map.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ given by

$$\mathscr{F}_0 = [\mathbb{O}/G], \quad \mathscr{F}_1 = [M/G], \quad \mathscr{F}_{\text{in}} = [\mathfrak{g}^\vee/G] = T^\vee_{BG}[1]$$

and maps $[\mathbb{O}/G] \hookrightarrow [\mathfrak{g}^\vee/G]$ and $\mu : [M/G] \to [\mathfrak{g}^\vee/G]$.

Hamiltonian reduction (iv)

Note: The **Kirillov-Kostant-Souriau** form on $\mathfrak{g}^\vee$ induces a relative 0-shifted symplectic structure ω on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbb{O})/G]$.

Hamiltonian reduction (iv)

Note: The **Kirillov-Kostant-Souriau** form on $\mathfrak{g}^\vee$ induces a relative 0-shifted symplectic structure ω on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbb{O})/G]$.

This is the **Marsden-Weinstein** Hamiltonian reduction theorem.

Quasi-Hamiltonian reduction (iii)

Setup:

- $X = S^2$ with strata $X_{\text{in}} = S^2 - \{S, N\}$, $X_0 = \{S\}$, $X_1 = \{N\}$.
- G - a complex reductive group, $\mathbf{C} \subset G$ - a conjugacy class.
- (M, ω) - an algebraic symplectic manifold equipped with a quasi-Hamiltonian G-action.
- $\mu : M \to G$ - a G-equivariant group valued moment map.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ given by

$$\mathscr{F}_0 = [\mathbf{C}/G], \quad \mathscr{F}_1 = [M/G], \quad \mathscr{F}_{\text{in}} = BG$$

and maps $[\mathbf{C}/G] \hookrightarrow [G/G]$ and $\mu : [M/G] \to [G/G]$.

Quasi-Hamiltonian reduction (iv)

Note:

- $K_X = (\text{extension by zero of } \mathbb{C}_{X_{\text{in}}}[2] \text{ from } X_{\text{in}} \text{ to } X)$.
- The standard 2-shifted symplectic form ω_κ on BG extends to a natural relative 0-shifted symplectic form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X}$ on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\text{tr}\,(\Gamma(\omega_X))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbf{C})/G]$.

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Quasi-Hamiltonian reduction (iv)

Note:

- $K_X = (\text{extension by zero of } \mathbb{C}_{X_{\text{in}}}[2] \text{ from } X_{\text{in}} \text{ to } X)$.
- The standard 2-shifted symplectic form ω_κ on BG extends to a natural relative 0-shifted symplectic form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X}$ on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\text{tr}\,(\Gamma(\omega_X))$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbf{C})/G]$.

This is the **Alexeev-Malkin-Meinrenken** quasi-Hamiltonian reduction theorem.

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