Constructing shifted symplectic structures part 1

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Outline

- joint with Dima Arinkin and Bertrand Toën
- shifted symplectic geometry
- relative shifted symplectic structures

Intrinsic geometry of moduli

Observation: Moduli spaces, when constructed as algebraic varieties, or schemes, or algebraic spaces, or analytic spaces, tend to be pathological.

Better setting: Use **derived algebraic stacks** as parameter spaces in a moduli problem. Leads to nicer solutions of the moduli problem:

- less singular,
- easier to describe.

The price to pay: Heavy technology relying on higher category theory, homotopical algebra, etc.

Derived stacks (i)

There is a category of derived schemes such that:

- It contains the category of schemes.
- Has fibered products and for affines $Spec(A) \times_{Spec(C)} Spec(B) = \mathbf{Spec}(A \otimes^L_C B)$, where $A \otimes^L_C B$ is the derived tensor product: a commutative dg-algebra.

Note:

- $H^0(A \otimes^L_C B) \cong A \otimes_C B$.
- $H^\bullet(A \otimes^L_C B)$ controlls the defect of transversality (excess intersection).

Derived stacks (ii)

Definition: A **derived scheme** (over $\mathbb{C}$) is a pair $\mathscr{X} = (X, \mathcal{O}^{\bullet}_{\mathscr{X}})$, consisting of a topological space X and a sheaf $\mathcal{O}^{\bullet}_{\mathscr{X}}$ of commutative non-positively graded $\mathbb{C}$-dg-algebras such that

- the ringed space $t_0\mathscr{X} = (X, \mathscr{H}^0(\mathcal{O}^{\bullet}_{\mathscr{X}}))$ is a scheme;
- for all $i \leqslant 0$ the sheaves $\mathscr{H}^i(\mathcal{O}^{\bullet}_{\mathscr{X}})$ are quasi-coherent $\mathscr{H}^0(\mathcal{O}^{\bullet}_{\mathscr{X}})$-modules.

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Intuition: $\mathscr{X}$ is an infinitesimal thickening of the ordinary scheme $t_0\mathscr{X}$ with extra nilpotent functions sitting in higher homological degrees.

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Definition: An algebraic **derived stack** is a quotient of a derived scheme by the action of a smooth groupoid.

Note: Derived stacks form a complicated ∞-category $\mathrm{dSt}_{\mathbb{C}}$.

Tangent complex

$X \in \mathrm{dSt}_{\mathbb{C}}, \quad x : \mathrm{Spec}(\mathbb{C}) \to X$ a point

$$\begin{pmatrix} \text{Stalk} & \mathbb{T}_{X,x} & \text{of} \\ \text{the} & \text{tangent} & \\ \text{complex} & & \end{pmatrix} = \begin{pmatrix} \text{normalized} & \text{chain} & \text{complex} \\ \text{of} & \text{the} & \text{homotopy fiber of} \\ X(\mathbb{C}[\varepsilon]) \to X(\mathbb{C}) & \text{over } x & \end{pmatrix}$$

simplicial abelian group

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When X is a moduli stack:

$H^{-1}(\mathbb{T}_{X,x}) =$ infinitesimal automorphisms of x;

$H^{0}(\mathbb{T}_{X,x}) =$ infinitesimal deformations of x;

$H^{1}(\mathbb{T}_{X,x}) \supseteq$ obstructions of x.

Examples (i):

- $X = BG = [\mathrm{pt}/G] \Rightarrow \mathbb{T}_{X,\mathrm{pt}} = \mathfrak{g}[1]$.
- $X =$ moduli of vector bundles E on a smooth projective $Y \Rightarrow \mathbb{T}_{X,E} = R\Gamma(Y, \mathrm{End}(E))[1]$.
- $X =$ moduli of maps f from C to $Y \Rightarrow$ $\mathbb{T}_{X,f} = R\Gamma(C, f^* T_Y)$.
- $X =$ moduli of local systems $\mathbb{E}$ on a compact manifold Y $\Rightarrow \mathbb{T}_{X,\mathbb{E}} = R\Gamma(Y, \mathrm{End}(\mathbb{E}))[1]$.

Examples (ii):

- $X =$ derived intersection

$$L_1 \overset{h}{\underset{M}{\times}} L_2 = \left(L_1 \cap L_2, \mathcal{O}_{L_1} \overset{L}{\underset{\mathcal{O}_M}{\otimes}} \mathcal{O}_{L_2} \right)$$

of smooth subvarieties $L_1, L_2 \subset M$ in a smooth $M \Longrightarrow$

$$\mathbb{T}_{X,x} = \begin{matrix} [\ T_{L_1,x} \oplus T_{L_2,x} & \longrightarrow & T_{M,x}\ ], \\ 0 & & 1 \end{matrix}$$

$H^0(\mathbb{T}_{X,x}) = T_{L_1 \cap L_2, x}$;
$H^1(\mathbb{T}_{X,x}) =$ failure of transversality.

Examples (iii):

Special case: $X =$ derived zero locus Rzero(s) of $s \in H^0(L, E)$.

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an algebraic vector bundle on a smooth variety L

Examples (iii):

Special case: $X =$ derived zero locus $\mathrm{Rzero}(s)$ of $s \in H^0(L, E)$.

Thus

$$X = L \overset{h}{\underset{M}{\times}} L = \left(Z,\ i_L^{-1}(\mathrm{Sym}^\bullet(E^\vee[1]), s^\flat)\right),$$

where:

- $Z = t_0 X = \mathrm{zero}(s)$ is the scheme theoretic zero locus of s,
- $i_L : Z \to L$ is the natural inclusion, and
- $s^\flat$ is the contraction with s.

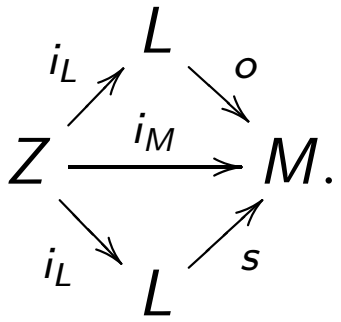
Examples (iv):

In particular $\mathbb{T}_X = \quad [\, i_L^* T_L \oplus i_L^* T_L \xrightarrow{i_L^* do + i_L^* ds} i_M^* T_M \,],$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \qquad\qquad\qquad\qquad 1$

where

- $M = \mathrm{tot}(E)$, and
- i_M, o, and s are the natural maps



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Remark: There is a natural quasi-isomorphism

$$\mathbb{T}_X = \left[i_L^* T_L \xrightarrow{(\nabla s)^\flat} i_L^* E \right] = \left[T_L \xrightarrow{(\nabla s)^\flat} E \right]_{|Z}.$$

Note: $\nabla : E \to E \otimes \Omega_L^1$ is an algebraic connection which exists only locally and is not unique. However $(\nabla s)_{|Z}$ is well defined globally and independent of the choice of ∇.

Cotangent complex

$A \in \mathrm{cdga}_{\mathbb{C}}$, $\quad X = \mathbf{Spec}(A) \in \mathrm{dSt}_{\mathbb{C}}$,
$QA \to A$ a cofibrant (quasi-free) replacement

$$\begin{pmatrix} \text{cotangent complex} \\ \mathbb{L}_X = \mathbb{L}_A \end{pmatrix} = \begin{pmatrix} \text{Kähler differentials} \\ \Omega^1_{QA} \text{ of } QA \end{pmatrix}$$

If $X \in \mathrm{dSt}_{\mathbb{C}}$ is a general derived Artin stack, then
$X = \mathrm{hocolim}\{\mathbf{Spec}\, A \to X\}$ (in the model category $\mathrm{dSt}_{\mathbb{C}}$) and

$$\mathbb{L}_X = \mathrm{holim}_{\mathbf{Spec}\, A \to X} \mathbb{L}_A$$

Note:

- $\mathbb{L}_X \in L_{qcoh}(X)$ - the dg category of quasi-coherent $\mathcal{O}_X$ modules.
- X is locally of finite presentation iff $\mathbb{L}_X$ is perfect. In this case $\mathbb{T}_X = \mathbb{L}_X^{\vee} = \mathrm{Hom}(\mathbb{L}_X, \mathcal{O}_X)$.

p-forms

$A \in \mathrm{cdga}_{\mathbb{C}}$, $\quad X = \mathbf{Spec}(A) \in \mathrm{dSt}_{\mathbb{C}}$,
$QA \to A$ a cofibrant (quasi-free) replacement. Then:

$\oplus_{p \geqslant 0} \bigwedge_A^p \mathbb{L}_A = \oplus_{p \geqslant 0} \Omega^p_{QA}$ - a fourth quadrant bicomplex with

vertical differential $d : \Omega^{p,i}_{QA} \to \Omega^{p,i+1}_{QA}$ induced by d_{QA}, and

horizontal differential $d_{DR} : \Omega^{p,i}_{QA} \to \Omega^{p+1,i}_{QA}$ given by the de Rham differential.

Hodge filtration: $F^q(A) := \oplus_{p>q} \Omega^p_{QA}$: still a fourth quadrant bicomplex.

(shifted) closed p-forms

Motivation: If X is a smooth scheme/$\mathbb{C}$, then $\Omega_X^{p,cl} \cong \left(\Omega_X^{\geqslant p}[p], d_{DR}\right)$. Use $(\Omega_X^{\geqslant p}[p], d_{DR})$ as a model for closed p forms in general.

Definition:

- **complex of closed p-forms on $X = \mathbf{Spec}\, A$:** $\mathbf{A}^{p,cl}(A) := \mathrm{tot}^{\Pi}(F^p(A))[p]$
- **complex of n-shifted closed p-forms on $X = \mathbf{Spec}\, A$:** $\mathbf{A}^{p,cl}(A; n) := \mathrm{tot}^{\Pi}(F^p(A))[n+p]$
- **Hodge tower:** $\cdots \to \mathbf{A}^{p,cl}(A)[-p] \to \mathbf{A}^{p-1,cl}(A)[1-p] \to \cdots \to \mathbf{A}^{0,cl}(A)$

(shifted) closed p-forms (ii)

Explicitly an n-shifted closed p-form ω on $X = \mathbf{Spec}\, A$ is an infinite collection

$$\omega = \{\omega_i\}_{i\geqslant 0}\,, \qquad \omega_i \in \Omega_A^{p+i,n-i}$$

satisfying

$$d_{DR}\omega_i = -d\omega_{i+1}.$$

Note: The collection $\{\omega_i\}_{i\geqslant 1}$ is the **key** closing ω.

p-forms and closed p-forms

Note:

- The complex $\mathbf{A}^{0,cl}(A)$ of closed 0-forms on $X = \mathbf{Spec}\, A$ is exactly Illusie's derived de Rham complex of A.
- There is an underlying p-form map

$$\mathbf{A}^{p,cl}(A; n) \to \bigwedge^p \mathbb{L}_{A/k}[n]$$

inducing

$$H^0(\mathbf{A}^{p,cl}(A)[n]) \to H^n(X, \bigwedge^p \mathbb{L}_{A/k}).$$

- The homotopy fiber of the underlying p-form map can be very complicated (complex of **keys**): being closed is **not** a property but rather a list of coherent data.

Functoriality and gluing:

Globally we have:

- the *n*-shifted *p*-forms ∞-functor $\mathcal{A}^p(-;n) : \mathrm{cdga}_{\mathbb{C}} \to \mathrm{SSets} : A \mapsto |\,\Omega^p_{QA}[n]\,|$, and
- the *n*-shifted closed *p*-forms ∞-functor $\mathcal{A}^{p,cl}(-;n) : \mathrm{cdga}_{\mathbb{C}} \to \mathrm{SSets} : A \mapsto |\,\mathbf{A}^{p,cl}(A)[n]\,|$.

Note: $\mathcal{A}^p(-;n)$ and $\mathcal{A}^{p,cl}(-;n)$ are **derived stacks** for the étale topology. underlying *p*-form map (of derived stacks)

$$\mathcal{A}^{p,cl}(-;n) \to \mathcal{A}^p(-;n)$$

Notation: $|-|$ denotes $Map_{\mathbb{C}-dgMod}(\mathbb{C},-) = \mathbf{DK}\tau^{\leqslant 0}(-)$ i.e. Dold-Kan applied to the $\leqslant 0$-truncation.

global forms and closed forms (i)

Fix a derived Artin stack X (locally of finite presentation $/\mathbb{C}$)

Definition:

- $\mathcal{A}^p(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-))$ - **space of p-forms** on X;
- $\mathcal{A}^{p,cl}(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,cl}(-))$ - **space of closed p-forms** on X;
- n-shifted versions : $\mathcal{A}^p(X;n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-;n))$ and $\mathcal{A}^{p,cl}(X;n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,cl}(-;n))$
- an n-shifted (respectively closed) p-form on X is an element in $\pi_0\mathcal{A}^p(X;n)$ (respectively in $\pi_0\mathcal{A}^{p,cl}(X;n)$)

global forms and closed forms (ii)

Note:

1) If X is a smooth scheme there are no negatively shifted forms.
2) When $X = \mathbf{Spec}\, A$ then there are no positively shifted forms.
3) For general X shifted forms may exist for any $n \in \mathbb{Z}$.

global forms and closed forms (ii)

- **underlying p-form** map (of simplicial sets)

$$\mathcal{A}^{p,cl}(X;n) \to \mathcal{A}^{p}(X;n)$$

- not a monomorphism for general X, its homotopy fiber at a given p-form ω_0 is the space of **keys** of ω_0.
- If X is a smooth and proper scheme then this map is indeed a mono (homotopy fiber is either empty or contractible) $\Rightarrow$ no new phenomena in this case.

global forms and closed forms (ii)

Theorem (PTVV): for X derived Artin, then forms satisfy smooth descent:

$$\mathcal{A}^p(X; n) \simeq \mathrm{Map}_{\mathrm{L}_{\mathrm{qcoh}}(X)}(\mathcal{O}_X, (\bigwedge^p \mathbb{L}_X)[n]).$$

In particular: an n-shifted p-form on X is an element in $H^n(X, \bigwedge^p \mathbb{L}_X)$

global forms and closed forms (iii)

Remark: If $A \in cdga$ is quasi-free, and $X = \mathbf{Spec}\, A$, then

$$\mathcal{A}^{p,cl}(X;n) = \left| \prod_{i \geqslant 0} \left(\Omega_A^{p+1}[n-i], d + d_{DR} \right) \right|$$

$$= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right|$$

$$= |NC(A)(p)[n+p]|$$

global forms and closed forms (iii)

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negative cyclic complex of weight p

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Hence

$$\boxed{\pi_0\mathcal{A}^{p,cl}(X;n) = HC_-^{n-p}(A)(p).}$$

global forms and closed forms (iv)

Definition: Given a higher Artin derived stack the n-th algebraic de Rham cohomology of X is defined to be $H^n_{DR}(X) = \pi_0 \mathcal{A}^{0,cl}(X; n)$.

Remark:

- agrees with Illusie's definition in the affine case.
- if X is a higher Artin derived stack locally f.p., then $H^\bullet_{DR}(X) \cong H^\bullet_{DR}(t_0 X) =$ algebraic de Rham cohomology of the underived higher stack $t_0 X$ defined by Hartschorne's completion formalism.

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Corollary: Let X be a locally f.p. derived stack and let ω be an n-shifted closed p-form on X with $n < 0$. Then ω is exact, i.e. $[\omega] = 0 \in H^{n+p}_{DR}(X)$.

Examples (i):

(1) If $X = \mathrm{Spec}(A)$ is an usual (underived) smooth affine scheme, then

$$\mathcal{A}^{p,cl}(X;n) = (\tau_{\leqslant n}(\underset{0}{\Omega^p_A} \xrightarrow{d_{DR}} \underset{1}{\Omega^{p+1}_A} \xrightarrow{d_{DR}} \cdots))[n],$$

and hence

$$\pi_0\mathcal{A}^{p,cl}(X;n) = \begin{cases} 0, & n<0 \\ \Omega^{p,cl}_A, & n=0 \\ H^{n+p}_{DR}(X), & n>0 \end{cases}$$

Remark: The standard notation for forms is inadequate - if $X = \mathbb{C}^\times$, then $dz/z \in \pi_0\mathcal{A}^{1,cl}(X;0)$ and also $dz/z \in \pi_0\mathcal{A}^{0,cl}(X;1)$.

Examples (ii):

(2) If X is a smooth and proper scheme, then $\pi_i \mathcal{A}^{p,cl}(X; n) = F^p H_{DR}^{n+p-i}(X)$.

(3) If X is a (underived) higher Artin stack, and $X_\bullet \to X$ is a smooth affine simplicial groupoid presenting X, then $\pi_0 \mathcal{A}^p(X; n) = H^n(\Omega^p(X_\bullet), \delta)$ with $\delta =$ Čech differential. In particular if G is a complex reductive group, then

$$\pi_0 \mathcal{A}^p(BG; n) = \begin{cases} 0, & n \neq p \\ (\mathrm{Sym}^\bullet \mathfrak{g}^\vee)^G, & n = p. \end{cases}$$

Examples (iii):

(4) Similarly

$$\mathcal{A}^{p,cl}(BG;n) = \left|\prod_{i\geqslant 0}\left(\mathrm{Sym}^{p+i}\,\mathfrak{g}^{\vee}\right)^{G}[n+p-2i]\right|,$$

and so

$$\pi_0\mathcal{A}^{p,cl}(BG;n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \left(\mathrm{Sym}^{p}\,\mathfrak{g}^{\vee}\right)^{G}, & \text{if } n \text{ is even.} \end{cases}$$

Examples (iv):

(5) If $X = \mathrm{Rzero}(s)$ for $s \in H^0(L, E)$ on a smooth L, then

$$\begin{array}{ccc} \Omega^1_X = & E^\vee_{|Z} & \xrightarrow{(\nabla s)^\flat} \Omega^1_{L|Z}, \\ & -1 & 0 \end{array}$$

and if we choose ∇ local flat algebraic connection on E we can rewrite Ω^1_X as a module over the Koszul complex:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \wedge^2 E^\vee \otimes \Omega^1_L & \xrightarrow{s^\flat} & E^\vee \otimes \Omega^1_L & \xrightarrow{s^\flat} & \Omega^1_L & \longrightarrow & \Omega^1_{L|Z} & & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle [\nabla, s^\flat]} & & \uparrow{\scriptstyle (\nabla s)^\flat} & & \\ \cdots & \longrightarrow & \wedge^2 E^\vee \otimes E^\vee & \xrightarrow{s^\flat} & E^\vee \otimes E^\vee & \xrightarrow{s^\flat} & E^\vee & \longrightarrow & E^\vee_{|Z} & & -1 \end{array}$$

Examples (v):

In the same way we can describe Ω^2_X as a module over the Koszul complex

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \wedge^2 E^\vee \otimes \Omega^2_L & \longrightarrow & E^\vee \otimes \Omega^2_L & \longrightarrow & \Omega^2_L & \longrightarrow & \Omega^2_{L|Z} \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots \to & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L & \to & E^\vee \otimes E^\vee \otimes \Omega^1_L & \to & E^\vee \otimes \Omega^1_L & \to & (E^\vee \otimes \Omega^1_L)_{|Z} \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \wedge^2 E^\vee \otimes S^2 E^\vee & \longrightarrow & E^\vee \otimes S^2 E^\vee & \longrightarrow & S^2 E^\vee & \longrightarrow & S^2 E^\vee|Z
\end{array}$$

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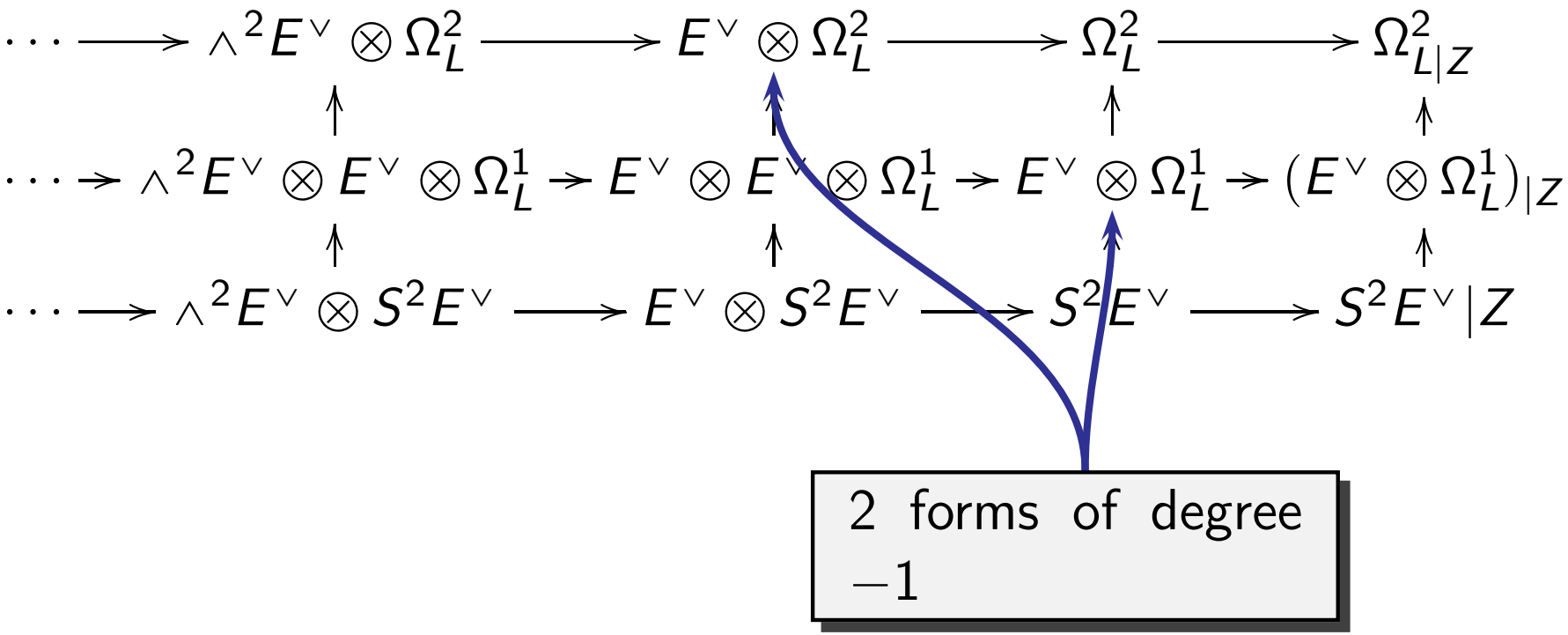

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& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots \to & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L & \to & E^\vee \otimes E^\vee \otimes \Omega^1_L & \to & E^\vee \otimes \Omega^1_L & \to & (E^\vee \otimes \Omega^1_L)_{|Z} \\
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\cdots \longrightarrow & \wedge^2 E^\vee \otimes S^2 E^\vee & \longrightarrow & E^\vee \otimes S^2 E^\vee & \longrightarrow & S^2 E^\vee & \longrightarrow & S^2 E^\vee|Z
\end{array}$$

Note: The de Rham differnetial $d_{DR} : \Omega^1_X \to \Omega^2_X$ is the sum $d_{DR} = \nabla + \kappa$, where κ is the Koszul contraction

$$\kappa : \wedge^a E^\vee \otimes S^b E^\vee \to \wedge^{a-1} E^\vee \otimes S^{b+1} E^\vee.$$

Examples (vi):

Lemma [Behrend] If $E = \Omega^1_L$ and so s is a 1-form, then a 2-form of degree -1 corresponds to a pair of elements

$\alpha \in (\Omega^1_L)^\vee \otimes \Omega^2_L$ and $\phi \in (\Omega^1_L)^\vee \otimes \Omega^1_L$ such that $[\nabla, s^\flat](\phi) = s^\flat(\alpha)$.

Take $\phi = \mathrm{id} \in (\Omega^1_L)^\vee \otimes \Omega^1_L$. Suppose the local ∇ is chosen so that $\nabla(\mathrm{id}) = 0$ (i.e. ∇ is torsion free). Then $[\nabla, s^\flat](\mathrm{id}) = ds$.

Conclusion: The pair (α, id) gives a 2-form of degree -1 iff $ds = s^\flat(\alpha)$, or equivalently $ds_{|Z} = 0$, i.e. is an almost closed 1-form on L.

Symplectic structures

Recall: For X a smooth scheme/$\mathbb{C}$ is a **symplectic structure** is an $\omega \in H^0(X, \Omega_X^{2,cl})$ such that its adjoint $\omega^\flat : T_X \to \Omega_X^1$ is a sheaf isomorphism.

Note: Does not work for X singular (or stacky or derived):

- T_X and Ω_X^1 are too crude as invariants and get promoted to complexes $\mathbb{T}_X$ and $\mathbb{L}_X$.
- A form being closed is not just a condition but rather an extra structure.

Definition: X derived Artin stack locally of finite presentation (so that $\mathbb{L}_X$ is perfect).

- A n-shifted 2-form $\omega : \mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]$ - i.e. $\omega \in \pi_0(\mathcal{A}^2(X; n))$ - is nondegenerate if its adjoint $\omega^\flat : \mathbb{T}_X \to \mathbb{L}_X[n]$ is an isomorphism (in $D_{qcoh}(X)$).
- The space of n-shifted symplectic forms $Sympl(X; n)$ on $X/\mathbb{C}$ is the subspace of $\mathcal{A}^{2,cl}(X; n)$ of closed 2-forms whose underlying 2-forms are nondegenerate i.e. we have a homotopy cartesian diagram of spaces

$$\begin{array}{ccc} Sympl(X, n) & \longrightarrow & \mathcal{A}^{2,cl}(X, n) \\ \downarrow & & \downarrow \\ \mathcal{A}^2(X, n)^{nd} & \longrightarrow & \mathcal{A}^2(X, n) \end{array}$$

Shifted symplectic structures: examples (i)

- Nondegeneracy: a duality between the stacky (positive degrees) and the derived (negative degrees) parts of $\mathbb{L}_X$.
- $G = GL_n \rightsquigarrow BG$ has a canonical 2-shifted symplectic form whose underlying 2-shifted 2-form is

$$k \to (\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^\vee[-1] \wedge \mathfrak{g}^\vee[-1])[2] = Sym^2\mathfrak{g}^\vee$$

  given by the dual of the trace map $(A, B) \mapsto tr(AB)$.
- Same as above (with a choice of G-invariant symm bil form on $\mathfrak{g}$) for G reductive over k.
- The n-shifted cotangent bundle $T^\vee X[n] := \mathrm{Spec}_X(\mathrm{Sym}(\mathbb{T}_X[-n]))$ has a canonical n-shifted symplectic form.

Shifted symplectic structures: examples (ii)

Theorem: [PTVV] Let F be a derived Artin stack equipped with an n-shifted symplectic form $\omega \in Symp(F, n)$. Let X be an $\mathcal{O}$-compact derived stack equipped with an $\mathcal{O}$-orientation $[X] : \mathbb{H}(X, \mathcal{O}_X) \longrightarrow k[-d]$ of degree d. If the derived mapping stack $Map(X, F)$ is a derived Artin stack locally of finite presentation over k, then, $Map(X, F)$ carries a canonical $(n-d)$-shifted symplectic structure.

Remark:

0) Analog to Alexandrov-Kontsevich-Schwarz-Zaboronsky result.
1) A d $\mathcal{O}$-orientation on X is a kind of Calabi-Yau structure of dimension d;
2) A compact oriented topological d-manifold has an $\mathcal{O}$-orientation of degree d (Poincaré duality).

Lagrangian structures

Let (Y, ω) be a n-shifted symplectic derived stack. A lagrangian structure on a map $f : X \to Y$ is

- path γ in $\mathcal{A}^{2,\mathrm{cl}}(X; n)$ from $f^*\omega$ to 0
- that is 'non-degenerate' (in a suitable sense), i.e. the induced map $\theta_\gamma : \mathbb{T}_f \to \mathbb{L}_X[n-1]$ is an equivalence.

Examples:

- usual smooth lagrangians $L \hookrightarrow (Y, \omega)$ where (Y, ω) is a smooth (0)-symplectic scheme.
- there is a bijection between lagrangian structures on the canonical map $X \to (\mathrm{Spec}\, k, \omega_{n+1})$ and n-shifted symplectic structures on X (thus lagrangian structures generalize shifted symplectic structures)

Shifted symplectic structures: examples (iii)

Theorem: [PTVV] Let (F, ω) be n-shifted symplectic derived Artin stack, and $L_i \to F$ a map of derived stacks equipped with a Lagrangian structure, $i = 1, 2$. Then the homotopy fiber product $L_1 \times_F L_2$ is canonically a $(n-1)$-shifted derived Artin stack.

In particular, if $F = Y$ is a smooth symplectic Deligne-Mumford stack (e.g. a smooth symplectic variety), and $L_i \hookrightarrow Y$ is a smooth closed lagrangian substack, $i = 1, 2$, then the derived intersection $L_1 \times_F L_2$ is canonically (-1)-shifted symplectic.

Remark: An interesting case is the derived critical locus $\mathbb{R}Crit(f)$ for f a global function on a smooth symplectic Deligne-Mumford stack Y. Here

$$\begin{array}{ccc} \mathbb{R}Crit(f) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle df} \\ Y & \xrightarrow[0]{} & T^{\vee}Y \end{array}$$

Sheaves of derived stacks

Problem: Define and construct symplectic structures on the stalks of a sheaf of derived stacks $\mathscr{F}$ over a space S.

Note:

- For this to make sense the sections of the sheaf $\mathscr{F}$ will have to satisfy representability conditions.
- The non-degeneracy condition on a stalkwise symplectic form will have to involve some notion of duality for complexes of sheaves of $\mathbb{C}$-vector spaces on S.

Diagrams of sheaves and stacks

Fix

- $\mathbf{I}$ - a finitely presentable ∞-category;
- C - any category with finite limits.

Notation:

- $F_\bullet$ - a **diagram of shape $\mathbf{I}$ in C**.

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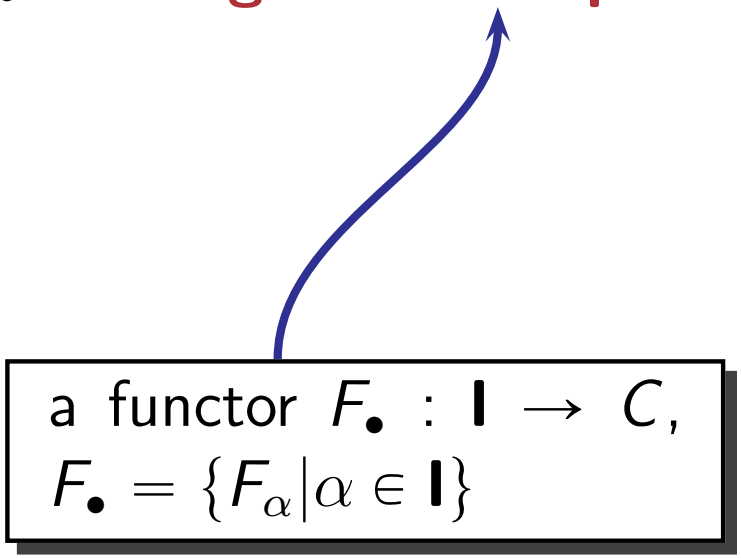

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- $F_{\mathbf{I}} = \lim_{\alpha \in \mathbf{I}} F_\alpha$ - **global sections of the diagram $F_\bullet$**.

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- $F_\bullet$ - a **diagram of shape $\mathbf{I}$ in C**.
- $F_{\mathbf{I}} = \lim_{\alpha \in \mathbf{I}} F_\alpha$ - **global sections of the diagram $F_\bullet$**.
- $\mathbf{I}^{\mathrm{tw}}$ - the category of **twisted arrows in $\mathbf{I}$**.

$\mathrm{ob}(\mathbf{I}^{tw})$: maps $x \xrightarrow{\gamma} y \in \mathrm{mor}(I)$;

$\mathrm{Hom}_{\mathbf{I}^{tw}} \begin{pmatrix} x_1 & , & x_2 \\ \downarrow\gamma_1 & & \downarrow\gamma_2 \\ y_1 & & y_2 \end{pmatrix}$: commutative diagrams $\begin{array}{ccc} x_1 & \xleftarrow{u} & x_2 \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2} \\ y_1 & \xrightarrow[v]{} & y_2 \end{array}$

Diagrams of sheaves and stacks

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- $F_{\mathbf{I}} = \lim_{\alpha \in \mathbf{I}} F_\alpha$ - **global sections of the diagram $F_\bullet$**.
- $\mathbf{I}^{\mathrm{tw}}$ - the category of **twisted arrows in $\mathbf{I}$**.
- $(t, s) : \mathbf{I}^{\mathrm{tw}} \to \mathbf{I} \times \mathbf{I}^{op}$ - the natural functor.

Closed forms on diagrams of stacks (i)

Fix $E_\bullet : \mathbf{I} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$, and $\mathscr{F}_\bullet : \mathbf{I} \longrightarrow \mathrm{dSt}_{\mathbb{C}}$.

Consider the functor

$$\mathcal{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} : \quad \mathbf{I}^{tw} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

$$\gamma \longrightarrow \mathcal{A}^{p,cl}(\mathscr{F}_{s(\gamma)}) \otimes E_{t(\gamma)},$$

and define the **complex of closed p-forms on $\mathscr{F}_\bullet$ with values in $E_\bullet$** as the complex

$$\mathbb{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} = \lim_{\gamma \in \mathbf{I}^{tw}} \mathcal{A}^{p,cl}(\mathscr{F}_{s(\gamma)}) \otimes E_{t(\gamma)}.$$

Closed forms on diagrams of stacks (iii)

The **space of closed p-forms on $\mathscr{F}_\bullet$ with values in $E_\bullet$** is defined as

$$\mathbf{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} = \left|\mathbf{DK}\left(\tau^{\leqslant 0}\mathbb{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet}\right)\right|$$

and an **$E_\bullet$-valued closed p-form on $\mathscr{F}_\bullet$** is an element in $\pi_0\mathbf{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} = H^0(\mathbb{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet})$.

Note: The space of forms comes equipped with a natural **global sections morphism**

$$\Gamma : \mathbb{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} \to \mathcal{A}^{p,cl}(\mathscr{F}_{\mathbf{I}}) \otimes E_{\mathbf{I}}.$$

Cospecialization and gluing (i)

Suppose

- X - nice topological space (e.g. a CW complex);
- $\imath : Z \hookrightarrow X$ - closed subspace;
- $\jmath : U \hookrightarrow X$ - the complementary open subspace.
- $\mathscr{C}$ -an ∞-category with all small limits and colimits.

For any $F \in \mathrm{Sh}(X, \mathscr{C})$ write $F_{|Z} = \imath^* F$ and $F_{|U} = \jmath^* F$. Applying $\imath^*$ to the unit of the adjunction $\jmath^* \dashv \jmath_*$ yields a **cospecialization map** in $\mathrm{Sh}(Z, \mathscr{C})$:

$$\mathrm{cosp}_Z : F_{|Z} \to \imath^* \jmath_* \left(F_{|U} \right).$$

Cospecialization and gluing (ii)

The assignment $F \longrightarrow \left(F_{|U}, F_{|Z}, \mathrm{cosp}_Z\right)$ provides an equivalence of ∞-categories:

$$\mathrm{Sh}(X, \mathscr{C}) \xrightarrow{\cong} \mathrm{lax}^{\mathrm{op}} \lim \left[\mathrm{Sh}(U, \mathscr{C}) \xrightarrow{i^* \jmath_*} \mathrm{Sh}(Z, \mathscr{C}) \right],$$

i.e. $\mathrm{Sh}(X, \mathscr{C})$ can be viewed as the $\mathrm{lax}^{\mathrm{op}}$ limit of the functor $i^* \jmath_*$.

Key observation: Applying this gluing description to strata in a stratification leads to a combinatorial picture for closed forms and symplectic structures on constructible sheaves of stacks over a space.

Sheaves on a stratified space (i)

Suppose

- X is a **good stratified space**.
- I is the finite poset labeling the strata of X.
- $X_\alpha \subset X$ - the stratum labeled by $\alpha \in I$.
- $\mathrm{Sh}^{\mathrm{str}}(X)$ - sheaves F of spaces, constructible for the given stratification.
- $F_\alpha \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha)$ - the restriction of F to X_α.

Sheaves on a stratified space (ii)

Construction: Fix $\alpha \in I$, $F \in \mathrm{Sh}^{\mathrm{str}}(X)$, and let

- $Z \subset X$ - a closed subset s.t. $X_\alpha \subset Z$ is open.
- $U \subset X$ - the complementary open to Z.
- $\overset{\circ}{F}_\alpha := \left(i^* j_* \left(F_{|U}\right)\right)_{|X_\alpha}$.
- $\mathrm{cosp}_\alpha : F_\alpha \to \overset{\circ}{F}_\alpha$ - the restriction of cosp_Z to X_α.

Note:

- $\overset{\circ}{F}_\alpha$ and cosp_α depend only on α and not on Z.
- $\overset{\circ}{F}_\alpha$ is the sheaf of nearby (co) cycles of F along X_α, and cosp_α is the integral of cospecialization maps over nearby points.

Sheaves on a stratified space (iii)

Note: As with gluing, nearby cycles, and integrated cospecializations make sense for constructible sheaves with values in any category C that admits finite limits.

Let $\mathbf{I}$ be the **∞-category of exit paths** for the stratification on X. Then

- If I is good, then $\mathbf{I}$ is finitely presentable.
- $\mathrm{Sh}^{\mathrm{str}}(X) = \mathrm{Fun}(\mathbf{I}, \mathrm{SSets})$.
- For any category C with finite products we define

$$\mathrm{Sh}^{\mathrm{str}}(X, C) = \mathrm{Fun}(\mathbf{I}, C),$$

i.e. C-valued constructible sheaves on X are $\mathbf{I}$-shaped diagrams in C.

Sheaves on a stratified space (iii)

Notation: Fix $F \in \mathrm{Sh}^{\mathrm{str}}(X, C)$. Then:

- Every $x \in X$ gives an object $\boldsymbol{\iota}(x) \in \mathbf{I}$. The value $F(\boldsymbol{\iota}(x)) \in C$ is called the **stalk of F at x** and is denoted by F_x.
- $\Gamma(X, F) := \lim_{\boldsymbol{\sigma} \in \mathbf{I}} F_{\boldsymbol{\sigma}} \in C$ is the **global sections** object of F. We have a natural **evaluation map** $\mathrm{ev}_x : \Gamma(X, F) \to F_x$ for every $x \in X$.
- For $\alpha \in I$ the fundamental groupoid $\Pi_1(X_\alpha)$ of the stratum X_α embeds in $\mathbf{I}$ as the full subcategory $\mathbf{I}_\alpha \subset \mathbf{I}$ spanned by $\boldsymbol{\iota}(x)$ for all $x \in X_\alpha$. We define the **value of F on X_α** as $F_\alpha := F_{|\mathbf{I}_\alpha} \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha, C)$.

Sheaves on a stratified space (iv)

Notation: Fix $\alpha \in I$, and $F \in \mathrm{Sh}^{\mathrm{str}}(X, C)$. Then consider:

- $\mathbf{I}_\alpha^{\geqslant} = \{a' \to a \mid a' \in \mathbf{I}, a \in \mathbf{I}_\alpha\}$
- $\mathbf{I}_\alpha^{>} \subset \mathbf{I}_\alpha^{\geqslant}$ - the full subcategory consisting of all $a' \to a$ which are **not** isomorphisms.
- $\overset{\circ}{F}_\alpha \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha, C)$ - the right Kan extension of $F_{|\mathbf{I}_\alpha^{>}}$ along the natural functor $\mathbf{I}_\alpha^{>} \to \mathbf{I}_\alpha$.
- $\mathrm{cosp}_\alpha : F_\alpha \to \overset{\circ}{F}_\alpha$ - the map guaranteed by the universal property of the right Kan extension.

Terminology: $\overset{\circ}{F}_\alpha$ is the **sheaf of nearby cycles** of F at X_α, and cosp_α is the (integrated) **cospecialization map**.

Sheaves on a stratified space (v)

Given a nice stratified space X with strata labeled by a poset I, and an exit path category $\mathbf{I}$, and
$E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}}) = \mathrm{Fun}(\mathbf{I}, \mathrm{Vect}_{\mathbb{C}})$,
$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) = \mathrm{Fun}(\mathbf{I}, \mathrm{dSt}_{\mathbb{C}})$, we get

- a **complex** $\mathbb{A}^{p,cl}_X(\mathscr{F})_E := \mathbb{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet}$ and a **space** $\mathbf{A}^{p,cl}_X(\mathscr{F})_E := \mathbf{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet}$ **of global closed E-valued p-forms on $\mathscr{F}$**;
- a natural **global sections map** of complexes

$$\Gamma : \mathbb{A}^{p,cl}_X(\mathscr{F})_E \to \mathcal{A}^{p,cl}(\Gamma(X, \mathscr{F})) \otimes \Gamma(X, E).$$

Sheaves on a stratified space (vi)

Let $\mathscr{F}$ be a constructible sheaf of locally f.p. derived Artin stacks. For any point $z \in \Gamma(X, \mathscr{F})$ the relative tangent complex $\mathbb{T}_{\mathscr{F},z} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$ is a constructible complex of vector spaces on X.

Given a point $z \in \Gamma(X, \mathscr{F})$, any closed form $\omega \in \mathbf{A}^{p,cl}_X(\mathscr{F})_E$ defines a map of constructible complexes on X

$$\omega^{\flat}_z : \bigwedge^p \mathbb{T}_{\mathscr{F},z} \to E.$$

Definition: The induced map $\Gamma(\omega^{\flat}_z)$ on global sections is the **value of ω at z**.

Sheaves on a stratified space (vi)

Note: All of this readily sheafifies:

- For reasonable open sets $U \subset X$ the assignment

$$U \longrightarrow \mathbb{A}^{p,cl}_U(\mathscr{F}_{|U})_{E_{|U}}$$

is a presheaf of complexes. Its sheafification $\underline{\mathcal{A}}^{p,cl}_X(\mathscr{F})_E$ is a sheaf of complexes on X equipped with a natural map

$$\Gamma : \Gamma(X, \underline{\mathcal{A}}^{p,cl}_X(\mathscr{F})_E) \to \mathcal{A}^{p,cl}(\Gamma(X, \mathscr{F})) \otimes \Gamma(X, E).$$

- In terms of diagrams $\underline{\mathcal{A}}^{p,cl}_X(\mathscr{F})_E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$ is constructed as the right Kan extension of $\mathcal{A}^{p,cl}_{\mathbf{I}} : \mathbf{I}^{tw} \to \mathrm{Vect}_{\mathbb{C}}$ along the functor $t : \mathbf{I}^{tw} \to \mathbf{I}$.

Non-degeneracy (v)

Setup:

- X - nicely stratified space with equidimensional strata.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ is a constructible sheaf of locally f.p. derived Artin stacks (or just locally formally representable derived stacks).
- $E = K_X[n] \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$, where $n \in \mathbb{Z}$ and K_X is the Verdier's dualizing complex of X.

Non-degeneracy (vi)

Definition:

(a) The **complex and space of relative n-shifted closed p-forms on $\mathscr{F}$** are defined to be

$$\mathbb{A}^{p,cl}_X(\mathscr{F}, n) := \mathbb{A}^{p,cl}_X(\mathscr{F})_{K_X[n]},$$
$$\mathbf{A}^{p,cl}_X(\mathscr{F}, n) := \mathbf{A}^{p,cl}_X(\mathscr{F})_{K_X[n]}.$$

(b) A closed relative n-shifted 2-form $\omega \in \mathbf{A}^{p,cl}_X(\mathscr{F}, n)$ is **symplectic** if for every $U \subset X$ and any point $z \in \Gamma(U, \mathscr{F})$, the map

$$\omega^{\flat}_z : \ \mathbb{T}_{\mathscr{F}_{|U},z} \longrightarrow \mathbb{L}_{\mathscr{F}_{|U},z} \otimes K_U[n]$$

is a quasi-isomorphism.

Non-degeneracy and nearby cycles (i)

Let $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$, $E = K_X[n]$, $\alpha \in I$.

- $\mathscr{F}_\alpha$ and $\overset{\circ}{\mathscr{F}}_\alpha$ are local systems of derived stacks on X_α, i.e. are derived stacks equipped with an action of $\Pi_1(X_\alpha) = \mathbf{I}_\alpha$.
- The fiber of $\mathrm{cosp}^E_\alpha : E_\alpha \to \overset{\circ}{E}_\alpha$ is equal to the !-restriction of $K_X[n]$ to X_α and so we have an exact triangle

$$(*) \qquad E_\alpha \xrightarrow{\mathrm{cosp}^E_\alpha} \overset{\circ}{E}_\alpha \longrightarrow \mathbb{C}\left[n + 1 + \dim X_\alpha\right].$$

Note: When $\alpha \in I$ is maximal we have $\overset{\circ}{E}_\alpha = 0$ and so $E_\alpha = \mathbb{C}\left[n + \dim X_\alpha\right]$.

Non-degeneracy and nearby cycles (ii)

Let $\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F}, n) = \mathbf{A}^{2,cl}_X(\mathscr{F})_E$ be a relative n-shifted closed 2-form on $\mathscr{F}$. Then ω induces

ω_α a relative closed E_α-valued 2-form on $\mathscr{F}_\alpha$, i.e. $\omega_\alpha \in \mathbf{A}^{2,cl}_{X_\alpha}(\mathscr{F}_\alpha)_{E_\alpha}$.

$\overset{\circ}{\omega}_\alpha$ a closed $\overset{\circ}{E}_\alpha$-valued 2-form on $\overset{\circ}{\mathscr{F}}_\alpha$, i.e. $\overset{\circ}{\omega}_\alpha \in \mathbf{A}^{2,cl}_{X_\alpha}(\overset{\circ}{\mathscr{F}}_\alpha)_{\overset{\circ}{E}_\alpha}$.

$\overline{\omega}_\alpha$ a closed $(n+1)$-shifted 2-form on $\overset{\circ}{\mathscr{F}}_\alpha$, i.e.

$$\overline{\omega}_\alpha \in \mathbf{A}^{2,cl}_{X_\alpha}(\overset{\circ}{\mathscr{F}}_\alpha, n+1) = \mathbf{A}^{2,cl}_{X_\alpha}(\overset{\circ}{\mathscr{F}}_\alpha)_{\mathbb{C}[n+1+\dim X_\alpha]}.$$

Non-degeneracy and nearby cycles (iii)

Note:

- $\overline{\omega}_\alpha$ is the pushout of $\mathring{\omega}_\alpha$ by the map

$$\mathring{E}_\alpha \to \mathbb{C}\left[n+1+\dim X_\alpha\right].$$

- Viewing $\mathring{\mathscr{F}}_\alpha$ as a constant derived Artin stack equipped with a $\Pi_1(X_\alpha)$ action, then we can view $\overline{\omega}_\alpha$ equivalently as an **absolute** $(n+1+\dim X_\alpha)$-shifted 2-form, i.e.

$$\overline{\omega}_\alpha \in \mathcal{A}^{2,cl}(\mathring{\mathscr{F}}_\alpha, n+1+\dim X_\alpha).$$

Non-degeneracy and nearby cycles (iv)

Key observation: Consider the $\Pi_1(X_\alpha)$-equivariant cospecialization map

$$\mathrm{cosp}_\alpha^{\mathscr{F}} : \mathscr{F}_\alpha \to \overset{\circ}{\mathscr{F}}_\alpha$$

of derived Artin stacks.
The exact triangle

$$E_\alpha \xrightarrow{\mathrm{cosp}_\alpha^E} \overset{\circ}{E}_\alpha \longrightarrow \mathbb{C}\left[n+1+\dim X_\alpha\right].$$

yields a natural path h_α between $\mathrm{cosp}_\alpha^{\mathscr{F}*}(\overline{\omega}_\alpha)$ and 0 in the space $\mathbf{A}^{2,cl}_{X_\alpha}(\mathscr{F}_\alpha, n+1)$ or equivalently a path between $\mathrm{cosp}_\alpha^{\mathscr{F}*}(\overline{\omega}_\alpha)$ and 0 in the space $\mathcal{A}^{2,cl}(\mathscr{F}_\alpha, n+1+\dim X_\alpha)$.

Non-degeneracy and nearby cycles (v)

Theorem: **[Arinkin-P-Toën]** Suppose

$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of derived Artin stacks, locally of f.p.;

$\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F}, n)$ - a relative closed n-shifted 2-form on $\mathscr{F}$.

Then ω is symplectic if and only if for every $\alpha \in I$ the following two conditions hold:

(a) $\overline{\omega}_\alpha$ is symplectic.

(b) $\mathrm{cosp}^{\mathscr{F}}_\alpha : (\mathscr{F}_\alpha, h_\alpha) \to (\overset{\circ}{\mathscr{F}}_\alpha, \overline{\omega}_\alpha)$ is Lagrangian.

Claim: **[Arinkin-P-Toën]** Fix $\beta \in I$ and assume that **(a)** and **(b)** hold for all $\alpha > \beta$. Then $\overline{\omega}_\beta$ is a shifted symplectic form on $\overset{\circ}{\mathscr{F}}_\beta$.

Sheaves and functors (i)

X - locally compact Hausdorff space;
$\mathscr{U}(X)$ - the poset of opens in X;
$\mathscr{C}$ - an ∞-category which admits all small limits.

Definition: A $\mathscr{C}$-valued **sheaf** on X is a functor $F : \mathscr{U}(X)^{op} \to \mathscr{C}$ satisfying the **sheaf condition**: for every open cover $\{U_\alpha\}$ of an open set U the natural map

$$F(U) \longrightarrow \varprojlim_{V} F(V)$$

is an equivalence in $\mathscr{C}$. Here the limit is taken over all open subsets $V \subset U$ which are contained in one of the U_α.

Sheaves and functors (ii)

Remark: Equivalently a $\mathscr{C}$-valued functor $F : \mathscr{U}(X)^{op} \to \mathscr{C}$ is a **sheaf** if

- For every sequence $U_0 \subset U_1 \subset \cdots \subset U_k \subset \cdots$ of opens in X the natural map $F(\cup U_i) \to \varprojlim_i F(U_i)$ is an equivalence in $\mathscr{C}$.
- The object $F(\varnothing)$ is terminal in $\mathscr{C}$.
- For every pair of opens $U, V \subset X$ the diagram

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

is a pullback square in $\mathscr{C}$.

Stratifications (i)

Let I be a poset viewed as a topological space for the **upward closed topology:** $U \subset I$ is open if and only if $x \in U$ implies $y \in U$ for all $y \geqslant x$ in I.

Definition: An **I-stratification** of a topological space X is a continuous map $a : X \to I$. The **stratum** in X corresponding to $\alpha \in I$ is the subset $X_\alpha = a^{-1}(\alpha)$.

Note: We will only be working with stratifications that satisfy a regularity condition.

Stratifications (ii)

Notation: If I is a poset, write $I^{\triangleleft}$ for the poset obtained from I by adjoining a new smallest element $-\infty$.

Definition: Let $a : X \to I$ be a stratified space. The **cone over** X is the $I^{\triangleleft}$-stratified space $C(X)$ defined as follows:

- As a set $C(X) = \{*\} \bigsqcup (X \times \mathbb{R}_{>0})$.
- A subset $U \subset C(X)$ is open if and only if $U \cap (X \times \mathbb{R}_{>0})$ is open, and if $* \in U$, then $X \times (0, \epsilon) \subset U$ for some positive real ϵ.
- $C(X)$ is stratified by the map $a^{\triangleleft} : C(X) \to I^{\triangleleft}$ given by $a^{\triangleleft}(*) = -\infty$, and $a^{\triangleleft}(x, t) = a(x)$ for $(x, t) \in X \times \mathbb{R}_{>0}$.

Stratifications (iii)

Definition: [Lurie] An I-stratification on X is **conical** if every point $x \in X_\alpha \subset X$ has a stratified neighborhood U_x which is stratified homeomorphic to a product $Z \times C(Y)$ where Z is a topological space, and Y is a $I_{<\alpha}$-stratified space.

Definition: An I-stratification on X is **good** if I is finite and every point $x \in X_\alpha \subset X$ has a stratified neighborhood U_x which is stratified homeomorphic to a product $\mathbb{R}^k \times C(Y)$ for some k and some compact $I_{<\alpha}$-stratified space Y.

Back

Exit paths (i)

Setup:

- X be a stratified space with strata labeled by a poset I.
- $|\Delta^n| = \left\{ (t_0, \ldots, t_n) \in [0,1]^{\times n} \;\middle|\; \sum_{i=0}^{1} = 1 \right\}$ is the standard simplex.

Definition: The **simplicial set of exit paths** of X is the simplicial subset $\mathbf{I} \subset \mathrm{Sing}(X)$ consisting of those simplices $\sigma : |\Delta^n| \to X$ that satisfy the condition

$$(*) \quad \left(\begin{array}{l} \text{There exists a chain of elements } \alpha_0 < \alpha_1 < \cdots < \alpha_n \in I \\ \text{so that for every point } (t_0, t_1, \ldots, t_i, 0, \ldots, 0) \in |\Delta^n| \\ \text{with } t_i \neq 0 \text{ we have that } \sigma(t_0, t_1, \ldots, t_i, 0, \ldots, 0) \in X_{\alpha_i}. \end{array} \right)$$

Exit paths (ii)

Theorem: [Lurie]

(a) If the stratification on X is conical, then $\mathbf{I}$ is an ∞-category.

(b) Let X be a paracompact topological space which is locally of a singular shape, and is equipped with a conical I-startification. Then the ∞-category of I-constructible sheaves of spaces on X is equivalent to the ∞-category $\mathrm{Fun}(\mathbf{I}, \mathrm{SSets})$.

Back