String duality and G_2 manifolds

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Motivation

- Manifolds with special holonomy X play important roles in the study of string theory and supersymmetric field theory. One can learn non-trivial lessons of physical systems from using the geometric property of X, and the physical methods can often provide unexpected insights about the geometry of X. An example is the discovery of mirror symmetry of Calabi-yau manifolds.
- This talk has two parts: the first part reviews the mirror construction of G_2 manifolds motivated by string duality (Gukov-Yau-Zaslow); The second part is mainly about the physical interpretation of K stability of the existence of three dimensional Ricci-flat conical metric (equivalently Sasaki-Einstein metric and Kahler-Einstein metric on Fano orbifolds) (Collins-Xie-Yau), and such metric can be used to construct non-compact G_2 manifolds.

Part I

Mirror symmetry for G_2 Manifolds and associative/co-associative fibration

String/M theory duality

String duality relates one type of string theory with another type of string theory. It typically involves the compactification of string theory on different manifolds. The basic string/M theory dualities that will be used in this talk are (Witten 1995):

- 11d M theory compactified on a circle is equivalent to 10d type IIA string theory, i.e. the strongly coupled limit of IIA string theory is 11d supergravity.
- Type IIA string theory compactified on a circle S_A is dual to type IIB string theory on a dual circle S_B. This is the T duality symmetry. More generally, Type IIA string theory compactified on odd dimensional torus is equivalent to type IIB string on dual torus.
- M theory compactified on K_3 manifold is equivalent to Heterotic string theory compactified on T^3.
- Type IIA string theory comapctified on K_3 surface is equivalent to Heterotic string theory on T^4.

D branes

One of major discovery in the development of string theory in the middle 90s is the realization of importance of higher dimensional objects. An important class of such objects are called D branes. In fact, D brane is the major motivation of the Strominger-Yau-Zaslow picture of mirror symmetry.

One of new feature of D brane is: previous studies of CY manifolds in string theory mainly explores the deformation space of KE metrics, but D brane probes geometry directly: for example, the physical theory on D0 brane has a moduli space which coincides with the manifold that it probes! String duality often maps D brane of one string theory to D brane of mirror string theory.

SYZ picture for mirror symmetry

The well-known conjecture of Strominger, Yau and Zaslow provides a geometric picture of mirror symmetry, at least in the so-called large complex structure limit. The conjecture proposes mirror pairs of Calabi-Yau manifolds which are special Lagrangian (slg) torus fibrations over the same base, but with dual fibres. This is motivated by T duality of type II string theory.

Consider $D3$ branes wrapping on T^3 fibre (type IIB theory), and we do T duality along the T^3 fibre and get type IIA string theory. D3 brane now becomes $D0$ brane on the mirror manifold. The quantum moduli space of $D3$ brane should be equivalent to D0 brane moduli space, which is equal to the mirror manifold. So by studying the quantum moduli space of D3 brane wrapping on Slg torus fibre, we could get the information of mirror manifold!

G_2 manifolds

SYZ picture of mirror symmetry is mainly about the string compactification on three dimensional Calabi-Yau manifolds. These manifolds preserves half of supersymmetry. They give four dimensional $\mathcal{N} = 2$ supersymmetric theory if we consider type II string theory on such manifolds. To have minimal four dimensional supersymmetric field theory, we should consider M theory compactified on seven dimensional G_2 manifolds (Another possibility is F theory compactified on elliptic four-folds).

G_2 manifolds is defined by a positive three form ϕ, which is closed and its dual four form is also closed. They admit covariant constant spinor and therefore will give supersymmetric theory if we compactify string/M theory on such manifolds.

Duality involving G_2 manifolds

We can get interesting string duality involving G_2 manifolds by using the basic string duality reviewed earlier. The basic idea is following:

- If our manifold has a fibration (In certain geometric limit) whose fibre is the geometric object appearing in the basic string duality, we can do the fibre-wise string duality to get a mirror manifold over the same base.
- The fibre should be calibrated cycle, and D brane wrapping on those cycles is supersymmetric and one can learn the mirror manifolds by studying the D brane moduli space.
- The fibration usually involves singular fibre, and it is conjectured that those singular fibres would not change the fibre-wise string duality picture.

Co-associative fibration and Calabi-Yau mirror

Now for the G_2 manifolds, there are two kinds of calibrated cycles: co-associative four-folds and associative three-folds. Consider co-associative fibration, we have the following possibilities (Gukov-Yau-Zaslow):

- We can consider M theory on G_2 manifold admitting co-associative T_4 fibration. We can first do reduction along one of T_4 direction and get type IIA string on CY manifolds with fluxes turned on, and then perform T duality along the remaining T_3 fibre to get type IIB string theory on a "mirror" CY3 fold with fluxes turned on.
- We can consider M theory on G_2 manifold admitting co-associative K_3 fibration. We can use the basic duality of M theory on K_3 and heterotic string on T_3, and the dual theory is heterotic string on T_3 fibred mirror Calabi-Yau manifold.

In each case, we can wrap M5 brane on co-assocative fibre, and the study of the quantum moduli space of it should give us the information of mirror manifolds.

Co-associative fibration and G_2 mirror

We can also study type II string theory on G_2 manifold (Acharya), and get mirror manifold which is also a G_2 manifold. There are several interesting scenarios

- Type II string theory on G_2 manifold admitting co-associative T_4 fibration.
- Type II string theory on G_2 manifold admitting co-associative K_3 fibration.
- Type II string theory on G_2 manifold admitting associative T_3 fibration.

One can use fibre-wise string duality to get type II string theory on mirror G_2 manifold which admits the same type of fibration. More interestingly, using type IIA on K_3 and heterotic string on T_4 duality, one can get duality between type IIA string on K_3 fibred G_2 manifold and heterotic string on T_4 fibered G_2 manifold.

K_3 fibration

Gukov-Yau-Zaslow had proposed some methods to construct G_2 manifolds admitting co-associative/associative fibration (see more recent discussions by Baraglia, Kovalev, Donaldon, etc). Here we review the basic ideas.

Imagine a G_2 manifold which is a K_3 fibration over a base S^3, with a discriminant locus Δ, which we assume to be a closed manifold of co-dimension two — a knot or link. If we consider the case of a non-satellite knot, then by Thurston's theorem there exists a hyperbolic metric on the complement S^3/Δ. We use this reasoning to look for a G_2 structure on a K_3 fibration X over a non-compact hyperbolic manifold. For simplicity, one use $B = SO(3,1)/SO(3)$ as the base, and use metric g_B left invariant by $SO(3,1)$.

We write

$$\pi : X \to B \tag{1}$$

for the projection to base. Note that at a point $p \in X$ the vertical vectors are defined as the kernel of π and span a sub-bundle $T_V X$ of TX, but there is no canonical notion of horizontal vectors until we have a connection, i.e. a choice of "horizontal" subbundle $T_H X$ of TX. We showed that there is a canonical way to decompose TX as $TX = T_H X \oplus T_V X$, and we write P_H and P_V for the corresponding projection operators. One can then construct a three form Φ using the projection operator P_H and P_V. The three form ϕ constructed there is not closed though, and it is interesting to further study the deformation of this ϕ so that one can find a G_2 structure.

Torus fibration

Hitchin has shown how certain functionals on differential forms in six dimensions generate metrics with G2 and weak SU(3) holonomy. Here, we outline his construction and use his result to construct new G2 metrics. The main point is to consider the Hamiltonian flow of a volume functional on a symplectic space of stable three- and four-forms on a six-manifold. When a group acts on the six-fold, the invariant differential forms can restrict the infinite-dimensional variational problem to a finite-dimensional set of equations governing the evolution. Including the "time" direction, one is able to create a closed and co-closed G2 three-form, thus a metric of G2 holonomy.

Gukov-Yau-Zaslow used Hitchin's method to produce non-compact G_2 manifolds admitting T^3 fibration.

Branes wrapping on transverse calibrated cycle

In the context of SYZ picture of mirror symmetry, one can also study the brane wrapping on slg sub-manifold intersecting with the fibre once. The mirror of it involves deformed Hermitian Yang-Mills (dHYM) equation. Now for the associative and co-associative fibration, we could also study the brane wrapping on calibrated cycles intersecting with the fibres once, and it would be interesting to study the analog of dHYM in this context.

String duality suggests that associative and co-associative fibration of G_2 manifolds are very useful to understand the mirror symmetry involving G_2 manifolds. Many details of these mirror symmetry involving G_2 manifolds are remained to be studied.

Up to now, the two main constructions of compact G_2 manifolds are Joyce's orbifold construction and twisted connected sum. Can we use above mirror symmetry of G_2 manifold to construct more G_2 manifolds?

Part 2

K stability and dynamics of supersymmetric field theory

The construction of non-compact G_2 manifold is easier than compact ones. One can construct non-compact G_2 manifolds starting with a six dimensional Ricci-flat conical metric (Foscolo, Haskins, Nordstrom). The existence of Ricci-flat conical metric is given by K stability, here we will discuss how to interpret K stability from string theory point of view.

Canonical singularity

If we deform a compact CY manifold, it could develop singularity which is proven to be canonical singularity. A canonical singularity X is normal and satisfies following conditions (Reid 81):

- The Weyl divisor K_X is Q-Cartier, i.e. there is an integer r such that rK_X is a Cartier divisor.
- For any resolution of singularity $f : Y \to X$, with exceptional divisors $E_i \in Y$, we have

$$K_Y = f^* K_X + \sum_i a_i E_i, \tag{2}$$

  with $a_i \geq 0$. r is called index of the singularity. If $a_i > 0$ for all exceptional divisors, it is called **terminal** singularity.

Two dimensional canonical singularity has a ADE classification:

$$\begin{aligned} &A_n : \quad x^2 + y^2 + z^n = 0, \quad D_n : \quad x^2 + y^{n-1} + zy^2 = 0 \\ &E_6 : \quad x^2 + y^3 + z^4 = 0, \; E_7 : \quad x^2 + y^3 + yz^3 = 0, \\ &E_8 : \quad x^2 + y^3 + z^5 = 0 \end{aligned} \tag{3}$$

We are mainly interested in three dimensional $\mathbb{Q}$-Gorenstein canonical singularity with a C^* action, so the index $r = 1$. There is no complete classification, and the space of such singularities is very large:

- Quotient singularity C^3/G, with $G \in SL(3)$.
- Toric Gorenstein singularity.
- Quasi-homogeneous isolated hypersurface singularity $f(z_1, z_2, z_3, z_4)$ satisfying the condition

$$f(\lambda^{q_i} z_i) = \lambda f(z_i), \quad \sum q_i > 1. \tag{4}$$

- Isolated complete intersection singularity defined by two polynomials f_1 and f_2 (5 is the maximal embedding dimension for 3d canonical singularity), the weights and degrees of f_1 and f_2 are $(w_1, \ldots, w_5; d_1, d_2)$, and the canonical condition is

$$\sum_{i=1}^{5} w_i - d_1 - d_2 > 0. \tag{5}$$

Here we also list 3d Gorenstein terminal singularity with C^* action

j	Singularity
A_{N-1}	$x_1^2+x_2^2+x_3^N+z^k=0$
	$x_1^2+x_2^2+x_3^N+x_3z^k=0$
D_N	$x_1^2+x_2^{N-1}+x_2x_3^2+z^k=0$
	$x_1^2+x_2^{N-1}+x_2x_3^2+z^kx_3=0$
E_6	$x_1^2+x_2^3+x_3^4+z^k=0$
	$x_1^2+x_2^3+x_3^4+z^kx_3=0$
	$x_1^2+x_2^3+x_3^4+z^kx_2=0$
E_7	$x_1^2+x_2^3+x_2x_3^3+z^k=0$
	$x_1^2+x_2^3+x_2x_3^3+z^kx_3=0$
E_8	$x_1^2+x_2^3+x_3^5+z^k=0$
	$x_1^2+x_2^3+x_3^5+z^kx_3=0$
	$x_1^2+x_2^3+x_3^5+z^kx_2=0$

For isolated three dimensional canonical singularity X with a chosen C^* action ζ, the link L_X of X is a smooth manifold and has a Sasakian structure (For the existence of Sasakian structure on the link L_X, the singularity can be relaxed to be isolated log terminal singularity). Sasakian manifold has a distinguished isometry which is related to chosen C^* action ζ on X.

We would like to determine whether L_X has Sasaki-Einstein metric, namely, determine the pair (X, ζ) such that L_X has SE metric whose Reeb vector field is given by ζ. Given (X, ζ), there are two situations:

- Given X, we can tune ζ such that L_X has SE metric.
- Given X, there is no choice of ζ such that L_X has SE metric.

Ricci-flat conical metric

Equivalently, we are interested in following Ricci-flat conical metric (it is also Kahler and has an isometry determined by ζ) on singularity X:

$$ds^2 = dr^2 + r^2 dg_{L_X}. \tag{6}$$

The existence of above metric on X implies:

- The link L_X has a Sasaki-Einstein metric.
- If the Sasakian manifold is quasi-regular, then it can be regarded as the total space of a circle bundle over a Fano orbifold S, and the existence of Ricci-flat conical metric is equivalent to the existence of the Kahler-Einstein metric on S.

K stability

The necessary and sufficient condition of the existence of Sasaki-Einstein metric on L_X is given by K stability (Chen-Donaldson-Sun, Collins-Szekelyhidi 15) of (X, ζ). We give definition of K stability relevant for Sasaki-Einstein metrics.

- Test configurations, or flat degenerations of the affine variety X embedded in some ambient space.
- Associated to such a degeneration, a number called the Futaki invariant.

To explain this in more detail, fix an affine variety X, defining a Gorenstein (for simplicity) affine variety of dimension $n+1$ with an isolated log-terminal singularity at $0 \in X$. The condition that the link of X admits a Sasakian structure guarantees there is a (maximal) torus $T \subset Aut(X)$ such that there is an element $\zeta \in \mathfrak{t}$ (generating a Reeb field) with the property that if f is any T-equivariant holomorphic function on X, then $L_\zeta f = \sqrt{-1}\lambda f$ for $\lambda = \lambda(f, \zeta) > 0$.

K-stability

More concretely, we can assume that $X \hookrightarrow \mathbb{C}^N$ in such a way that T is contained in the diagonal torus in $U(N)$. Write

$$X = Spec \frac{\mathbb{C}[z_1, \dots, z_N]}{(f_1, \dots, f_k)}$$

where $f_1, \dots, f_k$ are T-equivariant polynomials. Then there is an open (convex, polyhedral) cone $\mathcal{C}_R \subset \mathfrak{t} = Lie(T)$ such that, for any $\zeta \in \mathcal{C}_R$ we have

$$\mathcal{L}_\zeta z_i = \sqrt{-1}\lambda_i z_i$$

for $\lambda_i > 0$. The cone $\mathcal{C}_R$ is called the Reeb cone, and it's elements are called Reeb vector fields.

K-stability

Since X is Gorenstein there is a unique up to scale, non-vanishing holomorphic section $\Omega \in H^0(X, K_X)$. Consider the set

$$\Sigma_R = \{\zeta \in \mathcal{C}_R : \mathcal{L}_\zeta \Omega = \sqrt{-1}(n+1)\Omega\}.$$

It turns out this is a compact, affine slice (or cross section) of $\mathcal{C}_R$. We will call Reeb fields in Σ_R *normalized*.

The ring

$$\mathcal{H} = \frac{\mathbb{C}[z_1, \dots, z_N]}{(f_1, \dots, f_k)}$$

is positively multigraded by the Lie algebra $\mathfrak{t}$. More precisely, $\mathcal{H}$ decomposes under the T action into weight spaces

$$\mathcal{H} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathcal{H}_\alpha$$

and we can define the index character

$$\mathcal{C}_R \ni \zeta \mapsto F(\zeta, t) = \sum_{\alpha \in \mathfrak{t}^*} dim\, \mathcal{H}_\alpha e^{-\alpha(\zeta)t}.$$

According to (Collins-Székelyhidi) the index character admits a meromorphic expansion

$$\mathcal{C}_R \ni \zeta \mapsto F(\zeta, t) = \frac{a_0(\zeta)}{t^{n+1}} + \frac{a_1(\zeta)}{t^n} + O(t^{1-n})$$

where a_0, a_1 are smooth functions on $\mathcal{C}_R$. A key point is that all of this discussion still makes sense when X is only an affine scheme.

Define a test configuration, or degeneration to be a choice of T-equivariant holomorphic functions $f_1, \ldots, f_M$ generating $\mathcal{H}$, and a choice of weights $w_1, \ldots, w_M \in \mathbb{R}$. Associated to this data we get a degeneration of X by embedding X into $\mathbb{C}^M$, and then acting on the generators by $z_i \mapsto t^{w_i} z_i$. Taking a flat limit over zero yields a scheme X_0 acted on by a torus $T_0 \supset T$. We say the degeneration is **special** if X_0 is a normal affine variety.

Associated to this degeneration is the **Futaki invariant**. If $\eta \in \mathfrak{t}_0 \backslash \mathfrak{t}$ be chosen tangent to the normalized Reeb fields $\Sigma_{R,0}$ of X_0. Then the Futaki invariant is (up to a positive constant)

$$Fut(\zeta, \eta) = \frac{d}{ds}\bigg|_{s=0} a_0(\zeta + s\eta).$$

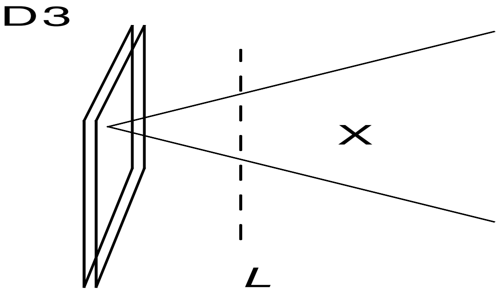
Define (X, ζ) to be K-stable if $Fut(\zeta, \eta) \geq 0$ for all *special* degenerations and $Fut(\zeta, \eta) = 0$ if and only if $X_0 = X$.

So how do we understand the ingredients of K stability from physical point of view? We will see that K stability can be naturally understood from the field theory associated with D branes probing X.

Now let's consider type IIB string theory on the following background

$$R^{1,3} \times X, \tag{7}$$

where X is a three dimensional canonical singularity, and we also add N D3 brane whose world volume is in the direction $R^{1,3}$, so D3 branes are points on X. One get four dimensional $\mathcal{N}=1$ supersymmetric field theory on D3 branes.



One can write down a gauge theory description for lots of X (mainly if X is toric). However, even without gauge theory description, we can learn two facts about the field theory from the geometry of X:

- The field theory has a $U(1)$ symmetry group which is identified with C^* action of X.
- The field theory has a branch of moduli space which is identified with Nth symmetric product of X.

We are concerned with the low energy behavior at the most singular point of the moduli space. At the deep IR, the low energy theory at the most singular point is a superconformal field theory (SCFT), which we might call it $\mathcal{T}_0$, whose property is crucial to understand K stability.

4d $\mathcal{N} = 1$ SCFT

Given the importance of $\mathcal{N} = 1$ SCFT, we review some relevant facts here:

- It has a bosonic $SO(2,4) \times U(1)_R$ symmetry group, here $SO(2,4)$ is the four dimensional conformal group.
- The representation theory of $\mathcal{N} = 1$ superconformal algebra has been well studied, a generic highest weight representation is labeled as $|\Delta, j_1, j_2, r\rangle$, here Δ is the scaling dimension, j_1, j_2 are left and right spins, and r is the $U(1)_R$ charge. Some representations are short (BPS representation which are annihilated by some number of supersymmetries.). A special class of operators are called chiral operators $\mathcal{B}_{r,(j,0)}$, whose scaling dimension is given as $\Delta = \frac{3}{2}r$.
- The chiral operators form a ring and is called chiral ring. For SCFT, there is a $U(1)_R$ symmetry group acting on this ring, so we actually have a graded ring. The determination of this chiral ring is crucial, for example, the coordinate ring of the moduli space can be read from chiral ring.

Let's emphasize that ordinary $\mathcal{N}=1$ theory has no distinguished global symmetry group as $U(1)_R$ symmetry. However, they also has a chiral ring which can also be used to determine the moduli space of vacua. There is no distinguished grading on the chiral ring though. There might be also some symmetry group acting on their chiral ring, so in general the chiral ring of a non-conformal $\mathcal{N}=1$ theory could also be graded. It is a difficult physical question to distinguish whether a graded chiral ring is the chiral ring of a SCFT!

K stability and stability of chiral ring

Now let's come back to the SCFT $\mathcal{T}_0$ which is defined as the IR theory of D3 branes probing the cone X, and the question is

- What is the graded chiral ring of $\mathcal{T}_0$?

There are following two possibilities:

- The chiral ring of $\mathcal{T}_0$ is determined by X and its C^* action.
- The chiral ring of $\mathcal{T}_0$ is not given by X.

Both possibilities can happen.

Example 1: Consider the singularity $X_A: \ x_1^2 + x_2^2 + x_3^2 = 0$, with x_4 free. It is well known that the theory $\mathcal{T}_{0A}$ has $\mathcal{N} = 2$ SUSY (Douglas-Moore) and its chiral ring is determined by X_A.
Example 2: Consider singularity $X_B: \ x_1^2 + x_2^2 + x_3^2 + x_4^4 = 0$. There is a Lagrangian description of this theory too, which is given by a deformation of theory associated with above singularity. The deformation is however marginal irrelevant, so the theory $\mathcal{T}_{0B}$ is actually the same as T_{0A}, whose chiral ring is given by X_A instead of X_B.

By Lagrangian description, I mean the theory can be described by an action formed by elementary fields. The two examples listed above have Lagrangian description, and one can use conventional field theory tools to study these theories. In general, the supersymmetric theory defined on D3 brane would not have a Lagrangian description though.

In the QFT framework, an operator near a CFT point is classified as relevant, irrelevant, and marginal (which is further classified as exact marginal, marginal irrelevant, and marginal relevant). Marginal irrelevant means the operator which is used to deform the theory has dimension 4 (for four dimensional theory), but the quantum correction changes it to be irrelevant, which means that at the IR, the theory flows back to the original theory.

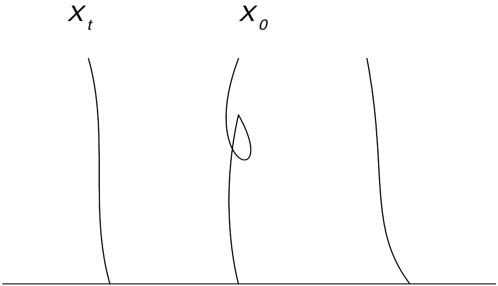
The geometric meaning of first possibility is that: There is a Ricci-flat conical metric on X, and in the large N limit, the superconformal field theory $\mathcal{T}_0$ is dual to type IIB string theory on following background

$$AdS_5 \times L_X, \tag{8}$$

where L_X is the link of X and has a Sasaki-Einstein metric.

Now the physical meaning of K stability is clear: X is K stable if the chiral ring of the corresponding SCFT $\mathcal{T}_0$ is given by X! The $U(1)_R$ symmetry is identified with a special automorphism group ζ of X whose determination will be discussed soon.

The next questions are what are the physical meaning of two ingredients of K stability? Test configuration and the central fibre X_0 is quite simple: X_0 is simply a candidate chiral ring for $\mathcal{T}_0$. The next question is what is the meaning of Futaki invariant?



Generalized a maximization and Futaki invariant

The Futaki invariant is related to the central charge a of $\mathcal{N}=1$ SCFT. A four dimensional $\mathcal{N}=1$ SCFT has an invariant called central charge a. This central charge is related to the anomaly of the $U(1)_R$ symmetry group:

$$a = \frac{3}{32} TrR^3 - \frac{1}{32} TrR. \tag{9}$$

In practice, we often do not know the $U(1)_R$ symmetry of a SCFT. However, if we know all the anomaly free symmetries and their 't hooft anomaly, Intriligator and Wecht proposed a very useful a maximization method to determine $U(1)_R$: the true $U(1)_R$ symmetry maximizes the central charge a.

For a QFT, the important information is exact global symmetries, and these are called anomaly free (as some classical symmetries can be broken by quantum anomalies). For such anomaly free symmetries, one can define some constants which is called 't hooft anomaly, which carries important information of the theory.

In our context, if X is K stable (namely there is no non-trivial destabilizing test configuration), a maximization has following simple geometric interpretation (Martelli-Sparks-Yau, '05). Consider the Hilbert series of the graded ring (X, ζ),

$$H(\zeta, t) = \sum_{\alpha} dim\, H_{\alpha} t^{\alpha} \tag{10}$$

and it has the following expansion:

$$H(\zeta, exp(-s)) = \frac{a_0(\zeta)}{s^3} + \frac{a_1(\zeta)}{s^2} + \dots \tag{11}$$

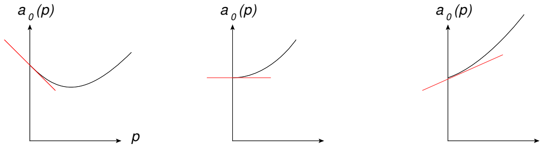
Now a_0 is proportional to the volume of Sasaki-Einstein manifold. It is also inverse proportional to the central charge a of the field theory. If the automorphism group of X is more than one dimensional, MSY proves that a maximization procedure is just equivalent to the minimization of the volume!

The volume minimization can be understood from the K stability point of view, namely, consider trivial test configuration where the central fibre X_0 is the same as X (Collins-Szekelyhidi). This suggests that the use of Futaki invariant in more general context might be related to a generalized volume minimization or *a* maximization.

The resolution comes from that the fact that the Reeb field ζ in pair (X, ζ) should be a candidate $U(1)_R$ symmetry, and this forces ζ to be normalized $\zeta \in \Sigma_R$. Recall that for a *normalized* test configuration generated by η the Futaki invariant has a very simple form:

$$Fut(X, \zeta, \eta) = \frac{d}{ds}\bigg|_{s=0} a_0(\zeta + s\eta).$$

The meaning of Futaki invariant should be clear from the following graphs:



Fut<0 *Fut=0* *Fut>0*

The interpretation of Futaki invariant is:

- In the case a, a_0 achieves its minimum at $p > 0$.
- In the case b, if X_0 is different from X, a_0 achieves its minimum at $p = 0$, but X_0 has more symmetries.
- In the case c, a_0 achieves its minimum at $p < 0$.

The destabilizing configuration gives less a_0 and therefore more central charge in case a. For case b, X and X_0 gives same central charge, but X_0 has more symmetry! We interpret this as the generalized **a** maximization, or a generalized volume minimization.

Unitarity bound

Example: There is a very general test configuration for every singularity X, which is to make one of the coordinate of X free. Physically, the Futaki invariant is to check whether the operator violate the unitarity bound. A scalar operator of a 4d $\mathcal{N} = 1$ SCFT has a bound on scaling dimension

$$\Delta \geq 1. \tag{12}$$

For chiral operator, its scaling dimension is determined by $U(1)_R$ symmetry: $\Delta = \frac{3}{2}r$. The elements in ring X is identified with scalar chiral operators of the field theory, so if we know ζ and its charge on the generators of X, we can check wether the generator satisfies the unitarity bound. In fact, Martelli-Sparks-Yau used this constraint to give obstruction to the existence of SE metric.

New obstruction

In the physics literature, there is very few known obstruction to the stability of chiral ring. K stability actually gives new obstruction. Consider a singularity $x_1^2 + x_2^2 + x_3^p + x_3 x_4^q = 0$, and the unitarity bound gives the constraint:

$$p < 2q + 1 \textit{ and } q < 2p - 2. \tag{13}$$

However, one can consider a test configuration whose central fibre is $x_1^2 + x_2^2 + x_3 x_4^q = 0$, and the bound is (Collins-Székelyhidi)

$$q > \frac{p^2 - 1}{2p - 1}. \tag{14}$$

Sometimes this bound is stronger than the unitarity bound.

Concluding remarks

- K stability is equivalent to generalized a maximization, and the new input is that one should consider new rings besides the given ring. Geometrically, it is a generalized volume minimization. Physically, K stability unifies many important concepts in field theory: a maximization, chiral ring, unitarity violation, etc.

- A crucial question is the reduction of number of test configurations. Some immediate physical inputs suggest that
  1. The central fibre has to have more automorphism groups, as the central charge is related to the anomaly of $U(1)_R$ symmetries, and if X_0 has more central charges, it must have more symmetries, this explains the use of test configuration generated by a symmetry group.
  2. It seems that the flatness in the definition of test configuration is related to supersymmetry preserving deformation.
  3. The a maximization is only sensible to the abelian symmetries (Intriligator, Wecht), so for a toric singularity, there is no nontrivial test configurations.
  4. The central fibre should be normal, as a non-normal singularity can not be the chiral ring of a $\mathcal{N} = 1$ SCFT. This has been proven by Collins-Szekelyhidi, Chen-Donaldson-Sun and Li-Xu.

  The hope is that the field theory inputs can further reduce the number of nontrivial test configurations. For the complete intersection singularities, known results suggest that we only need to check finite number of non-trivial test configurations associated with the defining equation of f. Is it possible to prove it?

- The field theory analysis is actually very general, so it can be generalized to other geometric contexts involving cones with special holonomy, and we have an analog of K stability. A straightforward generalization is the four dimensional canonical singularity and seven dimensional Sasaki-Einstein manifolds. A more interesting context is G_2 and $Spin_7$ cones.

Relation to G_2 manifolds

There is a general method to produce complete non-compact G_2 manifolds starting from a six dimensional cone with Ricci-flat conical metric (Foscolo, Haskins, Nordstrom). An interesting class involves the terminal singularity with crepant resolutions. On the one hand, using the existence of Ricci-flat conical metric on terminal singularity shown in table 1 (which is not completely solved), and the study of crepant resolution (not very difficult), one can produce many new G_2 manifolds. The relation between 6d cones and G_2 manifold might be interpreted from renormalization group flow of the field theory on D3 branes, and this is currently under study.

On the other hand, since not very three dimensional cone X admits a Ricci-flat conical metric, and it appears that there is also a K stability notion in this G_2 context too.