A family of Kähler flying wing steady Ricci solitons

Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics

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(joint with Pak-Yeung Chan and Yi Lai)

Kähler manifolds

- Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathrm{End}(TM), J^2 = -\mathrm{Id}$, + "integrability condition"
- A Riemannian metric g on (M, J) is *Kähler* if $g(J-, J-) = g(-, -)$ and $\nabla^g J = 0$
- In this case, the metric information is entirely encoded by the *Kähler form* $\omega = g(J-, -)$ which is a closed two-form
- We can introduce local complex coordinates $(z_1, \ldots, z_n)$ in which

$$\omega = ig_{\alpha\bar{\beta}}\, dz_\alpha \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha\bar{\beta}}) = \text{positive-definite hermitian matrix}$$

- Ricci form $\rho_\omega = \mathrm{Ric}(g)(J-, -)$ is a closed two-form.Locally,

$$\rho_\omega = -i\partial\bar{\partial}\,(\log\det(g_{k\bar{\ell}})) = -i\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}}\,(\log\det(g_{k\bar{\ell}}))\; dz_\alpha \wedge d\bar{z}_{\bar{\beta}}$$

- (M, ω) compact and Kähler $\implies [\rho_\omega] = c_1(M)$

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Ricci Solitons

Definition

A *Ricci soliton* is a triple (M, g, X), where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}$$

for some $\lambda \in \mathbb{R}$. The vector field X is called the *soliton vector field*. If $X = \nabla^g f$ for some smooth $f : M \to \mathbb{R}$, then we say that (M, g, X) is *gradient*,in which case (1) becomes

$$\mathrm{Ric}(g) + \mathrm{Hess}_g(f) = \lambda g,$$

and we call f the *soliton potential*.

$$\underbrace{\mathrm{Ric}(g) + \mathrm{Hess}_g(f)}_{\text{Bakry-Emery tensor}}$$

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Let ω denote the Kähler form of g. If $X = \nabla^g f$, then the soliton equation becomes

$$\rho_\omega + i\partial\bar{\partial} f = \lambda\omega, \qquad (2)$$

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Definition

A soliton is called *expanding* if $\lambda < 0$, *steady* if $\lambda = 0$, and *shrinking* if $\lambda > 0$.

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Why do we care?

Recall:

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.$$

- Generalisation of Einstein metrics

$$X = 0 \implies \mathrm{Ric}(g) = \lambda g \iff g \text{ is Einstein}$$

In the Kähler world:

Shrinking Kähler-Ricci solitons	$\rightsquigarrow$	KE Fano manifolds ($c_1 > 0$)
Steady Kähler-Ricci solitons	$\rightsquigarrow$	CY manifolds ($c_1 = 0$)
Expanding Kähler-Ricci solitons	$\rightsquigarrow$	KE manifolds with < 0 scalar curvature ($c_1 < 0$)

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- "Self-similar" solutions of the Ricci flow

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}$$

$$\Longrightarrow \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2\,\mathrm{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}$$

Shrinking $\lambda > 0 \quad \rightsquigarrow \quad t \in \left(-\infty, \frac{1}{2\lambda}\right)$ ancient solution

Steady $\lambda = 0 \quad \rightsquigarrow \quad t \in (-\infty, +\infty)$ eternal solution

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   - asymptotically conical
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We now focus on steady gradient Kähler-Ricci solitons

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Motivating conjecture

Recall:

Theorem (Cao 1996)

There exists a $U(n)$-invariant steady gradient Kähler-Ricci soliton on $\mathbb{C}^n$ with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on $\mathbb{C}^n$ with positive curvature.

- True for $n = 1 \rightsquigarrow$ the soliton is Hamilton's cigar soliton
- False for $n = 2 \rightsquigarrow$ $U(1) \times U(1)$-invariant examples constructed on $\mathbb{C}^2$ by Apostolov-Cifarelli '23
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Theorem A (Chan-C.-Lai '24)

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$-invariant, but non-$U(n)$-invariant, steady gradient Kähler-Ricci solitons on $\mathbb{C}^n$ with $\mathrm{Rm}^{1,1}_{\mathrm{op}} > 0$.

- This answers Cao's conjecture in the negative for all dimensions
- κ-collapsed
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Lemma (Collapsing lemma; Lai '20)

A sequence of expanding gradient Ricci solitons (M_i, g_i, p_i) *with* $\mathrm{Rm}(g_i) > 0$, *scal*$(p_i) = 1$, *and AVR*$(g_i) := \lim_{r\to\infty} \frac{B_{g_i}(x, r)}{\omega_{n-1} r^n} \to 0$ *as* $i \to \infty \implies (M_i, g_i, p_i) \to_{pCG} (M, g, p) =$ *a steady gradient Ricci soliton with scal*$(p) = 1$.

Proof.

$$\mathrm{Ric}(g_i) + \mathrm{Hess}_{g_i}(f_i) = \frac{1}{C_i} g_i$$

Show that $C_i \to 0$ as $i \to \infty$. □

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Reduction to construction of metric on links

Recall:

Definition

Let (L, g) be a compact connected Riemannian manifold. The *Riemannian cone* C with *link* L is defined to be $\mathbb{R}_{>0} \times L$ with metric $g_0 = dr^2 + r^2 g$.

General strategy

For a sequence of collapsing metrics on the links satisfying condition A, the corresponding cone metric satisfies condition B.We can then lift these to expanding Ricci solitons satisfying condition C.Then by the compactness lemma, the sequence of expanding solitons degenerates to a steady Ricci soliton satisfying condition C.

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We construct Kähler metrics on $\mathbb{P}^{n-1}$ with $\mathrm{Rm}^{1,1}_{\mathrm{op}} > 2$. The corresponding Kähler cone metric on $\mathbb{C}^n$ satisfies $\mathrm{Rm}^{1,1}_{\mathrm{op}} \geq 0$. By C.-Deruelle '16, we can lift these to expanding Kähler-Ricci solitons on $\mathbb{C}^n$ satisfying $\mathrm{Rm}^{1,1}_{\mathrm{op}} > 0$.

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Byproduct: Almost rigidity of $\mathbb{P}^n$

Theorem (Li-Wang '05)

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- (M, g, η, ξ) = Sasaki manifold
  - ξ = Reeb vector field
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  - $d\eta = g^T$ = transverse metric
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Consider $\widehat{M} = M \times (0, L)$,together with the doubly-warped product metric

$$\hat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T, \qquad a(r),\, b(r) > 0$$

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Step 2 (Use Kähler expander to smooth out the cone singularity):

Theorem (Cao '96)

There is a $U(n-1)$-invariant expanding Kähler-Ricci soliton on $\mathbb{C}^{n-1}$ asymptotic to Cone_k, $k > 1$.

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So we obtain:

1. a smooth Kähler metric close to the singular metric;
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In the curvature evolution equation

$$\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),$$

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