A family of Kähler flying wing steady Ricci solitons

Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics

Ronan Conlon

Department of Mathematical Sciences, The University of Texas at Dallas

May 15th 2024

(joint with Pak-Yeung Chan and Yi Lai)

 $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id}, +$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^g J = 0$
- In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-,+)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = ig_{\alpha\bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha\bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_{\omega} = -i\partial\bar{\partial} \left(\log \det(g_{k\bar{\ell}}) \right) = -i \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\bar{\beta}}} \left(\log \det(g_{k\bar{\ell}}) \right) dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}
$$

•
$$
(M, \omega)
$$
 compact and Kähler $\implies [\rho_{\omega}] = c_1(M)$.

R. Conlon (UT Dallas) Kähler-Ricci solitons

[M](#page-2-0)[ay](#page-0-0) [1](#page-1-0)[5t](#page-9-0)[h](#page-10-0) [2024](#page-0-0) [\(](#page-167-0)2021) 2024 (2021) 2024 (2021) 2024 (2021) 2024 (2021) 2022 (2021) 2022 (2021) 2022 (20

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id}, +$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^g J = 0$
- In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-,+)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = ig_{\alpha\bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha\bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_{\omega} = -i\partial\bar{\partial} \left(\log \det(g_{k\bar{\ell}}) \right) = -i \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\bar{\beta}}} \left(\log \det(g_{k\bar{\ell}}) \right) dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}
$$

•
$$
(M, \omega)
$$
 compact and Kähler $\implies [\rho_{\omega}] = c_1(M)$.

R. Conlon (UT Dallas) Kähler-Ricci solitons

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id},\, +\,$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^g J = 0$
- In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-,+)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = ig_{\alpha\bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha\bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_{\omega} = -i\partial\bar{\partial} \left(\log \det(g_{k\bar{\ell}}) \right) = -i \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\bar{\beta}}} \left(\log \det(g_{k\bar{\ell}}) \right) \, dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}
$$

•
$$
(M, \omega)
$$
 compact and Kähler $\implies [\rho_{\omega}] = c_1(M)$.

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id},\, +\,$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^{\mathcal{g}} J = 0$
- In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-,+)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = ig_{\alpha\bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha\bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_{\omega} = -i\partial\bar{\partial} \left(\log \det(g_{k\bar{\ell}}) \right) = -i \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\bar{\beta}}} \left(\log \det(g_{k\bar{\ell}}) \right) \, dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}
$$

•
$$
(M, \omega)
$$
 compact and Kähler $\implies [\rho_{\omega}] = c_1(M)$.

R. Conlon (UT Dallas) Kähler-Ricci solitons

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id},\, +\,$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^{\mathcal{g}} J = 0$
- \bullet In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-, -)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = ig_{\alpha\bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha\bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_{\omega} = -i\partial\bar{\partial} \left(\log \det(g_{k\bar{\ell}}) \right) = -i \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\bar{\beta}}} \left(\log \det(g_{k\bar{\ell}}) \right) \, dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}
$$

•
$$
(M, \omega)
$$
 compact and Kähler $\implies [\rho_{\omega}] = c_1(M)$

R. Conlon (UT Dallas) Kähler-Ricci solitons

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id},\, +\,$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^{\mathcal{g}} J = 0$
- \bullet In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-, -)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = i g_{\alpha \bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha \bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_{\omega} = -i\partial\bar{\partial} \left(\log \det(g_{k\bar{\ell}}) \right) = -i \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\bar{\beta}}} \left(\log \det(g_{k\bar{\ell}}) \right) \, dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}
$$

$$
\bullet \ \ (M, \ \omega) \ \text{compact and Kähler} \implies [\rho_\omega] = c_1 \big(\underline{M} \big)_{*, \text{square}} \ \text{for} \ \mathbb{R} \ \ \text{and} \ \math
$$

R. Conlon (UT Dallas) Kähler-Ricci solitons

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id},\, +\,$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^{\mathcal{g}} J = 0$
- \bullet In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-, -)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = i g_{\alpha \bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha \bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_{\omega} = -i\partial\bar{\partial} \left(\log \det(g_{k\bar{\ell}}) \right) = -i \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\bar{\beta}}} \left(\log \det(g_{k\bar{\ell}}) \right) \, dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}
$$

 $(M,\,\omega)$ $(M,\,\omega)$ $(M,\,\omega)$ compact and Kähler $\implies [\rho_\omega]=c_1(M)$ is equivalent wit[h](#page-10-0) α

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id},\, +\,$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^{\mathcal{g}} J = 0$
- \bullet In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-, -)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = i g_{\alpha \bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha \bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_\omega=-i\partial\bar\partial\left(\log{\rm det}(g_{k\bar\ell})\right)=-i\frac{\partial^2}{\partial z_\alpha\partial\bar z_{\bar\beta}}\left(\log{\rm det}(g_{k\bar\ell})\right)\,dz_\alpha\wedge d\bar z_{\bar\beta}
$$

$$
\bullet \ \ (M,\,\omega) \text{ compact and Kähler} \implies [\rho_\omega] = c_1(M) \text{ for all } \alpha \in \mathbb{N}^+, \text{ and } \alpha \in \mathbb{N}^+, \text{ and } \alpha \in \mathbb{N}^+ \text{ for all } \alpha \in \mathbb{N}^+ \text{ and } \alpha \in \mathbb{N}^+ \text{ for all } \alpha \in \mathbb{N}^+ \text{ and } \alpha \in \mathbb{N}^+ \text{ for all } \alpha \in \mathbb{N}^+ \text{ and } \alpha \in \mathbb{N}^+ \text{ for all } \alpha \in \mathbb{N}^+ \text{ for all } \alpha \in \mathbb{N}^+ \text{ and } \alpha \in \mathbb{N}^+ \text{ for all } \alpha \in \mathbb
$$

- • Let (M, J) be a complex manifold, $n = \dim_{\mathbb{C}} M$
- $J \in \mathsf{End}(\mathcal{T} \mathcal{M}), J^2 = -\mathsf{Id},\, +\,$ "integrability condition"
- A Riemannian metric g on (M, J) is Kähler if $g(J-, J-) = g(-, -)$ and $\nabla^{\mathcal{g}} J = 0$
- \bullet In this case, the metric information is entirely encoded by the Kähler form $\omega = g(J-, -)$ which is a closed two-form
- We can introduce local complex coordinates (z_1, \ldots, z_n) in which

 $\omega = i g_{\alpha \bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\bar{\beta}}, \quad (g_{\alpha \bar{\beta}}) =$ positive-definite hermitian matrix

• Ricci form $\rho_{\omega} = \text{Ric}(g)(J-, -)$ is a closed two-form. Locally,

$$
\rho_\omega=-i\partial\bar\partial\left(\log{\rm det}(g_{k\bar\ell})\right)=-i\frac{\partial^2}{\partial z_\alpha\partial\bar z_{\bar\beta}}\left(\log{\rm det}(g_{k\bar\ell})\right)\,dz_\alpha\wedge d\bar z_{\bar\beta}
$$

 (M, ω) (M, ω) (M, ω) compact and Kähler $\implies [\rho_\omega]=c_1(M)$ and $\text{rank}(p)$ and $\text{rank}(p)$

Definition

A Ricci soliton is a triple (M, g, X) , where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying

$$
Ric(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}
$$

for some $\lambda \in \mathbb{R}$ **.** The vector field X is called the *soliton vector field.* If $X = \nabla^g f$ for some smooth $f : M \to \mathbb{R}$, then we say that (M, g, X) is gradient,in which case [\(1\)](#page-10-1) becomes

 $Ric(g) + Hess_{\alpha}(f) = \lambda g$,

and we call f the soliton potential.

$$
Ric(g) + Hess_g(f)
$$

Bakry-Emery tensor

 $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$

Definition

A Ricci soliton is a triple (M, g, X) , where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying

$$
Ric(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}
$$

for some $\lambda \in \mathbb{R}$. The vector field X is called the soliton vector field. If $X = \nabla^g f$ for some smooth $f : M \to \mathbb{R}$, then we say that (M, g, X) is gradient,in which case [\(1\)](#page-10-1) becomes

 $Ric(g) + Hess_{\alpha}(f) = \lambda g$,

and we call f the soliton potential.

$$
Ric(g) + Hess_g(f)
$$

Bakry-Emery tensor

[M](#page-12-0)[ay](#page-9-0) [1](#page-10-0)[5t](#page-15-0)[h](#page-16-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2024 (15th 2024 Chan and Yi Lai)

Definition

A Ricci soliton is a triple (M, g, X) , where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying

$$
Ric(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}
$$

for some $\lambda \in \mathbb{R}$. The vector field X is called the *soliton vector field*. If $X = \nabla^g f$ for some smooth $f : M \to \mathbb{R}$, then we say that (M, g, X) is **gradient**, in which case [\(1\)](#page-10-1) becomes

 $Ric(g) + Hess_{\alpha}(f) = \lambda g$,

and we call f the soliton potential.

$$
Ric(g) + Hess_g(f)
$$

Bakry-Emery tensor

 $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$

Definition

A Ricci soliton is a triple (M, g, X) , where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying

$$
Ric(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}
$$

for some $\lambda \in \mathbb{R}$. The vector field X is called the *soliton vector field*. If $X = \nabla^g f$ for some smooth $f : M \to \mathbb{R}$, then we say that (M, g, X) is gradient,in which case [\(1\)](#page-10-1) becomes

$$
Ric(g) + Hess_g(f) = \lambda g,
$$

and we call f the soliton potential.

$$
Ric(g) + Hess_g(f)
$$

Bakry-Emery tensor

 $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$

Definition

A Ricci soliton is a triple (M, g, X) , where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying

$$
Ric(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}
$$

for some $\lambda \in \mathbb{R}$. The vector field X is called the *soliton vector field*. If $X = \nabla^g f$ for some smooth $f : M \to \mathbb{R}$, then we say that (M, g, X) is gradient,in which case [\(1\)](#page-10-1) becomes

$$
Ric(g) + Hess_g(f) = \lambda g,
$$

and we call f the soliton potential.

Definition

A Ricci soliton is a triple (M, g, X) , where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying

$$
Ric(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}
$$

for some $\lambda \in \mathbb{R}$. The vector field X is called the *soliton vector field*. If $X = \nabla^g f$ for some smooth $f : M \to \mathbb{R}$, then we say that (M, g, X) is gradient,in which case [\(1\)](#page-10-1) becomes

$$
Ric(g) + Hess_g(f) = \lambda g,
$$

and we call f the soliton potential.

 $Ric(g) + Hess_g(f)$ Bakry-Emery tensor R. Conlon (UT Dallas) Kähler-Ricci solitons $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$ 3 / 25

Definition

Let (M, g, X) be a Ricci soliton. If g is Kähler and X is real holomorphic (i.e., $\mathcal{L}_X J = 0$), then we say that (M, g, X) is a Kähler-Ricci soliton.

Let ω denote the Kähler form of g. If $X = \nabla^g f$, then the soliton equation becomes

$$
\rho_{\omega} + i\partial\bar{\partial}f = \lambda\omega,\tag{2}
$$

where ρ_{ω} is the Ricci form of ω .

A soliton is called *expanding* if $\lambda < 0$, steady if $\lambda = 0$, and shrinking if $\lambda > 0$.

Definition

Let (M, g, X) be a Ricci soliton. If g is Kähler and X is real holomorphic (i.e., $\mathcal{L}_X J = 0$), then we say that (M, g, X) is a Kähler-Ricci soliton.

Let ω denote the Kähler form of g. If $X = \nabla^g f$, then the soliton equation becomes

$$
\rho_{\omega} + i\partial\bar{\partial}f = \lambda\omega,\tag{2}
$$

where ρ_{ω} is the Ricci form of ω .

A soliton is called expanding if $\lambda < 0$, steady if $\lambda = 0$, and shrinking if $\lambda > 0$.

[M](#page-18-0)agnetic Robert Wit[h](#page-19-0) Pake-Yeung Chan and Yi Reserves The Pake-Yeung Chan and Yi

Definition

Let (M, g, X) be a Ricci soliton. If g is Kähler and X is real holomorphic (i.e., $\mathcal{L}_X J = 0$), then we say that (M, g, X) is a Kähler-Ricci soliton.

Let ω denote the Kähler form of g. If $X = \nabla^g f$, then the soliton equation becomes

$$
\rho_{\omega} + i\partial\bar{\partial}f = \lambda\omega,\tag{2}
$$

where ρ_{ω} is the Ricci form of ω .

Definition

A soliton is called *expanding* if $\lambda < 0$, steady if $\lambda = 0$, and shrinking if $\lambda > 0$.

 $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$

Recall:

$$
Ric(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

o Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \rightsquigarrow CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

scalar curvature $(c_1 < 0)$ [M](#page-20-0)ar [1](#page-19-0)[5t](#page-29-0)[h](#page-30-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 20

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

o Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \rightsquigarrow CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

scalar curvature $(c_1 < 0)$ [M](#page-21-0)ar [1](#page-19-0)[5t](#page-29-0)[h](#page-30-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 20

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

 $X = 0 \implies \text{Ric}(g) = \lambda g \iff g$ is Einstein

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \rightarrow CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

scalar curvature $(c_1 < 0)$ [M](#page-22-0)[ay](#page-18-0) [1](#page-19-0)[5t](#page-29-0)[h](#page-30-0) [2024](#page-0-0) [\(](#page-167-0)2021 - 2022 (2022) 2024 (2022 (2022) 2022 (2022) 2024 (2022) 20

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \rightsquigarrow CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

scalar curvature $(c_1 < 0)$ [M](#page-23-0)[ay](#page-18-0) [1](#page-19-0)[5t](#page-29-0)[h](#page-30-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \rightsquigarrow CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

scalar curvature $(c_1 < 0)$ [M](#page-24-0)[ay](#page-18-0) [1](#page-19-0)[5t](#page-29-0)[h](#page-30-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \rightsquigarrow CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0 scalar curvature $(c_1 < 0)$

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \rightsquigarrow CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0 scalar curvature $(c_1 < 0)$ [M](#page-26-0)[ay](#page-18-0) [1](#page-19-0)[5t](#page-29-0)[h](#page-30-0) [2024](#page-0-0) [\(](#page-167-0)2021 - 2022 (2022) 2024 (2022) 2022 (2022) 2024 (2022) 2024 (20

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) **Steady Kähler-Ricci solitons** \rightsquigarrow CY manifolds ($c_1 = 0$) Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

scalar curvature $(c_1 < 0)$

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \sim Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

CY manifolds
$$
(c_1 = 0)
$$

scalar curvature $(c_1 < 0)$

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \sim **Expanding Kähler-Ricci solitons** \rightsquigarrow KE manifolds with $\lt 0$

[M](#page-29-0)[ay](#page-18-0) [1](#page-19-0)[5t](#page-29-0)[h](#page-30-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

$$
\rightarrow \quad \text{CY manifolds } (c_1 = 0)
$$

scalar curvature $(c_1 < 0)$

Recall:

$$
\mathsf{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g, \qquad \lambda \in \mathbb{R}.
$$

• Generalisation of Einstein metrics

$$
X = 0 \implies \text{Ric}(g) = \lambda g \iff g \text{ is Einstein}
$$

In the Kähler world:

Shrinking Kähler-Ricci solitons \rightarrow KE Fano manifolds ($c_1 > 0$) Steady Kähler-Ricci solitons \leadsto CY manifolds $(c_1 = 0)$ Expanding Kähler-Ricci solitons \rightarrow KE manifolds with < 0

-
- -

scalar curvature $(c_1 < 0)$

"Self-similar" solutions of the Ricci flow

$$
\operatorname{Ric}(g) + \frac{1}{2} \mathcal{L}_X g = \lambda g \rightsquigarrow \left\{ \begin{array}{c} \sigma(t) = 1 - 2\lambda t \\ \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{array} \right.
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \left\{ \begin{array}{c} \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \phi \quad \tilde{g}(0) = g. \end{array} \right.
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n

\n\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n

\n\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

"Self-similar" solutions of the Ricci flow

$$
\text{Ric}(g) + \frac{1}{2} \mathcal{L}_X g = \lambda g \leadsto \begin{cases} \sigma(t) = 1 - 2\lambda t \\ \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \leadsto \begin{cases} \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \phi(\tilde{g}(0)) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n

\n\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n

\n\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

"Self-similar" solutions of the Ricci flow

$$
\text{Ric}(g) + \frac{1}{2} \mathcal{L} \times g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} \times \end{cases}
$$
\n
$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n

\n\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n

\n\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

"Self-similar" solutions of the Ricci flow

$$
\text{Ric}(g) + \frac{1}{2}\mathcal{L} \times g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} \times \end{cases}
$$
\n
$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n

\n\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n

\n\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

"Self-similar" solutions of the Ricci flow

$$
\text{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \leadsto \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \leadsto \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \text{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n

\n\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n

\n\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

• "Self-similar" solutions of the Ricci flow

$$
\text{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \leadsto \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \leadsto \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \text{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

Shrinking $\lambda > 0$ \rightarrow $t \in \left(-\infty, \frac{1}{\infty}\right)$ 2λ) ancient solution Steady $\lambda = 0 \implies t \in (-\infty, +\infty)$ eternal solution Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2}\right)$ $\left(\frac{1}{2\lambda}, +\infty\right)$ immortal solution
"Self-similar" solutions of the Ricci flow

$$
\operatorname{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 ancient solution

\nSteady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ **eternal solution**

\nExpanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ **immortal solution**

"Self-similar" solutions of the Ricci flow

$$
\operatorname{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n

\n\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n

\n\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

"Self-similar" solutions of the Ricci flow

$$
\operatorname{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n

\n\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n

\n\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

"Self-similar" solutions of the Ricci flow

$$
\operatorname{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

"Self-similar" solutions of the Ricci flow

$$
\operatorname{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \rightsquigarrow \begin{cases} \bullet & \sigma(t) = 1 - 2\lambda t \\ \bullet & \psi_t = \text{flow of } \frac{1}{\sigma(t)} X \end{cases}
$$

$$
\implies \tilde{g}(t) := \sigma(t) \cdot \psi_t^*(g) \rightsquigarrow \begin{cases} \bullet & \frac{\partial \tilde{g}(t)}{\partial t} = -2 \operatorname{Ric}(\tilde{g}(t)) \\ \bullet & \tilde{g}(0) = g. \end{cases}
$$

\n Shrinking
$$
\lambda > 0 \quad \leadsto \quad t \in \left(-\infty, \frac{1}{2\lambda} \right)
$$
 \n ancient solution\n Steady $\lambda = 0 \quad \leadsto \quad t \in \left(-\infty, +\infty \right)$ \n eternal solution\n Expanding $\lambda < 0 \quad \leadsto \quad t \in \left(\frac{1}{2\lambda}, +\infty \right)$ \n immortal solution\n

Song-Tian \rightsquigarrow "Kähler-Ricci flow through singularities" on compact Kähler manifolds

Finite time "Type I" singularity \sim modelled on a non-flat shrinking soliton (Naber, Enders-Muller-Topping)

[M](#page-42-0)[ay](#page-40-0) [1](#page-41-0)[5t](#page-46-0)[h](#page-47-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2022 (15th 2022 (15th 2022 (15th 2022 (15th 2022)

Song-Tian \rightarrow "Kähler-Ricci flow through singularities" on compact Kähler manifolds

Finite time "Type I" singularity \sim modelled on a non-flat shrinking soliton (Naber, Enders-Muller-Topping)

[M](#page-43-0)[ay](#page-40-0) [1](#page-41-0)[5t](#page-46-0)[h](#page-47-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2022 (15th 2022 (15th 2021 (15th 2022 (15th 2022 (15th 20

Song-Tian \rightarrow "Kähler-Ricci flow through singularities" on compact Kähler manifolds

Finite time "Type I" singularity \sim modelled on a non-flat shrinking soliton (Naber, Enders-Muller-Topping)

[M](#page-44-0)[ay](#page-40-0) [1](#page-41-0)[5t](#page-46-0)[h](#page-47-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2022 (15th 2022 (15th 2021 (15th 2022 (15th 2022 (15th 20

Song-Tian \rightarrow "Kähler-Ricci flow through singularities" on compact Kähler manifolds

Finite time "Type I" singularity \sim modelled on a non-flat shrinking soliton (Naber, Enders-Muller-Topping)

[M](#page-45-0)[ay](#page-40-0) [1](#page-41-0)[5t](#page-46-0)[h](#page-47-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

Song-Tian \rightarrow "Kähler-Ricci flow through singularities" on compact Kähler manifolds

Finite time "Type I" singularity \sim modelled on a non-flat shrinking soliton (Naber, Enders-Muller-Topping)

[M](#page-46-0)[ay](#page-40-0) [1](#page-41-0)[5t](#page-46-0)[h](#page-47-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2022 (15th 2022 (15th 2021 (15th 2022 (15th 2022 (15th 20

Song-Tian \rightsquigarrow "Kähler-Ricci flow through singularities" on compact Kähler manifolds

Finite time "Type I" singularity \sim modelled on a non-flat shrinking soliton (Naber, Enders-Muller-Topping)

[M](#page-47-0)[ay](#page-40-0) [1](#page-41-0)[5t](#page-46-0)[h](#page-47-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2022 (15th 2022 (15th 2021 (15th 2022 (15th 2022 (15th 20

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- Rm > 0

 \bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- Rm > 0

 \bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- Rm > 0

 \bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- Rm > 0

\bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- Rm > 0

\bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

[M](#page-53-0)[ay](#page-46-0) [1](#page-47-0)[5t](#page-58-0)[h](#page-59-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- $Rm > 0$

\bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- \bullet Rm > 0

\bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical

2 Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)

- $SO(n)$ -invariant \rightsquigarrow ODE method
- Rm > 0

\bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3 \,\, (2020)$ \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical
- **2** Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)
	- $SO(n)$ -invariant \rightsquigarrow ODE method
	- \bullet Rm >0

 \bullet Lai's Riemannian flying wings on $\mathbb{R}^n, \, n \geq 3$ (2020) \rightsquigarrow collapsing

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical
- **2** Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)
	- $SO(n)$ -invariant \rightsquigarrow ODE method
	- Rm > 0

 \bullet Lai's Riemannian flying wings on \mathbb{R}^n , $n\geq 3$ (2020) \rightsquigarrow collapsing method

 \bullet Rm > 0

- • $SO(n)$ -invariant \rightsquigarrow ODE method
- **•** asymptotically conical
- **2** Bryant steady soliton on \mathbb{R}^n , $n \geq 3$ (2005)
	- $SO(n)$ -invariant \rightsquigarrow ODE method
	- Rm > 0
- \bullet Lai's Riemannian flying wings on \mathbb{R}^n , $n\geq 3$ (2020) \rightsquigarrow collapsing method
	- \bullet Rm > 0

We now focus on steady gradient Kähler-Ricci solitons

[M](#page-60-0)[ay](#page-58-0) [15t](#page-59-0)[h](#page-60-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

1 Hamilton 1980s, "cigar" soliton on \mathbb{C}

- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
- positive scalar curvature
- **•** linear volume growth
- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **6** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-61-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

1 Hamilton 1980s, "cigar" soliton on \mathbb{C}

- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
- positive scalar curvature
- **•** linear volume growth
- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-62-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- **•** linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-63-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- **•** linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **6** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-64-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **6** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-65-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **6** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-66-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **6** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-67-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-68-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-69-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-70-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-71-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- **·** positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-72-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak-Yeung Chan and Yi Lai)
- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- **3 Yang 2008, Dancer-Wang 2011, on** K_M **, M KE Fano** \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-73-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 20

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$

\bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method

- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-74-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 20

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ **finite** \rightarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-75-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 20

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-76-0)[ay](#page-59-0) [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 20

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- **•** linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-77-0)an [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)View Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- **•** linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow ODE method
- **6** C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method [M](#page-78-0)an [1](#page-60-0)[5t](#page-79-0)[h](#page-80-0) [2024](#page-0-0) [\(](#page-167-0)View Chan and Yi Lai)

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow ODE method
- C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$

- **1** Hamilton 1980s, "cigar" soliton on \mathbb{C}
	- \bullet $U(1)$ -invariant \rightsquigarrow ODE method
	- positive scalar curvature
	- linear volume growth
	- asymptotically cylindrical
- 2 Cao, Koiso 1990s, on \mathbb{C}^n
	- \bullet $U(n)$ -invariant \rightsquigarrow ODE method
	- positive sectional curvature
	- volume growth $\sim O(t^n)$
	- curvature $\sim O(t^{-1})$
- \bullet Yang 2008, Dancer-Wang 2011, on K_M , M KE Fano \rightsquigarrow ODE method
- \bullet Macbeth-Bicquard 2017, on crepant resolutions of \mathbb{C}^n/Γ , $\Gamma \subset SU(n)$ finite \rightsquigarrow gluing method
- **•** Schäfer 2020, on K_M , (M, ω) , $\rho_\omega \geq 0$ has constant eigenvalues \rightsquigarrow ODE method
- C.-Deruelle 2020, every Kähler class of a crepant resolution of a CY cone \rightsquigarrow PDE method $\mathbb{P}^{\mathbb{P}}\triangleq \left\{ \mathbb{P}^{\mathbb{P}}\right\} \times \left\{ \mathbb{P}^{\mathbb{P}}\right\$

 \bullet Schäfer 2021, ACyl \rightsquigarrow PDE method

3 Apostolov-Cifarelli 2023, continuous families on \mathbb{C}^n using Hamiltonian two-forms and toric geometry \rightarrow ODE method

7 Schäfer 2021, ACyl \rightsquigarrow PDE method

3 Apostolov-Cifarelli 2023, continuous families on \mathbb{C}^n using Hamiltonian two-forms and toric geometry \rightarrow ODE method

- **7** Schäfer 2021, ACyl \rightsquigarrow PDE method
- **•** Apostolov-Cifarelli 2023, continuous families on \mathbb{C}^n using Hamiltonian two-forms and toric geometry \sim ODE method

- **7** Schäfer 2021, ACyl \rightsquigarrow PDE method
- **•** Apostolov-Cifarelli 2023, continuous families on \mathbb{C}^n using Hamiltonian two-forms and toric geometry \rightsquigarrow ODE method

Recall:

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

- True for $n = 1 \rightsquigarrow$ the soliton is Hamilton's cigar soliton
- False for $n=2$ $\mapsto U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23
- \bullet n $> 3 \rightsquigarrow$ open

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

- True for $n = 1 \rightsquigarrow$ the soliton is Hamilton's cigar soliton
- False for $n=2$ $\mapsto U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23
- \bullet n $> 3 \rightsquigarrow$ open

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

- True for $n = 1 \rightsquigarrow$ the soliton is Hamilton's cigar soliton
- False for $n=2$ $\mapsto U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23
- $n > 3 \rightarrow$ open

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

• True for $n = 1$ the soliton is Hamilton's cigar soliton

False for $n=2$ $\mapsto U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23

• $n > 3 \rightarrow$ open

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

• True for $n = 1 \rightarrow \infty$ the soliton is Hamilton's cigar soliton

False for $n=2$ $\mapsto U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23

• $n > 3 \rightarrow$ open

[M](#page-89-0)[ay](#page-83-0) [1](#page-84-0)[5t](#page-92-0)[h](#page-93-0) 2021 [\(](#page-167-0)15th 2022 (15th 2022 Chan and Yi Lai)

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

- True for $n = 1 \rightarrow \infty$ the soliton is Hamilton's cigar soliton
- False for $n = 2 \rightarrow U(1) \times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23
- $n > 3 \rightarrow$ open

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

- True for $n = 1 \rightarrow \infty$ the soliton is Hamilton's cigar soliton
- False for $n=2$ $\rightsquigarrow U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23
- $n > 3 \rightarrow$ open

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

- True for $n = 1 \rightarrow \infty$ the soliton is Hamilton's cigar soliton
- False for $n=2$ $\rightsquigarrow U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23
- $n > 3$ open

Recall:

Theorem (Cao 1996)

There exists a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive sectional curvature.

Conjecture (Cao 1996)

This is the unique steady gradient Kähler-Ricci soliton on \mathbb{C}^n with positive curvature.

- True for $n = 1 \rightarrow \infty$ the soliton is Hamilton's cigar soliton
- False for $n=2$ $\rightsquigarrow U(1)\times U(1)$ -invariant examples constructed on \mathbb{C}^2 by Apostolov-Cifarelli '23
- $n > 3 \rightarrowfty$ open

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$ -invariant, but non- $U(n)$ -invariant, steady gradient Kähler-Ricci solitons on \mathbb{C}^n with

- This answers Cao's conjecture in the negative for all dimensions
- \bullet κ -collapsed
- vol $(B_r(x)) > c r^n$
- Zero AVR
- Not isometric to examples of Apostolov-Cifarelli

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$ -invariant, but non- $U(n)$ -invariant, steady gradient Kähler-Ricci solitons on \mathbb{C}^n with $\mathsf{Rm}^{1,1}_{\mathsf{op}} > 0.$

- This answers Cao's conjecture in the negative for all dimensions
- \bullet κ -collapsed
- vol $(B_r(x)) > c r^n$
- Zero AVR
- Not isometric to examples of Apostolov-Cifarelli

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$ -invariant, but non- $U(n)$ -invariant, steady gradient Kähler-Ricci solitons on \mathbb{C}^n with $\mathsf{Rm}^{1,1}_{\mathsf{op}} > 0.$

- This answers Cao's conjecture in the negative for all dimensions
- \bullet κ -collapsed
- vol $(B_r(x)) > c r^n$
- Zero AVR
- Not isometric to examples of Apostolov-Cifarelli

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$ -invariant, but non- $U(n)$ -invariant, steady gradient Kähler-Ricci solitons on \mathbb{C}^n with $\mathsf{Rm}^{1,1}_{\mathsf{op}} > 0.$

- This answers Cao's conjecture in the negative for all dimensions
- \bullet κ -collapsed
- vol $(B_r(x)) > c r^n$
- Zero AVR
- Not isometric to examples of Apostolov-Cifarelli

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$ -invariant, but non- $U(n)$ -invariant, steady gradient Kähler-Ricci solitons on \mathbb{C}^n with $\mathsf{Rm}^{1,1}_{\mathsf{op}} > 0.$

- This answers Cao's conjecture in the negative for all dimensions
- \bullet κ -collapsed
- vol $(B_r(x)) > c r^n$
- Zero AVR
- Not isometric to examples of Apostolov-Cifarelli

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$ -invariant, but non- $U(n)$ -invariant, steady gradient Kähler-Ricci solitons on \mathbb{C}^n with $\mathsf{Rm}^{1,1}_{\mathsf{op}} > 0.$

- This answers Cao's conjecture in the negative for all dimensions
- \bullet κ -collapsed
- vol $(B_r(x)) > c r^n$
- 7ero AVR

Not isometric to examples of Apostolov-Cifarelli

For $n \geq 2$, there exists a family of $U(1) \times U(n-1)$ -invariant, but non- $U(n)$ -invariant, steady gradient Kähler-Ricci solitons on \mathbb{C}^n with $\mathsf{Rm}^{1,1}_{\mathsf{op}} > 0.$

- This answers Cao's conjecture in the negative for all dimensions
- \bullet κ -collapsed
- vol $(B_r(x)) > c r^n$
- 7ero AVR
- Not isometric to examples of Apostolov-Cifarelli

- **ODE** methods
- **PDE** methods
- Collapsing methods (Lai '20)

o ODE methods

- **PDE methods**
- Collapsing methods (Lai '20)

- • ODE methods
- PDE methods
- Collapsing methods (Lai '20)

- • ODE methods
- PDE methods
- Collapsing methods (Lai '20)

A sequence of expanding gradient Ricci solitons $(M_i,\, g_i,\, p_i)$ with $\mathsf{Rm}(g_i) > 0$, scal $(p_i) = 1$, and $\mathsf{AVR}(g_i) := \lim\limits_{r \to \infty}$ $B_{g_i}(x, r)$ $\frac{\omega_{B_i}(n, r)}{\omega_{n-1}r^n} \rightarrow 0$ as $\bm{i}\rightarrow\bm{\infty}\Longrightarrow~(M_{i},\,g_{i},\,p_{i})\rightarrow_{pCG}(M,\,g,\,p)=$ a steady gradient Ricci soliton with $scal(p)=1$.

$$
Ric(g_i) + Hess_{g_i}(f_i) = \frac{1}{C_i}g_i
$$

Show that $C_i \rightarrow 0$ as $i \rightarrow \infty$.

A sequence of expanding gradient Ricci solitons $(M_i,\, g_i,\, p_i)$ with $\mathsf{Rm}(g_i) > 0$, scal $(p_i) = 1$, and $\mathsf{AVR}(g_i) := \lim\limits_{r \to \infty}$ $B_{g_i}(x, r)$ $\frac{\omega_{B_i}(n, r)}{\omega_{n-1}r^n} \rightarrow 0$ as $i\rightarrow\infty\Longrightarrow\; (M_i,\, g_i,\, p_i)\rightarrow_{pCG} (M,\, g,\, p)=$ a steady gradient Ricci soliton with $scal(p)=1$.

$$
Ric(g_i) + Hess_{g_i}(f_i) = \frac{1}{C_i}g_i
$$

Show that $C_i \rightarrow 0$ as $i \rightarrow \infty$.

A sequence of expanding gradient Ricci solitons $(M_i,\, g_i,\, p_i)$ with $\mathsf{Rm}(g_i) > 0$, scal $(p_i) = 1$, and $\mathsf{AVR}(g_i) := \lim\limits_{r \to \infty}$ $B_{g_i}(x, r)$ $\frac{\omega_{B_i}(n, r)}{\omega_{n-1}r^n} \rightarrow 0$ as $i\rightarrow\infty\Longrightarrow\; (M_i,\, g_i,\, p_i)\rightarrow_{pCG} (M,\, g,\, p)=$ a steady gradient Ricci soliton with scal(p) = 1.

Proof.

$$
Ric(g_i) + Hess_{g_i}(f_i) = \frac{1}{C_i}g_i
$$

Show that $C_i \rightarrow 0$ as $i \rightarrow \infty$.

A sequence of expanding gradient Ricci solitons $(M_i,\, g_i,\, p_i)$ with $\mathsf{Rm}(g_i) > 0$, scal $(p_i) = 1$, and $\mathsf{AVR}(g_i) := \lim\limits_{r \to \infty}$ $B_{g_i}(x, r)$ $\frac{\omega_{B_i}(n, r)}{\omega_{n-1}r^n} \to 0$ as $i\rightarrow\infty\Longrightarrow\; (M_i,\, g_i,\, p_i)\rightarrow_{pCG} (M,\, g,\, p)=$ a steady gradient Ricci soliton with $scal(p)=1$.

Proof.

$$
Ric(g_i) + Hess_{g_i}(f_i) = \frac{1}{C_i}g_i
$$

Show that $C_i \rightarrow 0$ as $i \rightarrow \infty$.
Let (L, g) be a compact connected Riemannian manifold. The Riemannian *cone* C with link L is defined to be $\mathbb{R}_{>0} \times L$ with metric $g_0 = dr^2 + r^2g$.

General strategy

For a sequence of collapsing metrics on the links satisfying condition A, the corresponding cone metric satisfies condition B.We can then lift these to expanding Ricci solitons satisfying condition C.Then by the compactness lemma, the sequence of expanding solitons degenerates to a steady Ricci soliton satisfying condition C.

Definition

Let (L, g) be a compact connected Riemannian manifold. The Riemannian *cone* C with link L is defined to be $\mathbb{R}_{>0} \times L$ with metric $g_0 = dr^2 + r^2g$.

General strategy

For a sequence of collapsing metrics on the links satisfying condition A, the corresponding cone metric satisfies condition B.We can then lift these to expanding Ricci solitons satisfying condition C.Then by the compactness lemma, the sequence of expanding solitons degenerates to a steady Ricci soliton satisfying condition C.

Definition

Let (L, g) be a compact connected Riemannian manifold. The Riemannian *cone* C with link L is defined to be $\mathbb{R}_{>0} \times L$ with metric $g_0 = dr^2 + r^2g$.

General strategy

For a sequence of collapsing metrics on the links satisfying condition A, the corresponding cone metric satisfies condition B.We can then lift these to expanding Ricci solitons satisfying condition C.Then by the compactness lemma, the sequence of expanding solitons degenerates to a steady Ricci soliton satisfying condition C.

Definition

Let (L, g) be a compact connected Riemannian manifold. The Riemannian *cone* C with link L is defined to be $\mathbb{R}_{>0} \times L$ with metric $g_0 = dr^2 + r^2g$.

General strategy

For a sequence of collapsing metrics on the links satisfying condition A, the corresponding cone metric satisfies condition B. We can then lift these to expanding Ricci solitons satisfying condition C.Then by the compactness lemma, the sequence of expanding solitons degenerates to a steady Ricci soliton satisfying condition C.

Definition

Let (L, g) be a compact connected Riemannian manifold. The Riemannian *cone* C with link L is defined to be $\mathbb{R}_{>0} \times L$ with metric $g_0 = dr^2 + r^2g$.

General strategy

For a sequence of collapsing metrics on the links satisfying condition A, the corresponding cone metric satisfies condition B.We can then lift these to expanding Ricci solitons satisfying condition C. Then by the compactness lemma, the sequence of expanding solitons degenerates to a steady Ricci soliton satisfying condition C.

Definition

Let (L, g) be a compact connected Riemannian manifold. The Riemannian *cone* C with link L is defined to be $\mathbb{R}_{>0} \times L$ with metric $g_0 = dr^2 + r^2g$.

General strategy

For a sequence of collapsing metrics on the links satisfying condition A, the corresponding cone metric satisfies condition B.We can then lift these to expanding Ricci solitons satisfying condition C.Then by the compactness lemma, the sequence of expanding solitons degenerates to a steady Ricci soliton satisfying condition C.

We construct Kähler metrics on \mathbb{P}^{n-1} with $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 2.$ The corresponding Kähler cone metric on \mathbb{C}^n satisfies $\mathsf{Rm}_{\mathsf{op}}^{1,1} \geq 0. \mathsf{By}$ C.-Deruelle '16, we can lift these to expanding Kähler-Ricci solitons on \mathbb{C}^n satisfying $\mathsf{Rm}^{1,1}_{\mathsf{op}}>0.$

We construct Kähler metrics on \mathbb{P}^{n-1} with $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 2.$ The corresponding Kähler cone metric on \mathbb{C}^n satisfies $\mathsf{Rm}^{1,1}_{\mathsf{op}} \ge 0$ By C.-Deruelle '16, we can lift these to expanding Kähler-Ricci solitons on \mathbb{C}^n satisfying $\mathsf{Rm}^{1,1}_{\mathsf{op}}>0.$

We construct Kähler metrics on \mathbb{P}^{n-1} with $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 2.$ The corresponding Kähler cone metric on \mathbb{C}^n satisfies $\mathsf{Rm}_{\mathsf{op}}^{1,1} \geq 0.$ By C.-Deruelle '16, we can lift these to expanding Kähler-Ricci solitons on \mathbb{C}^n satisfying $\mathsf{Rm}^{1,1}_{\mathsf{op}}>0.$

A Kähler manifold with bisectional curvature $BK > 2$ must satisfy diam $(M,\, g)\leq \frac{\pi}{2}=\mathsf{diam}(\mathbb{P}^n,\, \frac{1}{2})$

If a Kähler manifold has bisectional curvature $BK > 2$ and $\textsf{diam}(M,\,g)=\frac{\pi}{2}.$ then the Kähler manifold is holomorphically isometric to

Is this theorem almost rigid?

Theorem (Li-Wang '05)

A Kähler manifold with bisectional curvature $BK \geq 2$ must satisfy diam $(M,\, g)\leq \frac{\pi}{2}=\mathsf{diam}(\mathbb{P}^n,\, \frac{1}{2})$ $\frac{1}{2}\omega_{FS}$).

If a Kähler manifold has bisectional curvature $BK > 2$ and $\textsf{diam}(M,\,g)=\frac{\pi}{2}.$ then the Kähler manifold is holomorphically isometric to

Is this theorem almost rigid?

Theorem (Li-Wang '05)

A Kähler manifold with bisectional curvature $BK > 2$ must satisfy diam $(M,\, g)\leq \frac{\pi}{2}=\mathsf{diam}(\mathbb{P}^n,\, \frac{1}{2})$ $\frac{1}{2}\omega_{FS}$).

Theorem (Diameter Rigidity; Datar-Seshadri '23)

If a Kähler manifold has bisectional curvature $BK > 2$ and diam $(M, g) = \frac{\pi}{2}$, then the Kähler manifold is holomorphically isometric to $(\mathbb{P}^n, \frac{1}{2})$ $\frac{1}{2}\omega_{FS}$).

Is this theorem almost rigid?

[M](#page-120-0)[ay](#page-116-0) [1](#page-117-0)[5t](#page-120-0)[h](#page-121-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2024 (15th 2024 Chan and Yi Lai)

Theorem (Li-Wang '05)

A Kähler manifold with bisectional curvature $BK > 2$ must satisfy diam $(M,\, g)\leq \frac{\pi}{2}=\mathsf{diam}(\mathbb{P}^n,\, \frac{1}{2})$ $\frac{1}{2}\omega_{FS}$).

Theorem (Diameter Rigidity; Datar-Seshadri '23)

If a Kähler manifold has bisectional curvature $BK > 2$ and diam $(M, g) = \frac{\pi}{2}$, then the Kähler manifold is holomorphically isometric to $(\mathbb{P}^n, \frac{1}{2})$ $\frac{1}{2}\omega_{FS}$).

Is this theorem almost rigid?

Let $n > 1$. Then for all $\varepsilon > 0$, there exists a $U(n)$ -invariant Kähler metric g on \mathbb{P}^n with $\mathsf{Rm}^{1,1}_{\rm op} \geq 2$ (in particular, the holomorphic bisectional curvature $BK > 2$) such that

$d_{GH}\left((\mathbb{P}^n, d_g), [0, \frac{\pi}{2} \right)$

• This implies Theorem A by the general strategy.

Theorem B (Chan-C.-Lai '24)

Let $n \geq 1$. Then for all $\varepsilon > 0$, there exists a $U(n)$ -invariant Kähler metric ${\sf g}\,$ on $\mathbb{P}^n\,$ with $\mathsf{Rm}^{1,1}_{\rm op}\geq 2$ (in particular, the holomorphic bisectional curvature $BK > 2$) such that

> $d_{GH}\left((\mathbb{P}^n, d_g), [0, \frac{\pi}{2} \right)$ $\frac{\pi}{2}$]) $\leq \varepsilon$.

• This implies Theorem A by the general strategy.

Theorem B (Chan-C.-Lai '24)

Let $n \geq 1$. Then for all $\varepsilon > 0$, there exists a $U(n)$ -invariant Kähler metric ${\sf g}\,$ on $\mathbb{P}^n\,$ with $\mathsf{Rm}^{1,1}_{\rm op}\geq 2$ (in particular, the holomorphic bisectional curvature $BK > 2$) such that

> $d_{GH}\left((\mathbb{P}^n, d_g), [0, \frac{\pi}{2} \right)$ $\frac{\pi}{2}$]) $\leq \varepsilon$.

• This implies Theorem A by the general strategy.

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n $\bullet \xi =$ Reeb vector field\n $\bullet \eta =$ contact form\n $\bullet \ d\eta = g^T =$ transverse metric\n $\bullet \ g = \eta \otimes \eta + g^T$ \n
\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
\n

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \leadsto \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler
\n- $a = bb'$
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

$$
\text{ on } S^{2n-1}\times \left(0,\,\frac{\pi}{2}\right)
$$

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

 $(M, g, \eta, \xi) =$ Sasaki manifold

- $\bullet \xi$ = Reeb vector field
- \bullet η = contact form

•
$$
d\eta = g^T
$$
 = transverse metric

$$
\bullet \ \ g = \eta \otimes \eta + g^{\mathsf{T}}
$$

Consider $M = M \times (0, L)$, together with the doubly-warped product metric

 $\hat g=dr^2+a^2(r)\eta\otimes \eta+b^2(r)g^{\sf T},\qquad$ a(r), b(r) >0

•
$$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
 is a complex manifold
• \widehat{g} Kähler $\Longleftrightarrow a = bb'$

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

R. Conlon (UT Dallas) Kähler-Ricci solitons

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$

2 \setminus

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

•
$$
(M, g, \eta, \xi)
$$
 = Sasaki manifold

 $\bullet \xi$ = Reeb vector field

•
$$
\eta
$$
 = contact form

$$
\bullet \ \ d\eta = g^{\mathcal{T}} = \text{transverse metric}
$$

$$
\bullet \ \ g=\eta\otimes \eta+g^{\,7}
$$

Consider $M = M \times (0, L)$, together with the doubly-warped product metric

 $\hat g=dr^2+a^2(r)\eta\otimes \eta+b^2(r)g^{\sf T},\qquad$ a(r), b(r) >0

•
$$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
 is a complex manifold
• \widehat{g} Kähler $\Longleftrightarrow a = bb'$

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$ 2 \setminus

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

•
$$
(M, g, \eta, \xi)
$$
 = Sasaki manifold

- $\bullet \xi$ = Reeb vector field
- \bullet η = contact form

$$
\bullet \ \ d\eta = g^{\mathcal{T}} = \text{transverse metric}
$$

$$
\bullet \ \ g = \eta \otimes \eta + g^{\mathsf{T}}
$$

Consider $M = M \times (0, L)$, together with the doubly-warped product metric

 $\hat g=dr^2+a^2(r)\eta\otimes \eta+b^2(r)g^{\sf T},\qquad$ a(r), b(r) >0

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \leadsto \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler
\n- $a = bb'$
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^{T}
$$

$$
\text{ on } S^{2n-1}\times \left(0,\,\frac{\pi}{2}\right)
$$

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

•
$$
(M, g, \eta, \xi)
$$
 = Sasaki manifold

- $\bullet \xi$ = Reeb vector field
- \bullet η = contact form

$$
\bullet \ \ d\eta=g^\mathcal{T}=\text{transverse metric}
$$

 $\mathbb{g} = \eta \otimes \eta + \mathbb{g}^{\mathsf{T}}$ Consider $M = M \times (0, L)$, together with the doubly-warped product metric $\hat g=dr^2+a^2(r)\eta\otimes \eta+b^2(r)g^{\sf T},\qquad$ a(r), b(r) >0

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \leadsto \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler
\n- $a = bb'$
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^{T}
$$

$$
\text{ on } S^{2n-1}\times \left(0,\,\frac{\pi}{2}\right)
$$

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

•
$$
(M, g, \eta, \xi)
$$
 = Sasaki manifold

- $\bullet \xi$ = Reeb vector field
- \bullet η = contact form

$$
\bullet \ \ d\eta = g^{\mathcal{T}} = \text{transverse metric}
$$

 $\mathsf{g} = \eta \otimes \eta + \mathsf{g}^\mathsf{T}$

Consider $M = M \times (0, L)$, together with the doubly-warped product metric $\hat g=dr^2+a^2(r)\eta\otimes \eta+b^2(r)g^{\sf T},\qquad$ a(r), b(r) >0

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \leadsto \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler
\n- $a = bb'$
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$ 2 \setminus

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

•
$$
(M, g, \eta, \xi)
$$
 = Sasaki manifold

- $\bullet \xi$ = Reeb vector field
- \bullet η = contact form

$$
\bullet \ \ d\eta=g^\mathcal{T}=\text{transverse metric}
$$

$$
\bullet \ \ \mathsf{g} = \eta \otimes \eta + \mathsf{g}^{\, \mathsf{T}}
$$

Consider $M = M \times (0, L)$, together with the doubly-warped product metric

 $\hat g=dr^2+a^2(r)\eta\otimes \eta+b^2(r)g^{\sf T},\qquad$ a(r), b(r) >0

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler
\n- \iff $a = bb'$
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$ 2 \setminus

 M [ay](#page-123-0) [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) P a \bm{k}

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n
	\n- \n $\xi =$ Reeb vector field\n
		\n- \n $\eta =$ contact form\n
			\n- \n $d\eta = g^T$ = transverse metric\n
			\n\n
		\n

$$
\bullet \ \ g = \eta \otimes \eta + g^{\mathsf{T}}
$$

Consider $M = M \times (0, L)$, together with the doubly-warped product metric

$$
\hat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T, \qquad a(r), b(r) > 0
$$

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler
\n- \iff $a = bb'$
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

on
$$
S^{2n-1} \times \left(0, \frac{\pi}{2}\right)
$$

 M [ay](#page-123-0) [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) P a \bm{k}

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n $\bullet \xi =$ Reeb vector field\n $\bullet \eta =$ contact form\n $\bullet \eta = g^T =$ transverse metric\n $\bullet \ g = \eta \otimes \eta + g^T$ \n
\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
\n

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler
\n- $a = bb'$
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$ 2 \setminus

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n $\bullet \xi =$ Reeb vector field\n $\bullet \eta =$ contact form\n $\bullet \eta = g^T =$ transverse metric\n $\bullet \ g = \eta \otimes \eta + g^T$ \n
\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
\n

\n- \n
$$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
\n is a complex manifold\n
\n- \n \widehat{g} \n Kähler\n
\n- \n $a = bb'$ \n
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

R. Conlon (UT Dallas) Kähler-Ricci solitons

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$

2 \setminus

 Mav Mav Mav [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) Pak

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n
	\n- \n $\epsilon =$ Reeb vector field\n
		\n- \n $\eta =$ contact form\n
			\n- \n $d\eta = g^T =$ transverse metric\n
				\n- \n $g = \eta \otimes \eta + g^T$ \n
				\n\n
			\n\n
		\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
		\n

\n- \n
$$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
 is a complex manifold\n
\n- \n \widehat{g} Kähler\n
\n- \n $a = bb'$ \n
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

R. Conlon (UT Dallas) Kähler-Ricci solitons

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$

2 \setminus

 M [ay](#page-123-0) [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) P a \bm{k}

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n $\bullet \xi =$ Reeb vector field\n $\bullet \eta =$ contact form\n $\bullet \eta = g^T =$ transverse metric\n $\bullet \, g = \eta \otimes \eta + g^T$ \n
\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
\n

\n- \n
$$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
\n is a complex manifold\n
\n- \n \widehat{g} \n Kähler\n
\n- \n $a = bb'$ \n
\n

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$.

R. Conlon (UT Dallas) Kähler-Ricci solitons

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$

2 \setminus

 M [ay](#page-123-0) [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) P a \bm{k}

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n
	\n- \n $\epsilon =$ Reeb vector field\n
		\n- \n $\eta =$ contact form\n
			\n- \n $d\eta = g^T =$ transverse metric\n
				\n- \n $g = \eta \otimes \eta + g^T$ \n
				\n\n
			\n\n
		\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
		\n

\n- \n
$$
\widehat{J}(a(r)\partial_r) = \xi \rightsquigarrow \widehat{M}
$$
 is a complex manifold\n
\n- \n \widehat{g} Kähler\n
\n- \n $a = bb'$ \n
\n

Example (Fubini-Study metric on \mathbb{P}^n)

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

$$
\underline{\text{Note:}} \ \mathsf{Rm}_{\text{op}}^{1,1} \geq 2.
$$

R. Conlon (UT Dallas) Kähler-Ricci solitons

20 / 25

on $S^{2n-1}\times\left(0,\frac{\pi}{2}\right)$

2 \setminus

 M [ay](#page-123-0) [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) $\mathsf{Pa} \mathsf{k}$

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n
	\n- \n $\epsilon =$ Reeb vector field\n
		\n- \n $\eta =$ contact form\n
			\n- \n $d\eta = g^T =$ transverse metric\n
				\n- \n $g = \eta \otimes \eta + g^T$ \n
				\n\n
			\n\n
		\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
		\n

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \leadsto \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler $\Longleftrightarrow a = bb'$
\n

Example (Fubini-Study metric on \mathbb{P}^n)

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

$$
\mathrm{on}\ S^{2n-1}\times\left(0,\,\frac{\pi}{2}\right)
$$

 M [ay](#page-123-0) [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) $\mathsf{Pa} \mathsf{k}$

<u>Note:</u> Rm $_{\text{op}}^{1,1} \geq 2$. R. Conlon (UT Dallas) Kähler-Ricci solitons

20 / 25

\n- \n
$$
(M, g, \eta, \xi) =
$$
 Sasaki manifold\n
	\n- \n $\epsilon =$ Reeb vector field\n
		\n- \n $\eta =$ contact form\n
			\n- \n $d\eta = g^T =$ transverse metric\n
				\n- \n $g = \eta \otimes \eta + g^T$ \n
				\n\n
			\n\n
		\n- \n Consider $\widehat{M} = M \times (0, L)$, together with the doubly-warped product metric\n $\widehat{g} = dr^2 + a^2(r)\eta \otimes \eta + b^2(r)g^T$,\n $a(r), b(r) > 0$ \n
		\n

\n- $$
\widehat{J}(a(r)\partial_r) = \xi \leadsto \widehat{M}
$$
 is a complex manifold
\n- \widehat{g} Kähler $\Longleftrightarrow a = bb'$
\n

Example (Fubini-Study metric on \mathbb{P}^n)

$$
\frac{1}{2}\omega_{FS} = dr^2 + \frac{\sin^2(2r)}{4}\eta \otimes \eta + \sin^2(r)g^T
$$

$$
\text{ on } S^{2n-1}\times \left(0,\,\frac{\pi}{2}\right)
$$

 M [ay](#page-123-0) [1](#page-124-0)[5t](#page-138-0)[h](#page-139-0) [2024](#page-0-0) [\(](#page-167-0)[joi](#page-0-0)[nt](#page-167-0) [w](#page-0-0)[ith](#page-167-0) $\mathsf{Pa} \mathsf{k}$

<u>Note:</u> Rm $_{\text{op}}^{1,1}$ \geq 2. R. Conlon (UT Dallas) Kähler-Ricci solitons

20 / 25

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

 \bullet h_k is $U(n-1)$ -invariant;

 $2\,$ h_{k} satisfies $\, \mathsf{Rm}^{1,1}_{\mathsf{op}} > 2$ on the smooth part;

- 3 h_k collapses to $[0,\frac{\pi}{2}]$ $\frac{\pi}{2}$ as $k \to \infty$;
- \bullet has a Kähler cone singularity at $r = 0$:

$$
Cone_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$

[M](#page-140-0)[ay](#page-138-0) [1](#page-139-0)[5t](#page-146-0)[h](#page-147-0) 2021 [\(](#page-167-0)15th 2022 (15th 2022 Chan and Yi Lai)

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

 \bullet h_k is $U(n-1)$ -invariant;

- $2\,$ h_{k} satisfies $\, \mathsf{Rm}^{1,1}_{\mathsf{op}} > 2$ on the smooth part;
- 3 h_k collapses to $[0,\frac{\pi}{2}]$ $\frac{\pi}{2}$ as $k \to \infty$;
- \bullet has a Kähler cone singularity at $r = 0$:

$$
Cone_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$

[M](#page-141-0)[ay](#page-138-0) [1](#page-139-0)[5t](#page-146-0)[h](#page-147-0) 2021 [\(](#page-167-0)15th 2022 (15th 2022 Chan and Yi Lai)

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

 \bullet h_k is $U(n-1)$ -invariant;

 $2\,$ h_{k} satisfies $\, \mathsf{Rm}^{1,1}_{\mathsf{op}} > 2$ on the smooth part;

- 3 h_k collapses to $[0,\frac{\pi}{2}]$ $\frac{\pi}{2}$ as $k \to \infty$;
- \bullet has a Kähler cone singularity at $r = 0$:

$$
Cone_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$

[M](#page-142-0)[ay](#page-138-0) [1](#page-139-0)[5t](#page-146-0)[h](#page-147-0) 2021 [\(](#page-167-0)15th 2022 (15th 2022 Chan and Yi Lai)

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

\bullet h_k is $U(n-1)$ -invariant;

 $2\,$ h_{k} satisfies $\, \mathsf{Rm}^{1,1}_{\mathsf{op}} > 2$ on the smooth part;

- 3 h_k collapses to $[0,\frac{\pi}{2}]$ $\frac{\pi}{2}$ as $k \to \infty$;
- \bullet has a Kähler cone singularity at $r = 0$:

$$
Cone_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

1 h_k is $U(n-1)$ -invariant;

 \bullet h_k satisfies $\, \mathsf{Rm}^{1,1}_{\rm op} > 2$ on the smooth part;

3 h_k collapses to $[0,\frac{\pi}{2}]$ $\frac{\pi}{2}$ as $k \to \infty$;

 \bullet has a Kähler cone singularity at $r = 0$:

$$
Cone_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$
$\mathsf{Step\ 1}$ (The model metrics from Fubini-Study on \mathbb{P}^{n-1}):For any $k > 1$, consider the metric

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

1 h_k is $U(n-1)$ -invariant;

 \bullet h_k satisfies $\, \mathsf{Rm}^{1,1}_{\rm op} > 2$ on the smooth part;

3 h_k collapses to $[0, \frac{\pi}{2}]$ $\frac{\pi}{2}$] as $k \to \infty$;

 \bullet has a Kähler cone singularity at $r = 0$:

$$
Cone_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$

 $\mathsf{Step\ 1}$ (The model metrics from Fubini-Study on \mathbb{P}^{n-1}):For any $k > 1$, consider the metric

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

1 h_k is $U(n-1)$ -invariant;

- \bullet h_k satisfies $\, \mathsf{Rm}^{1,1}_{\rm op} > 2$ on the smooth part;
- **3** h_k collapses to $[0, \frac{\pi}{2}]$ $\frac{\pi}{2}$] as $k \to \infty$;
- \bullet has a Kähler cone singularity at $r = 0$:

$$
Cone_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$

[M](#page-146-0)at[h](#page-147-0) [\(](#page-167-0)Biki Ban Yeung Chan and Yi Lai)

 $\mathsf{Step\ 1}$ (The model metrics from Fubini-Study on \mathbb{P}^{n-1}):For any $k > 1$, consider the metric

$$
h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^{\mathsf{T}}, \qquad r \in [0, \frac{\pi}{2}].
$$

1 h_k is $U(n-1)$ -invariant;

- \bullet h_k satisfies $\, \mathsf{Rm}^{1,1}_{\rm op} > 2$ on the smooth part;
- **3** h_k collapses to $[0, \frac{\pi}{2}]$ $\frac{\pi}{2}$] as $k \to \infty$;
- \bullet has a Kähler cone singularity at $r = 0$:

$$
\mathsf{Cone}_k = dr^2 + \frac{r^2}{k^2} \eta \otimes \eta + \frac{r^2}{k} g^T, \qquad r \in [0, \infty).
$$

[M](#page-147-0)at[h](#page-147-0) [\(](#page-167-0)Biki Ban Yeung Chan and Yi Lai)

There is a $U(n-1)$ -invariant expanding Kähler-Ricci soliton on \mathbb{C}^{n-1} asymptotic to Cone_k, $k > 1$.

Gluing: Cut off the conical singularity of h_k and glue in a suitable expanding soliton in Cao's family at a small scale $\delta_i > 0$.

Theorem (Cao '96)

There is a $U(n-1)$ -invariant expanding Kähler-Ricci soliton on \mathbb{C}^{n-1} asymptotic to Cone_k, $k > 1$.

Gluing: Cut off the conical singularity of h_k and glue in a suitable expanding soliton in Cao's family at a small scale $\delta_i > 0$.

Theorem (Cao '96)

There is a $U(n-1)$ -invariant expanding Kähler-Ricci soliton on \mathbb{C}^{n-1} asymptotic to Cone_k, $k > 1$.

Gluing: Cut off the conical singularity of h_k and glue in a suitable expanding soliton in Cao's family at a small scale $\delta_i > 0$.

Theorem (Cao '96)

There is a $U(n-1)$ -invariant expanding Kähler-Ricci soliton on \mathbb{C}^{n-1} asymptotic to Cone_k, $k > 1$.

Gluing: Cut off the conical singularity of h_k and glue in a suitable expanding soliton in Cao's family at a small scale $\delta_i > 0$.

- a smooth Kähler metric close to the singular metric;
- smooths out the Kähler conical singularity;
- Θ $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 0$ everywhere and $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 2$ holds outside a tiny region.

Letting the gluing scale $\delta_i \rightarrow 0$, we obtain a sequence of such metrics $h_{k,i}$ converging to h_k .

- a smooth Kähler metric close to the singular metric;
- smooths out the Kähler conical singularity;
- Θ $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 0$ everywhere and $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 2$ holds outside a tiny region.

Letting the gluing scale $\delta_i \rightarrow 0$, we obtain a sequence of such metrics $h_{k,i}$ converging to h_k .

■ a smooth Kähler metric close to the singular metric;

smooths out the Kähler conical singularity;

 Θ $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 0$ everywhere and $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 2$ holds outside a tiny region.

Letting the gluing scale $\delta_i \rightarrow 0$, we obtain a sequence of such metrics $h_{k,i}$ converging to h_k .

[M](#page-154-0)[ay](#page-150-0) [1](#page-151-0)[5t](#page-156-0)[h](#page-157-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2022 (15th 2022 (15th 2021 (15th 2022 (15th 2022 (15th 20

■ a smooth Kähler metric close to the singular metric;

- 2 smooths out the Kähler conical singularity;
- Θ $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 0$ everywhere and $\mathsf{Rm}_{\mathsf{op}}^{1,1} > 2$ holds outside a tiny region.

Letting the gluing scale $\delta_i \rightarrow 0$, we obtain a sequence of such metrics $h_{k,i}$ converging to h_k .

[M](#page-155-0)[ay](#page-150-0) [1](#page-151-0)[5t](#page-156-0)[h](#page-157-0) [2024](#page-0-0) [\(](#page-167-0)15th 2024 (15th 2022 (15th 2022 (15th 2022 (15th 2021 (15th 2022 (15th 2022 (15th 20

- ■ a smooth Kähler metric close to the singular metric;
- 2 smooths out the Kähler conical singularity;
- Θ $\mathsf{Rm}^{1,1}_{\mathsf{op}}>0$ everywhere and $\mathsf{Rm}^{1,1}_{\mathsf{op}}>2$ holds outside a tiny region.

Letting the gluing scale $\delta_i \rightarrow 0$, we obtain a sequence of such metrics $h_{k,i}$ converging to h_k .

[M](#page-156-0) 제 H [2024](#page-0-0) [\(](#page-167-0)Pak-Yeung Chan And Alex Chan And And Chan And Young Chan and Young Chan and Young Chan and Young Ch

- ■ a smooth Kähler metric close to the singular metric;
- 2 smooths out the Kähler conical singularity;
- Θ $\mathsf{Rm}^{1,1}_{\mathsf{op}}>0$ everywhere and $\mathsf{Rm}^{1,1}_{\mathsf{op}}>2$ holds outside a tiny region.

Letting the gluing scale $\delta_i \rightarrow 0$, we obtain a sequence of such metrics $h_{k,i}$ converging to h_k .

[M](#page-157-0) 제 H 시 파 H - 미 파 - KO Q Ch

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(\text{Rm}) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2.$ Let $k \to \infty$. This implies Theorem B.

[M](#page-158-0)[ay](#page-156-0) [1](#page-157-0)[5t](#page-166-0)[h](#page-167-0) 2021 (15th 2021) [2024](#page-0-0) [\(](#page-167-0)15th 2022) 2022 (15th 2022) 2022 20

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(\text{Rm}) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2.$ Let $k \to \infty$. This implies Theorem B.

[M](#page-159-0)[ay](#page-156-0) [1](#page-157-0)[5t](#page-166-0)[h](#page-167-0) 2021 [\(](#page-167-0)15th 2022 (15th 2022 Chan and Yi Lai)

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(\text{Rm}) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2.$ Let $k \to \infty$. This implies Theorem B.

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(\text{Rm}) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2.$ Let $k \to \infty$. This implies Theorem B.

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(\text{Rm}) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2.$ Let $k \to \infty$. This implies Theorem B.

R. Conlon (UT Dallas) Kähler-Ricci solitons

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(Rm) \geq 0$ if $Rm \geq 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \geq 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \geq 2.$ Let $k \to \infty$. This implies Theorem B.

[M](#page-163-0)any [1](#page-157-0)[5t](#page-166-0)[h](#page-167-0) [2024](#page-0-0) [\(](#page-167-0)15th 2021) 2024 (15th 2022) 2024 (15th 2021) 2022 (15th 2021) 20

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(Rm) \geq 0$ if Rm $\geq 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \geq 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \geq 2.$ Let $k \to \infty$. This implies Theorem B.

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(Rm) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2$. Let $k \to \infty$. This implies Theorem B.

[M](#page-165-0)agnet Konzept Wit[h](#page-167-0) Pake-Yeung Chan And Chan and Yi Reserves The Pake-Yeung Chan and Yi

In the curvature evolution equation

$$
\partial_t \operatorname{Rm} = \Delta_{g_t} \operatorname{Rm} + Q(\operatorname{Rm}),
$$

 $Q(Rm) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2$. Let $k \to \infty$. This implies Theorem B.

[M](#page-166-0)agnet Konzept Wit[h](#page-167-0) Pake-Yeung Chan And Chan and Yi Reserves The Pake-Yeung Chan and Yi

In the curvature evolution equation

$$
\partial_t Rm = \Delta_{g_t} Rm + Q(Rm),
$$

 $Q(Rm) \ge 0$ if Rm $\ge 0 \Longrightarrow Rm_{h_{k,i}(t)}^{1,1} \ge 2 - \delta_i \Longrightarrow Rm_{h_k(t)}^{1,1} \ge 2$. Let $k \to \infty$. This implies Theorem B.

[M](#page-167-0)agnet Konzept Wit[h](#page-167-0) Pake-Yeung Chan And Chan and Yi Reserves The Pake-Yeung Chan and Yi

Thank you for your attention!