Toric Nearly Kähler 6-manifolds

Andrei Moroianu
CNRS - Paris-Sud University

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– joint work with Paul-Andi Nagy –
Nearly Kähler manifolds were originally introduced as the class $W_1$ in the Gray-Hervella classification of almost Hermitian manifolds.

More precisely, an almost Hermitian manifold $(M^{2n}, g, J)$ is called nearly Kähler (NK) if

$$(\nabla_X J)(X) = 0$$

for every vector field $X$ on $M$, where $\nabla$ denotes the Levi-Civita covariant derivative of $g$. A NK manifold is called strict if $(\nabla J)_p \neq 0$ for every $p \in M$.

Remark. In dimension $2n = 4$, NK = Kähler.
Examples:

- Kähler manifolds.

- twistor spaces over positive QK manifolds, endowed with the non-integrable almost com-
  plex structure and with the metric rescaled by a factor 2 on the fibres.

- naturally reductive 3-symmetric spaces $G/H$ where $G$ is compact, $H$ is the invariant
group of an automorphism $\sigma$ of $G$ of or-
der 3, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, and $\mathfrak{p}$ has a scalar product
such that for every $X, Y, Z \in \mathfrak{p}$:

  $$\langle [X, Y]_p, Z \rangle + \langle [X, Z]_p, Y \rangle = 0.$$ 

The almost complex structure is determined by the endomorphism $J$ of $\mathfrak{p}$ satisfying

$$\sigma_* = -\frac{1}{2} \text{Id}_p + \frac{\sqrt{3}}{2} J.$$
A product of NK manifolds is again NK. Conversely, the factors of the de Rham decomposition of a NK manifold are NK.

**Theorem.** (Nagy 2002): Every simply connected, complete, de Rham irreducible NK manifold is either one of the above examples, or a strict NK 6-manifold.

From now on, we restrict our attention to strict NK 6-manifolds. These are interesting for several reasons:
Properties of strict NK 6-manifolds:

• carry real Killing spinors \( \rightsquigarrow \) positive Einstein; after rescaling the metric, one can normalize them to having scalar curvature 30 (like the round \( S^6 \)).

• \( \nabla J \) has constant norm \( \rightsquigarrow \) SU(3)-structure

• carry a connection with parallel and skew-symmetric torsion

\[
\tilde{\nabla}_X = \nabla_X - \frac{1}{2} J \circ \nabla_X J
\]

• the Riemannian cone \((M \times \mathbb{R}^*, t^2 g + dt^2)\) of a normalized NK 6-manifold \((M, g, \omega)\) has holonomy contained in \( G_2 \), defined by the positive 3-form

\[
\varphi = \frac{1}{3} d(t^3 \omega) = \frac{1}{3} t^3 d\omega + t^2 dt \wedge \omega
\]
Main problem: lack of examples.

3-symmetric spaces were classified by Gray. In dimension 6:

- $S^6 = G_2 / SU(3)$
- $SU(2) \times SU(2) \times SU(2) / \Delta \sim S^3 \times S^3$
- $Sp(2) / U(2) \sim \mathbb{C}P^3$
- $SU(3) / U(1) \times U(1) \sim F(1, 2)$.

**Theorem.** (Butruille 2004) These are all homogeneous SNK 6-manifolds.
Foscolo and Haskins (2017): 2 new examples (of cohomogeneity 1) on $S^6$ and $S^3 \times S^3$, both with isometry group $SU(2) \times SU(2)$.

Deformations of SNK 6-manifolds were studied by –, Nagy, Semmelmann (2008, 2010, 2011).

The moduli space is isomorphic to the space of co-closed primitive $(1,1)$-forms which are eigenforms of the Laplace operator for the eigenvalue 12.

Using representation theory one can compute this space on the homogeneous examples. It vanishes except on $F(1,2)$ where it has dimension 8. However, these infinitesimal deformations are obstructed (Foscolo 2017).
SU(3)-structures on SNK 6-manifolds

Let $M^6$ be an oriented manifold. An SU(3)-structure on $M$ is a triple $(g, J, \psi)$, where

- $g$ is a Riemannian metric,
- $J$ is a compatible almost complex structure (i.e. $\omega := g(J\cdot, \cdot)$ is a 2-form),
- $\psi = \psi^+ + i\psi^-$ is a $(3, 0)$ complex volume form satisfying
  \[ \psi^+ \wedge \psi^- = 4\text{vol}_g = \frac{2}{3} \omega^3. \]

It is possible to characterize SU(3)-structures in terms of exterior forms only (Hitchin).
Lemma 1 A pair $(\omega, \psi^+)$ $\in C^\infty(\Lambda^2 M \times \Lambda^3 M)$ defines an SU(3)-structure on $M$ provided that:

- $\omega^3 \neq 0$ (i.e. $\omega$ is non-degenerate).
- $\omega \wedge \psi^+ = 0$.
- If $K \in \text{End}(TM) \otimes \Lambda^6 M$ is defined by $K(X) := (X \downarrow \psi^+) \wedge \psi^+ \in \Lambda^5 M \cong TM \otimes \Lambda^6 M$, then $\text{tr} K^2 = -\frac{1}{6}(\omega^3)^2 \in (\Lambda^6 M)^\otimes 2$
- $\omega(X, K(X))/\omega^3 > 0$ for every $X \neq 0$.

“Proof”: Define $J := 6K/\omega^3$, $g(\cdot, \cdot) := \omega(\cdot, J \cdot)$, $\psi^- := -\psi^+(J \cdot, \cdot, \cdot)$. 
A normalized SNK structure \((g, J, \omega)\) on \(M^6\) \(\leadsto\) SU(3)-structure \((g, J, \omega, \psi^+, \psi^-)\) where
\[
\psi^+ := \nabla \omega, \quad \psi^- := -\psi^+(J \cdot, \cdot, \cdot).
\]

This satisfies the exterior differential system
\[
\begin{cases}
\text{d}\omega = 3\psi^+ \\
\text{d}\psi^- = -2\omega^2.
\end{cases}
\]

Conversely, an SU(3)-structure satisfying this system is a normalized SNK structure (Hitchin).

This is similar to the case of \(G_2\) structures, where a stable 3-form is parallel if and only if it is harmonic.
Toric NK 6-manifolds

An infinitesimal automorphism of a normalized SNK 6-manifold \((M, g, J, \omega, \psi^\pm)\) is a vector field \(\xi\) whose flow preserves the whole structure (enough to have \(\mathcal{L}_\xi \omega = 0 = \mathcal{L}_\xi \psi^\pm\)).

**Lemma.** \(\text{rk}(\text{aut}(M, g, J)) \leq 3\).

If equality holds, \((M, g, J)\) is called toric. The only homogeneous example is \(S^3 \times S^3\).

Assume from now on that \((M, g, J)\) is toric and let \(\xi_1, \xi_2, \xi_3\) be a basis of \(\text{aut}(M, g, J)\).

**Lemma.** The vector fields

\[
\xi_1, \xi_2, \xi_3, J\xi_1, J\xi_2, J\xi_3
\]

are linearly independent on a dense open subset \(M_0\) of \(M\).

\(\leadsto\) dual basis \(\{\theta^1, \theta^2, \theta^3, \gamma^1, \gamma^2, \gamma^3\}\) of \(\Lambda^1 M_0\).
Define the functions
\[ \mu_{ij} := \omega(\xi_i, \xi_j), \quad \varepsilon := \psi^-(\xi_1, \xi_2, \xi_3). \]

The Cartan formula and
\[
\begin{align*}
\text{d} \omega &= 3 \psi^+ \\
\text{d} \psi^- &= -2 \omega^2
\end{align*}
\]

\[ \text{d} \mu_{ij} = \text{d}(\xi_j \llcorner \xi_i \llcorner \omega) = -\xi_j \llcorner \text{d}(\xi_i \llcorner \omega) = \xi_j \llcorner \xi_i \llcorner \text{d} \omega = -3 \xi_i \llcorner \xi_j \llcorner \psi^+. \]

Similarly,
\[ \text{d} \varepsilon = \text{d}(\xi_3 \llcorner \xi_2 \llcorner \xi_1 \llcorner \psi^-) = -\xi_3 \llcorner \xi_2 \llcorner \xi_1 \llcorner \text{d} \psi^- = 2 \xi_3 \llcorner \xi_2 \llcorner \xi_1 \llcorner \omega^2. \]

Remarks:

1. \[ \psi^+(\xi_1, \xi_2, \xi_3) = 0 \text{ on } M. \]

2. \[ \varepsilon \text{ does not vanish on } M_0. \]
It follows that the map $\mu : M \to \Lambda^2 \mathbb{R}^3 \cong \mathfrak{so}(3)$ defined by

$$\mu := \begin{pmatrix} 0 & \mu_{12} & \mu_{13} \\ \mu_{21} & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{pmatrix}$$

is the multi-moment map of the strong geometry $(M, \psi^+)$ defined by Madsen and Swann (and studied further by Dixon in the particular case where $M = S^3 \times S^3$).

Similarly, the function $\varepsilon$ is the multi-moment map associated to the stable closed 4-form $d\psi^-$. 
Consider the symmetric $3 \times 3$ matrix
\[ C := (C_{ij}) = (g(\xi_i, \xi_j)). \]

In terms of the basis $\{\theta^1, \theta^2, \theta^3, \gamma^1, \gamma^2, \gamma^3\}$ of $\Lambda^1 M_0$ we can write
\[
\psi^+ = \varepsilon(\gamma^{123} - \theta^{12} \wedge \gamma^3 - \theta^{31} \wedge \gamma^2 - \theta^{23} \wedge \gamma^1),
\]
\[
\psi^- = \varepsilon(\theta^{123} - \gamma^{12} \wedge \theta^3 - \gamma^{31} \wedge \theta^2 - \gamma^{23} \wedge \theta^1),
\]
where $\gamma^{123} = \gamma^1 \wedge \gamma^2 \wedge \gamma^3$ etc. Similarly,
\[
\omega = \sum_{1 \leq i < j \leq 3} \mu_{ij}(\theta^{ij} + \gamma^{ij}) + \sum_{i,j=1}^{3} C_{ij} \theta^i \wedge \gamma^j
\]

The normalization condition
\[
\psi^+ \wedge \psi^- = \frac{2}{3} \omega^3
\]
translates into
\[
\det(C) = \varepsilon^2 + \sum_{i,j=1}^{3} C_{ij} y_i y_j,
\]
where
\[
y_1 := \mu_{23}, \quad y_2 := \mu_{31}, \quad y_3 := \mu_{12}.
\]
The previous formula $d\mu_{ij} = -3\xi_i \cdot \xi_j \cdot \psi^+$ can be restated as

$$dy_i = -3\varepsilon \gamma^i, \quad i = 1, 2, 3.$$  

Similarly, $d\varepsilon = 2\xi_3 \cdot \xi_2 \cdot \xi_1 \cdot \omega^2$ is equivalent to

$$d\varepsilon = 4 \sum_{i,j=1}^{3} C_{ij} y_i \gamma^j.$$  

Remark also that $\xi_j \cdot d\theta^i = 0 \iff$ explicit expression of $d\theta^i$ in terms of $\gamma_j, y_j, \varepsilon$ and $C$.

Let $U := M_0/T^3$ be the set of orbits of the $T^3$-action generated by the vector fields $\xi_i$.

All invariant functions and basic forms descend to $U \leadsto y_i, \varepsilon, \gamma^i, C_{ij}$, etc. Since $\varepsilon$ does not vanish on $M_0 \leadsto \{y_i\}$ define a local coordinate system on $U$.

**Key point:** The system

$$\begin{cases}
  d\omega = 3\psi^+ \\
  d\psi^- = -2\omega^2
\end{cases} \leadsto \exists \varphi \text{ on } U \text{ such that } \text{Hess}(\varphi) = C \text{ in the coordinates } \{y_i\}.$$
Let us introduce the operator $\partial_r$ of radial differentiation, acting on functions on $U$ by

$$
\partial_r f := \sum_{i=1}^{3} y_i \frac{\partial f}{\partial y_i}.
$$

**Claim:** The function $\varphi$ can be chosen in such a way that

$$
\varepsilon^2 = \frac{8}{3}(\varphi - \partial_r \varphi).
$$

**Proof:** It is enough to show that the exterior derivatives of the two terms coincide. Since

$$
\frac{\partial (\partial_r \varphi)}{\partial y_j} = \sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial y_i \partial y_j} y_i + \frac{\partial \varphi}{\partial y_j},
$$

we get:

$$
d(\partial_r \varphi - \varphi) = \sum_{i,j=1}^{3} C_{ij} y_i d y_j = -3 \sum_{i,j=1}^{3} C_{ij} y_i \varepsilon \gamma^j
$$

$$
= -\frac{3}{4} \varepsilon d \varepsilon = -\frac{3}{8} d(\varepsilon^2).
$$
On the other hand,

\[ \partial^2_r \varphi = \partial_r \left( \sum_{i=1}^{3} y_i \frac{\partial f}{\partial y_i} \right) = \sum_{i,j=1}^{3} C_{ij} y_i y_j + \partial_r \varphi. \]

Summing up, the previous relation

\[ \det(C') = \varepsilon^2 + \sum_{i,j=1}^{3} C_{ij} y_i y_j \]

becomes:

\[ \boxed{\det(\text{Hess}(\varphi)) = \frac{8}{3} \varphi - \frac{11}{3} \partial_r \varphi + \partial^2_r \varphi.} \]

This Monge-Ampère equation is enough to recover (locally) the full structure of the toric SNK manifold provided some positivity constraints hold.
The inverse construction

We will show that a solution $\varphi$ of

$$\det(\text{Hess}(\varphi)) = \frac{8}{3}\varphi - \frac{11}{3}\partial_r \varphi + \partial_r^2 \varphi$$

on some open set $U \subset \mathbb{R}^3$ defines a toric SNK structure on $U_0 \times \mathbb{T}^3$, where $U_0$ is some open subset of $U$.

Let $y_1, y_2, y_3$ be the standard coordinates on $U$ and let $\mu$ be the $3 \times 3$ skew-symmetric matrix

$$\mu := \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}.$$  

Define the $6 \times 6$ symmetric matrix

$$D := \begin{pmatrix} \text{Hess}(\varphi) & -\mu \\ \mu & \text{Hess}(\varphi) \end{pmatrix}.$$  

Let $U_0 \subset U$ denote the open subset

$$U_0 := \{ x \in U | \varphi(x) - \partial_r \varphi(x) > 0 \text{ and } D > 0 \}.$$
Note that the matrix \( D \) is positive definite if and only if \( C := \text{Hess}(\varphi) > 0 \) and \( \langle \mu a, b \rangle^2 < \langle Ca, a \rangle \langle Cb, b \rangle \) for all \((a, b) \in (\mathbb{R}^3 \times \mathbb{R}^3) \setminus (0, 0)\).

On \( U_0 \) we define a positive function \( \varepsilon \) by
\[
\varepsilon^2 = \frac{8}{3}(\varphi - \partial_r \varphi),
\]
and 1-forms \( \gamma^i \) by \( dy_i = -3\varepsilon \gamma^i \).

We pull-back \( \varepsilon, y_i, \) and \( \gamma_i \) to \( U_0 \times \mathbb{T}^3 \) and define \( \theta_i \) on \( U_0 \times \mathbb{T}^3 \) as connection forms whose curvature is given by the explicit expression of \( d\theta_i \) in the direct construction in terms of \( C, \varepsilon, y_i, \) and \( \gamma_i \).

It remains to check that \( \omega \) and \( \psi^\pm \) defined by the previous expressions form indeed an SU(3)-structure on \( U_0 \times \mathbb{T}^3 \).
Example

Let $K := \text{SU}_2$ with Lie algebra $\mathfrak{k} = \mathfrak{su}_2$ and $G := K \times K \times K$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k}$. We consider the 6-dimensional manifold $M = G/K$, where $K$ is diagonally embedded in $G$. The tangent space of $M$ at $o = eK$ can be identified with

$$\mathfrak{p} = \{(X,Y,Z) \in \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} | X + Y + Z = 0\}.$$ 

The Killing form $B$ on $\mathfrak{su}_2$ induces a scalar product on $\mathfrak{g}$ by

$$|(X,Y,Z)|^2 := B(X,X) + B(Y,Y) + B(Z,Z)$$

which defines a 3-symmetric nearly Kähler metric $g$ on $M = S^3 \times S^3$.

The $G$-automorphism $\sigma$ of order 3 defined by $\sigma(a_1,a_2,a_3) = (a_2,a_3,a_1)$ induces a canonical almost complex structure on the 3-symmetric space $M$ by the relation

$$\sigma = \frac{-\text{Id} + \sqrt{3}J}{2} \quad \text{on } \mathfrak{p}.$$
\[ J(X, Y, Z) = \frac{2}{\sqrt{3}}(Y, Z, X) + \frac{1}{\sqrt{3}}(X, Y, Z). \]

Let \( \xi \) be a unit vector in \( \mathfrak{su}_2 \) with respect to \( B \). The right-invariant vector fields on \( G \) generated by the elements
\[
\tilde{\xi}_1 = (\xi, 0, 0), \quad \tilde{\xi}_2 = (0, \xi, 0), \quad \tilde{\xi}_3 = (0, 0, \xi)
\]
of \( \mathfrak{g} \), define three commuting Killing vector fields \( \xi_1, \xi_2, \xi_3 \) on \( M \).

Let us compute \( g(\xi_1, J\xi_2) \) at some point \( aK \in M \), where \( a = (a_1, a_2, a_3) \) is some element of \( G \). By the definition of \( J \) we have
\[
g(\xi_1, J\xi_2)_{aK} = \frac{1}{\sqrt{3}}B(a_1^{-1}\xi a_1, a_2^{-1}\xi a_2).
\]
We introduce the functions \( y_1, y_2, y_3 : G \to \mathbb{R} \) defined by
\[
y_i(a_1, a_2, a_3) = \frac{1}{\sqrt{3}}B(a_j^{-1}\xi a_j, a_k^{-1}\xi a_k),
\]
for every permutation \((i, j, k)\) of \((1, 2, 3)\).

A similar computation yields

\[
C_{ij} \coloneqq g(\xi_i, \xi_j)_{aK} = 2\delta_{ij} + \frac{1}{\sqrt{3}}y_k(a).
\]

The function \(\varphi\) in the coordinates \(y_i\) such that \(\text{Hess}(\varphi) = C\) is determined by

\[
\varphi(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 + \frac{1}{\sqrt{3}}y_1y_2y_3 + h,
\]

up to some affine function \(h\) in the coordinates \(y_i\). On the other hand, since

\[
\det(C) = -\frac{2}{3}(y_1^2 + y_2^2 + y_3^2) + \frac{2}{3\sqrt{3}}y_1y_2y_3 + 8,
\]

the above function \(\varphi\) satisfies the Monge–Ampère equation

\[
\det(\text{Hess}(\varphi)) = \frac{8}{3}\varphi - \frac{11}{3}\partial_r\varphi + \partial_r^2\varphi
\]

for \(h = 3\).
Radial solutions

We search here radial solutions to the Monge–Ampère equation on (some open subset of) $\mathbb{R}^3$ with coordinates $y_1, y_2, y_3$.

Write $\varphi(y_1, y_2, y_3) := x(\frac{r^2}{2})$ where $x$ is a function of one real variable and $r^2 = y_1^2 + y_2^2 + y_3^2$. A direct computation yields

$$\text{Hess}(\varphi) = \begin{pmatrix} y_1^2x'' + x' & y_1y_2x'' & y_1y_3x'' \\ y_1y_2x'' & y_2^2x'' + x' & y_2y_3x'' \\ y_1y_3x'' & y_2y_3x'' & y_3^2x'' + x' \end{pmatrix}$$

$$= x'\text{Id} + x''(\frac{r^2}{2})V \cdot tV$$

where $V := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. In particular,

$$\det \text{Hess}(\varphi) = (x')^2x''r^2 + (x')^3$$

$$\partial_r \varphi = r^2x', \quad \partial_r^2 \varphi = r^4x'' + 2r^2x',$$

whence after making the substitution $t := \frac{r^2}{2}$ we get:
Proposition 1  **Radial solutions to the Monge-Ampère equation are given by solutions of the second order ODE**

\[ x'' = F(t, x, x') \]

where \( F(t, p, q) := \frac{8p - (10tq + 3q^3)}{6(q^2t - 2t^2)} \).

To decide which solutions of this equation yield genuine Riemannian metrics in dimension six, we observe that

Proposition 2  **For any radial solution \( \varphi = x(r^2/2) \), the set**

\[ U_0 := \{ x \in U \mid \varphi(x) - \partial_r \varphi(x) > 0 \text{ and } D > 0 \} \]

**defined above is given by**

\[ U_0 = \{ t > 0 \mid x(t) > 2tx'(t) > 2t\sqrt{2t} \}. \]
Remark 1 The solutions of the above ODE of the form \( x = kt^l \) with \( k, l \in \mathbb{R} \) are \( x_{1,2} = \pm \frac{2\sqrt{2}}{9} t^\frac{3}{2} \) and \( x_3 = kt^\frac{1}{2} \), corresponding to

\[
\varphi_{1,2} = \pm \frac{r^3}{9}, \quad \varphi_3 = \frac{k}{\sqrt{2}} r.
\]

However, they do not satisfy the positivity requirements from Proposition 2.

Admissible solutions can be obtained by solving the Cauchy problem with initial data

\[
(t_0, x(t_0), x'(t_0)) \in \mathcal{S}
\]

where

\[
\mathcal{S} := \{(t, p, q) \in \mathbb{R}^3 : t > 0, \ p > 2tq > 2t\sqrt{2t}\}.
\]