Global aspects of the 3d − 3d correspondence

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based on work w/ J. Eckhard, H. Kim, and S. Schäfer-Nameki (arXiv:1909.xxxxx)
Special holonomy manifolds play an important role in string and M-theory, describing compactifications that preserve some supersymmetry.

In particular, compactification of $M$-theory on special holonomy manifolds, $X_{11-D}$, with singularities can give rise to supersymmetric quantum field theories (SQFTs) with non-abelian gauge symmetry and matter in $D$ dimensions.

We may also consider wrapping M5 branes on a calibrated cycle $M_{6-d}$ in a special holonomy manifold.

This describes a $d$ dimensional SQFT, which can exist inside the $D$-dimensional theory as a “defect theory.”

This potentially gives a dictionary between calibrated cycles in special holonomy manifolds and defect theories in SQFTs.
We may also directly start with the low energy description of $N$ M5 branes.

This is believed to be described by the $A_{N-1}$-type $6d$ $\mathcal{N} = (2, 0)$ SCFT.

More generally, $6d$ $\mathcal{N} = (2, 0)$ SCFTs are classified by an ADE Lie algebra, $\mathfrak{g}$, or $\mathfrak{g} = \mathfrak{u}(1)$ (free tensor multiplet). We consider these on a spacetime:

$$M_{6-d} \times \mathbb{R}^{d-1,1} \xrightarrow{\text{low energies}} T[M_{6-d}, \mathfrak{g}] \rightarrow d\text{-dim'1 SQFT}$$

where we perform a “topological twist” along $M_{6-d}$ to preserve some supersymmetry.

This perspective also generalizes to $6d$ $\mathcal{N} = (1, 0)$ SCFTs, which have a much richer classification (e.g., [Heckman-Morrison-Vafa]), and can describe M5 branes probing singularities in special holonomy manifolds.
Example: $4d \mathcal{N} = 4$ super Yang-Mills theory and S-duality

- Compactification gives a useful perspective on many lower dimensional theories.
- For example, the compactification of the theory on $T^2$ gives the celebrated $4d \mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, eg, arising in Vafa-Witten invariants and geometric Langlands program.
- The coupling strength of the theory is identified with the complex structure parameter of the torus:
  \[
  \tau = \frac{\theta}{2\pi} + \frac{i}{4\pi g^2}
  \]
- Then Montonen-Olive/electric-magnetic duality corresponds to modular invariance:
  \[
  \tau \rightarrow -\frac{1}{\tau}
  \]
Example: $4d \mathcal{N} = 4$ super Yang-Mills theory and $S$-duality

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- Then Montonen-Olive/electric-magnetic duality corresponds to modular invariance:
  $$\tau \rightarrow -\frac{1}{\tau}$$
- However, the SYM theory is determined by a choice of Lie group, $G$, not just Lie algebra.
- Moreover, electric-magnetic duality acts on this choice, e.g.:
  $$SU(N) \leftrightarrow SU(N)/\mathbb{Z}_N$$
This extra data comes from the fact that the 6d theory is a “relative QFT” [Witten, Freed-Teleman].

Due to anomalies, it is ill-defined by itself, but requires the specification of a 7d topological QFT, and observables of the 6d theory are valued in this 7d TQFT.

For example, the partition function on $M_6$ is an element in the Hilbert space of the 7d theory on $M_6$; it is only defined after we choose a “polarization:”

$$\Lambda \subset H^3(M_6, \Gamma_g) \rightarrow Z_{M_6}(\omega), \quad \omega \in \Lambda$$

where $\Gamma_g$ is the center of $G_{simp}$. 
6d theory as a “relative QFT”

- This extra data comes from the fact that the 6d theory is a “relative QFT” [Witten, Freed-Teleman].
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- For example, the partition function on \( M_6 \) is an element in the Hilbert space of the 7d theory on \( M_6 \); it is only defined after we choose a “polarization:”

\[
\Lambda \subset H^3(M_6, \Gamma_g) \rightarrow Z_{M_6}(\omega), \quad \omega \in \Lambda
\]

where \( \Gamma_g \) is the center of \( G_{simp} \).
- For the compactification on \( T^2 \), this amounts to a choice of Lagrangian subgroup of \( H^1(T^2, \Gamma_g) \). These correspond to the different choices of global form of \( G \).
- Electric-magnetic duality acts on this choice, and maps between different choices.
Similar issues arise for compactifications of the $6d$ $\mathcal{N} = (2, 0)$ theory on higher dimensional manifolds, $M_{6-d}$, where there is a richer dependence on the topology of $M_{6-d}$.

This structure can be conveniently summarized in terms of higher form symmetries of the theories $T[M_{6-d}, g]$. 
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In this talk we will focus on compactifications on three-manifolds, which lead to the 3d-3d correspondence of [Dimofte-Gaiotto-Gukov].

We will see there are additional choices that must be made in defining the theories $T[M_3, g]$, which leads to a refinement of the 3d – 3d correspondence dictionary.

We use this structure to gain new insights into these theories, and perform several tests by computing observables which are sensitive to it.
Outline

- $T[M_3, g]$ for Seifert manifolds
- 6$d$ theory and higher form symmetries
- Witten index and the Bethe-ansatz equations
$T[M_3, g]$ for Seifert manifolds
There are two natural classes of three manifolds on which we can wrap M5 branes:

\[
\begin{align*}
\text{M5 branes} \quad &\rightarrow \quad M_3 \times \mathbb{R}^{2,1} \quad \rightarrow \quad 3d \mathcal{N} = 1 \\
\cap \text{assoc.} \quad &\rightarrow \quad \mathbb{R}^{3,1} \quad \rightarrow \quad 4d \mathcal{N} = 1 \\
X_7 \quad &\rightarrow \quad G_2
\end{align*}
\]

\[
\begin{align*}
\text{M5 branes} \quad &\rightarrow \quad M_3 \times \mathbb{R}^{2,1} \quad \rightarrow \quad 3d \mathcal{N} = 2 \\
\cap \text{sLag} \quad &\rightarrow \quad \mathbb{R}^{4,1} \quad \rightarrow \quad 5d \mathcal{N} = 1 \\
X_6 \quad &\rightarrow \quad CY_3
\end{align*}
\]

Field theoretically, starting from the \( \mathcal{N} = (2, 0) \) SCFT of type \( g \), which has a \( Sp(4)_R \) \( R \)-symmetry, we identify subgroups:

\[
\begin{align*}
\left\{\begin{array}{l}
SU(2)_{\text{diag}} \subset SU(2)_\ell \times SU(2)_r \subset Sp(4)_R \\
\text{“DGG twist”} \quad \rightarrow \quad \mathcal{N} = 2 \\
SU(2)_\ell \subset SU(2)_\ell \times SU(2)_r \subset Sp(4)_R \\
\text{“minimal twist”} \quad \rightarrow \quad \mathcal{N} = 1
\end{array}\right.
\end{align*}
\]

which we use to twist the \( SU(2) \) symmetry acting on the tangent bundle of \( M_3 \) to preserve supersymmetry [Blau-Thompson].

We denote the resulting 3d SQFTs by \( T_{\mathcal{N}=2}[M_3, g] \) and \( T_{\mathcal{N}=1}[M_3, g] \).
3d – 3d correspondence

- We expect $T[M_3, g]$ to depend topologically on $M_3$, and physical observables of this theory to compute topological invariants of $M_3$.

\[ 6d \mathcal{N} = (2, 0) \text{ type } g \text{ theory on } M_3 \times W_3 \]

- In particular, we may consider the supersymmetric partition function on a spacetime manifold $W_3$.

- The complex BF and BFH theories compute the Euler characteristic of moduli spaces, respectively, of flat $g^C$ connections and solutions to the generalized Seiberg Witten equations on $M_3$[Eckhard-Schäfer-Nameki-Wong]:

\[ \mathcal{D} \phi^{\alpha \hat{\alpha}} = 0, \quad \epsilon_{abc} F^{bc} - \frac{i}{2} [\phi^{\alpha \hat{\alpha}}, (\sigma_a)^{\alpha \beta} \phi^{\beta \hat{\alpha}}] = 0 \]

which also arise in the study of associatives in $G_2$ manifolds [Doan, Walpuski].
Constructions of $T[M_3, g]$

- To compute these observables, we need a Lagrangian description of the $T[M_3]$ theories. There are several approaches.

**DGG ($\mathcal{N} = 2$) twist:**

- For $M_3$ a hyperbolic manifold, we may decompose into ideal hyperbolic tetrahedron and construct $T_{\mathcal{N}=2}[M_3]$ by assigning gluing rules as an abelian Chern-Simons-matter theory. [Dimofte,Gaiotto,Gukov].

- For $M_3$ a more general Seifert manifold (or more generally, graph manifold) we can find a quiver gauge theory description by studying boundary conditions of the $4d$ $\mathcal{N} = 4$ theory [Gadde-Gukov-Putrov,Gukov-Putrov-Pei-Vafa,Alday-Bullimore-Genolini-van Loon].

**Minimal ($\mathcal{N} = 1$) twist:**

- For $M_3 = L(k, 1)$, by reduction of the $6d$ theory along the Hopf fiber, one finds a $3d$ $\mathcal{N} = 1$ $g$ Chern-Simons theory [Acharya-Vafa,Eckhard-Schäfer-Nameki-Wong].
A graph manifold can be cut along disjoint embedded tori to form pieces $\Sigma \times S^1$, where $\Sigma$ is a compact surface with boundary.

It can also be constructed as the boundary of a 4-manifold obtained by plumbing with a link of unknots.

It can be represented by a graph, with vertices labeled by $\Sigma \times S^1$ and edges by $S$-transformations.
A special case are *Seifert manifolds*, which are $S^1$ fibrations over orbifold Riemann surfaces:

$$M_{\text{Seifert}} \to \{g, (p_1, q_1), \ldots, (p_n, q_n)\}$$

These can be labeled by the underlying genus, $g$, and a list of special fibers:

Examples with $g = 0$:

- $\{0, (p, q)\} = L(p, q) = S^3/\mathbb{Z}_p$ - lens spaces
- $\{0, (-1, 2), (1, 2), (1, n)\} = S^3/\Gamma_{D_n}$ - prism manifolds
- $\{0, (-1, 2), (1, 3), (1, n)\} = S^3/\Gamma_{E_n}$ - including Poincaré homology sphere.
- $\{0, (1, p), (1, q), (1, r)\} = \Sigma(p, q, r)$ - Brieskorn manifolds.

Many known examples of associative 3-cycles in $G_2$ manifolds are Seifert manifolds (eg, $S^3/\Gamma$).
The theory $T[M_3, g]$ can be directly read off from the graph defining $M_3$.

For $T[M_3, u(1)]$, the theory is a Chern-Simons theory. It can be conveniently described by defining the linking matrix, $L_{ij}$, of the graph manifold, which has:

\[ L_{ii} = k_i, \quad L_{ij} = -1 \text{ if nodes } i \text{ and } j \text{ connected by } S\text{-gluing} \]

Then the action is given by:

\[ S_{T[M_3, u(1)]} = \frac{1}{4\pi} \sum_{i,j} \int L_{ij} A_i \wedge dA_j + \cdots \]

where there are additional matter fields to complete $N = 1$ or $N = 2$ supermultiplets [Gadde-Gukov-Putrov, Eckhard-Schäfer-Nameki-Wong].
For the non-abelian case, the quiver can be derived by reducing the $4d$ $\mathcal{N} = 4$ theory.

- To each node, we assign a gauge group with Lie algebra $\mathfrak{g}$ and Chern-Simons level $k_i$
  \[
  \frac{k_i}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \cdots
  \]

- To each edge, we assign a copy of the duality wall theory $T[\mathfrak{g}]$ of [Gaiotto-Witten], with $\mathfrak{g} \oplus \mathfrak{g}$ symmetry:

The different twists (minimal vs DGG) correspond to taking 1/2 or 1/4 BPS boundary conditions for the $\mathcal{N} = 4$ theory, which we give rise to different vector multiplets ($\mathcal{N} = 1$ or $\mathcal{N} = 2$) when gauging. We mostly focus on the $\mathcal{N} = 2$ twist.
Example: \( T[L(p, q), \mathfrak{su}(N)] \)

- The theory \( T[L(k, 1), \mathfrak{su}(N)] \) is given by a degree \( k \) fibration over \( S^2 \), and leads to a \( \mathfrak{su}(N) \) gauge theory with level \( k \) CS term, which we denote:

  \[
  k
  \]

  This agrees with the derivation by first reducing to 5\( d \).

- For \( L(p, q) \), we write:

  \[
  \varphi = \begin{pmatrix} s \\ p \end{pmatrix} = \mathcal{T}^{k_1} S \mathcal{T}^{k_2} ... S \mathcal{T}^{k_n}, \quad \frac{p}{q} = k_1 - \frac{1}{k_2 - \ldots - \frac{1}{k_n}}
  \]

  Then the quiver gauge theory description is:

  \[
  \begin{array}{c}
  k_1 \\
  \text{ } \\
  k_2 \\
  \text{ } \\
  \text{...} \\
  \text{ } \\
  k_n
  \end{array}
  \]
Symmetries of the quiver

- The description is not unique, due to relations in $SL(2, \mathbb{Z})$ (e.g., $(ST)^3 = 1$).
- For example, we find the following dual description of the $T[g]$ theory:

![Diagram of quiver transformations](image)

where we used the star-shaped quiver duality of [Benini-Tachikawa-Xie].

- For $g = su(2)$ it gives a useful new description of $T[SU(2)]$ (see also [Teschner-Vartanov]):

$$\mathcal{N} = 4 \ U(1) \text{ with two hypermultiplets} \quad \leftrightarrow \quad \mathcal{N} = 2 \ SU(2)_{k=1} \text{ with two fundamental flavors}$$

- There should be analogous $3d \ \mathcal{N} = 1$ dualities related to these relations for the minimal twist.
So far we have not specified the global form of in the various nodes in the quiver.

This turns out to be intimately related to the relative nature of the 6d theory.

Moreover, there are several versions of $T[M_3, \mathfrak{g}]$, analogous to $SU(N)$ vs $SU(N)/\mathbb{Z}_N$ in 4d. These lead to different prescriptions for the global form of the quiver gauge group.
6d theory and

higher form symmetries
6d theory on a boundary

The 6d $\mathcal{N} = (2,0)$ theory is a “relative QFT” meaning it only exists as the boundary of a 7d topological theory.

This is given by a certain 7d “Wu-Chern-Simons theory” [Witten,Monnier]. Roughly, for $g = A_{N-1}$:

$$S_{WCS} = \frac{N}{4\pi} \int C \wedge dC, \quad C \in \Omega^3(M_7)$$

Then the partition function of the 6d SCFT on $M_6$ is replaced by a **partition vector:**

$$|Z^{M_6}\rangle \in \mathcal{H}^{WCS}_{M_6}$$
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Then the partition function of the $6d$ SCFT on $M_6$ is replaced by a **partition vector**: 

$$|Z^{M_6}\rangle \in \mathcal{H}_{WCS}^{M_6}$$

The Hilbert space on $M_6$ is acted on by Wilson surfaces supported on 3-cycles, but with commutation relation:

$$[O_\omega, O_{\omega'}] = e^{\frac{2\pi i}{N} \omega \cap \omega'}, \quad \omega, \omega' \in H_3(M_6, \Gamma_g)$$

A basis of the Hilbert space depends on a choice of “polarization,” roughly, a decomposition into “position” vs “momentum” operators:

$$H_3(M_6, \Gamma_g) = \Lambda \oplus \tilde{\Lambda} \quad \text{such that} \quad \Lambda \cap \Lambda = \tilde{\Lambda} \cap \tilde{\Lambda} = 0$$

$$\Rightarrow |Z^{M_6}\rangle = \sum_{\lambda \in \Lambda} Z^{M_6}_{\lambda} |\lambda\rangle$$

Changing polarization is implemented by a discrete Fourier transform, eg:

$$Z^{M_6}_{\tilde{\lambda}} = \frac{1}{|H_3(M_6, \Gamma_g)|^{1/2}} \sum_{\lambda \in \Lambda} e^{\frac{2\pi i}{N} \lambda \cap \tilde{\lambda}} Z^{M_6}_{\lambda}, \quad \tilde{\lambda} \in \tilde{\Lambda}$$
Compactification to $4d\ \mathcal{N} = 4$ theory

For example, suppose $M_6 = W_4 \times T^2$, and assume $W_4$ is simply connected. Then (for $g = \text{su}(N)$):

$$H_3(M_6, \mathbb{Z}_N) = H^3(M_6, \mathbb{Z}_N) = H^2(W_4, \mathbb{Z}_N) \otimes H^1(T^2, \mathbb{Z}_N) \cong H^2(W_4, \mathbb{Z}_N)_A \oplus H^2(W_4, \mathbb{Z}_N)_B$$

Then one finds taking $\Lambda = H^2(W_4, \mathbb{Z}_N)_A$, we have:

$$Z_{W_4 \times T^2}^{\lambda} = \left\{ \begin{array}{l}
\text{contribution to path integral of } 4d\ \text{SU}(N)/\mathbb{Z}_N\ \text{SYM from bundles } P \text{ with } w_2(P) = \lambda \in H^2(W_4, \mathbb{Z}_N) \\
\end{array} \right\}$$

In particular:

$$Z_{W_4 \times T^2}^0 = \text{partition function of } 4d\ \text{SYM with } \text{SU}(N)\ \text{group}$$

while, for the other polarization

$$Z_{W_4 \times T^2}^0 \propto \sum_{\lambda} Z_{W_4 \times T^2}^{\lambda} = \text{partition function of } 4d\ \text{SYM with } \text{SU}(N)/\mathbb{Z}_N\ \text{group}$$

More general $SL(2, \mathbb{Z})$ transformations take us to more complicated choices of polarization, leading to many forms of the $\mathcal{N} = 4$ theory differing by global structure and topological “discrete theta angle” terms [Aharony, Seiberg, Tachikawa],[Tachikawa]
Compactification to 3d

Now consider $T[M_3, g]$ on the spacetime $W_3$ (assume $W_3$ torsionless).

Then we have $M_6 = W_3 \times M_3$, and:

$$H^3(M_6, \mathbb{Z}_N) = (H^2(W_3, \Gamma_g) \otimes H^1(M_3, \Gamma_g)) \oplus (H^1(W_3, \Gamma_g) \otimes H^2(M_3, \Gamma_g)) \oplus ...$$

$$\equiv A \oplus B$$

For simplicity, assume $\Gamma_g$ is a field (eg, $g = A_{N-1}$ for $N$ prime). Then we may use the universal coefficient theorem to write:

$$A = H^2(W_3, \gamma), \quad B = H^1(W_3, \gamma)$$

where:

$$\gamma = H^1(M_3, \Gamma_g) \cong H^2(M_3, \Gamma_g)$$

There two corresponding polarizations:

$A$-polarization: $Z_\lambda, \quad \lambda \in H^2(M_3, \gamma)$, $B$-polarization: $\tilde{Z}_{\tilde{\lambda}}, \quad \tilde{\lambda} \in H^1(M_3, \gamma)$

**Upshot**: the theory $T[M_3, g]$ is not uniquely defined, as it depends on a choice of polarization, and even then, its partition function is labeled by a cohomology class.
Higher form symmetries

- This structure can be naturally interpreted in terms of higher form symmetries of $T[M_3, g]$ [Gaiotto-Kapustin-Seiberg-BW].
- While a global symmetry of a quantum field theory is a group acting on point-like operators, a $q$-form symmetry group acts on extended $q$-dimensional operators.
- It can be characterized by codimension $(q + 1)$ topological surfaces labeled by the group, and can be related to $n$-group structures acting on the QFT [Kapustin-Thorngren, Cordova-Dumitrescu-Intriligator].
- E.g., in $d = 3$, a 1-form symmetry is both carried and measured by line operators:

\[
\begin{array}{c}
\text{"charge measuring operator"}
\\
\downarrow
\\
U^g[\Sigma]
\\
\end{array}
\quad
\begin{array}{c}
\text{"physical line operator"}
\\
\leftarrow
\\
\end{array}
\quad
\begin{array}{c}
\mathcal{O}[\rho]
\\
\chi_\mathcal{O}(g)\mathcal{O}[\rho]
\\
\end{array}
\]

\[\Gamma = \text{"1-form symmetry group"} \quad g \in \Gamma\]

- Many useful properties of ordinary symmetries hold for higher-form symmetries, such as:
  - Selection rules
  - Spontaneous symmetry breaking
  - ’t Hooft anomalies
Higher form symmetries (cont’d)

- As with ordinary symmetries, we can couple the theory to a classical background $q+1$-form gauge field. This gives an observable:

$$Z_W^d, \quad \omega \in H^{q+1}(W_d, \Gamma)$$

- We may “gauge” this symmetry by summing over gauge fields:

$$\tilde{Z}_W^d = \sum_{\omega \in H^{q+1}(W_d, \Gamma)} Z_{W_d}^d [\omega]$$

- We may naturally interpret the observables in compactifications of the 6$d$ theory as the partition function of a theory with higher form symmetry, and change of polarization as gauging these symmetries!
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- We may naturally interpret the observables in compactifications of the $6d$ theory as the partition function of a theory with higher form symmetry, and change of polarization as gauging these symmetries!

**3d − 3d correspondence** - Now we see the two choices of polarization above correspond to theories with either 1-form or 0-form symmetry $\Upsilon$.

- More generally, there is a polarization for every choice of $H \subset \Upsilon = H^1(M_3, \Gamma_g)$, which defines a theory we denote:

$$T[M_3, g, H]$$

where:

- A-polarization: $T[M_3, g, H = 1]$, B-polarization: $T[M_3, g, H = \Upsilon]$

and other choices are obtained by gauging symmetries.

- This is true for both minimal and DGG twists.
Eg, $T[L(k, 1), \mathfrak{su}(2)]$

- Recall this is a level $k$ CS theory with gauge algebra $\mathfrak{su}(2)$. More precisely, the theory is specified by:

$$H \subset \mathcal{Y} = H^1(L(k, 1), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k \text{ even} \\ 1 & k \text{ odd} \end{cases}$$

- Then we claim:

$$T[L(k, 1), \mathfrak{su}(2), 1] \rightarrow (\mathcal{N} = 1, 2) \quad SU(2) \text{ level } k \text{ CS theory}$$

$$T[L(k, 1), \mathfrak{su}(2), \mathbb{Z}_2] \rightarrow (\mathcal{N} = 1, 2) \quad SO(3) \text{ level } k \text{ CS theory (only for } k \text{ even!)}$$
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The $SO(3)$ theory is ill-defined when $k$ is odd due to an 't Hooft anomaly in the corresponding 1-form symmetry.

Anomalies represent an obstruction to gauging a symmetry, occurring when the charge measuring operators, $U^g[\Sigma]$, are themselves charged under the symmetry.

For a 3d 1-form symmetry, they can be characterized by an “anomaly form” describing two linked charge-measuring operators:

$$\mathcal{A} : \Gamma \times \Gamma \rightarrow \mathbb{R}/\mathbb{Z}$$

A subgroup $H \subset \Gamma$ can be gauged iff $\mathcal{A}(H, \Gamma) = 0$. 
Next take $M_3 = L(p, q)$ and $g = su(N)$ (for $N$ prime):

$$
\gamma = H^1(L(p, q), \mathbb{Z}_N) = \begin{cases} 
\mathbb{Z}_N & p|N \\
1 & p \not| N 
\end{cases}
$$

For our quiver description from before, each $SU(N)$ gauge group has a potential $\mathbb{Z}_N$ 1-form symmetry acting on its Wilson loops, but most of these are anomalous. The anomaly matrix can be computed as:

$$
\mathcal{A} = \begin{pmatrix}
k_1 & -1 & 0 & \cdots & 0 \\
-1 & k_2 & -1 & \cdots & 0 \\
0 & -1 & k_3 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & & \cdots & -1 & k_n
\end{pmatrix} \mod N,
$$

This has $\det \mathcal{A} = p$ and a unique null-vector if and only if $p = 0 \mod N$, corresponding to the generator of the $\mathbb{Z}_N$ 1-form symmetry in this case.

For general graph $M_3$, the anomaly matrix is the linking matrix of the graph manifold, which has a null-vector for each generator of $H^1(M_3, \mathbb{Z}_N)$. 


Explicitly, we can write the null vector as:

$$\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{Z}_N^n$$

Then the gauge group determined by $\delta_j$ in the $j$th node is $SU(N)/\mathbb{Z}_d$, $d = N/\langle \delta_j, N \rangle$.

Geometrically, $\delta$ corresponds to an element of $H^1 \cong H_2$ as follows:

The cycle of $T^2$ wrapped in the $j$th interval then picks a polarization of $T^2$. This in turn determines the gauge group for this node by the same rule as for the 4d SYM theory.
Summary so far

- So far we have seen that the correct definition of the theory $T[M_3, g]$ depends on some extra data:

  $$\rightarrow T[M_3, g, H], \quad H \subset \Upsilon \equiv H^1(M_3, \Gamma_g)$$

- These theories have higher form symmetries, and can naturally be coupled to higher form gauge fields, leading to more refined observables on a spacetime $W_3$. eg:

  $$Z^{W_3}_\lambda, \quad \lambda \in H^2(W_3, \Upsilon)$$

- Next we compute some explicit observables which depend on this data, and discuss the implications for the $3d - 3d$ correspondence.
Witten index and

the Bethe-ansatz equations
We can test some of these properties by studying the Hilbert space of $T[M_3, g]$ on $T^2$, and the associated $T^3$ partition function.

For supersymmetric theories, we consider the **Witten index**:

$$Z_{T^3} = \text{Tr}(-1)^F = N_{vac}^{bos} - N_{vac}^{ferm}$$

By the $3d - 3d$ correspondence, this maps to:

$$\begin{cases} 
\text{DGG twist} & \rightarrow \text{complex BF theory} & \rightarrow \text{flat } g^C \text{ connections} \\
\text{minimal twist} & \rightarrow \text{BFH theory} & \rightarrow \text{moduli space of gSW equations}
\end{cases}$$

$3d \mathcal{N} = 1$ theories may exhibit wall-crossing where the Witten index jumps. This corresponds to the presence of twisted harmonic spinors on $M_3$, where the Euler characteristic of the moduli space of gSW equations may jump.

We focus on the case of **rational homology spheres**, where we do not expect twisted harmonic spinors to arise. Then the BFH theory counts flat $g$ connections.
Recall the $T[M_3, u(1)]$ theory is a Chern-Simons theory:

$$S_{T[M_3, u(1)]} = \frac{1}{4\pi} \int \sum_{i,j} L_{i,j} A_i \wedge dA_j$$

where $L$ is the linking matrix of the graph manifold.

In general, this TQFT has:

$$\dim(\mathcal{H}_{T^2}) = |\det L_{ij}| = |H^1(M_3, \mathbb{Z})|$$

which agrees with the counting of (real and complex) flat $U(1)$ connections.
Supersymmetric vacua and Bethe equations

- For most other examples, $T[M_3, g]$ is an interacting QFT, and it is more difficult to study its Hilbert space on $T^2$.

- We study the supersymmetric vacua of the $3d \mathcal{N} = 2$ $T[M_3]$ theory, using the effective twisted superpotential.

- For a $3d$ $G$ gauge theory with matter in representation $R$ and CS levels $k^{ab}$, this is:

  \[ W(x) = \text{Tr}_R \text{Li}_2(x) + \sum_{a, b=1}^{r_G} \frac{1}{2} k^{ab} \log x_a \log x_b, \quad x \in \mathbb{T}_G^\mathbb{C} = (\mathbb{C}^*)^{r_G} \]

- The vacua lie at critical points of the twisted superpotential, which are given by Bethe-ansatz type equations [Nekrasov-Shatashvili]:

  \[ \exp \left( x_a \partial_{x_a} W(u) \right) = \prod_{\rho \in R} (1 - x^\rho)^{\rho^a x_b k^{ab}} = 1, \quad a = 1, \ldots, r_G \]

- The solutions to these equations define SUSY ground states on $T^2$:

  \[ \mathcal{H}^{\text{SUSY}}_{T^2} = \text{span} \left\{ |x_a\rangle \mid x_a \text{ sol'n to Bethe eqn's} \right\} \]

  where all states are bosonic.
Example: $T[L(k, 1), u(N)]$

**Minimal twist**
- Here we must first take an $\mathcal{N} = 2$ deformation [Bashmakov-Gomis-Komargodski-Sharon]. Then we find:

\[ x_i^k = 1 \]

with:

\[ N_{vac} = \binom{k}{N} \]

which counts the flat $U(N)$ connections.
- This agrees with the result of [Acharya-Vafa], and counts domain walls in the 4d $\mathcal{N} = 1$ $SU(k)$ gauge theory.

**DGG twist**
- Here the Bethe equations are given, for the DGG twist, by:

\[ x_i^k \prod_{j \neq i} \frac{x_i - tx_j}{x_j - tx_i} = 1, \quad i = 1, \ldots, N \]

- This gives a system of coupled polynomial equations, with:

\[ N_{vac} = \binom{N + k - 1}{N} \]

- This agrees with the number of flat $GL(N, \mathbb{C})$ connections on $L(k, 1)$. 
Recall we have (for the DGG twist)

\[ T[L(k, 1), \mathfrak{su}(2), 1] \rightarrow (N = 2) \quad SU(2) \text{ level } k \text{ CS theory} \]
\[ T[L(k, 1), \mathfrak{su}(2), \mathbb{Z}_2] \rightarrow (N = 2) \quad SO(3) \text{ level } k \text{ CS theory} \quad \text{(only for } k \text{ even)} \]

We may always take the \( H = 1 \), which implies the gauge group is \( SU(2) \). The Bethe equations are:

\[ x^{2k} \left( \frac{x^2 t - 1}{t - x^2} \right)^2 = 1 \]

and we find

\[ N_{\text{vac}}[T[L(k, 1), \mathfrak{su}(2), 1]] = k + 1 \]
To gauge the $\mathbb{Z}_2$ 1-form symmetry acting on the $T^2$ Hilbert space, we must project onto the trivial subspace of the charge-measuring operators for each cycle of $T^2$:

$$U_A^\gamma = 1, \quad U_B^\gamma = 1$$

In general, these act on the Bethe vacua as:

$$U_A^\gamma |x_a\rangle = e^{\gamma a x_a \partial_a \mathcal{W}(x)} |x_a\rangle, \quad U_B^\gamma |x_a\rangle = |e^{2\pi i \gamma a} x_a\rangle$$

In the present case, we find:

$$U_A^\gamma |x\rangle = x^k \left( \frac{x^2 t - 1}{t - x^2} \right) |x\rangle, \quad U_B^\gamma |x\rangle = |-x\rangle$$

Note that when $k$ is odd, it is impossible to solve $U_A = U_B = 1$ as they do not commute, reflecting the anomaly.

The number of solutions in this case is given by:

$$N_{vac}[T[L(k, 1), \mathfrak{su}(2), \mathbb{Z}_2]] = \left\lfloor \frac{k}{4} \right\rfloor + 1$$
Example: $T[L(p, q), (s)u(2)]$

- For quiver gauge theories, the Bethe equations are more complicated. For example:

\[ M_3 = L(p, q), \quad \frac{p}{q} = k_1 - \frac{1}{k_2 - \frac{1}{k_3}}, \quad p = k_1 k_2 k_3 - k_1 - k_3, \quad q = k_1 k_2 - 1 \]

- This leads to a system of 5 coupled equations:

\[
\begin{align*}
    x_1^{2(k_1 - 1)} \frac{(x_1 - tx_2 x_3)(x_1 x_2 - tx_3)(x_1 x_3 - tx_2)(x_1 x_2 x_3 - t)}{(1 - tx_1 x_2 x_3)(x_2 - tx_1 x_3)(x_3 - tx_1 x_2)(x_2 x_3 - tx_1)} &= 1, \\
    x_2^{2(k_2 - 2)} \frac{(x_2 - tx_1 x_3)(x_1 x_2 - tx_3)(x_2 x_3 - tx_1)(x_1 x_2 x_3 - t)}{(1 - tx_1 x_2 x_3)(x_1 - tx_2 x_3)(x_3 - tx_1 x_2)(x_1 x_3 - tx_2)} &= 1
\end{align*}
\]

These can be solved by finding a Gröbner basis, $g_a(x_i)$, for the ideal generated by these polynomial equations. Then:

\[
N_{solutions} = \# \left\{ \text{monomials } m = \prod_i x_i^{\ell_i} \mid \text{Im}(g_a) \not\mid m \right\}
\]

- We find Gröbner bases using SINGULAR. In this example we find:

\[
N_{solutions} = 4(k_1 k_2 k_3 - k_1 - k_3 + 1) = 4(p + 1)
\]
Example: $T[L(p, q), (s)u(2)]$ (cont’d)

To form the $T[L(p, q), u(2)]$ theory, we take the tensor product with the $T[M_3, u(1)]$ theory, and then take the quotient corresponding to:

$$U(1) \times SU(2) \to (U(1) \times SU(2))/\mathbb{Z}_2 \cong U(2)$$

at each node, and we find:

$$N_{\text{vac}}[T[L(p, q), u(2)]] = \frac{p(p+1)}{2} = \binom{p+1}{2}$$

which agrees with the flat connection counting.

For the $T[L(p, q), su(2)]$ theory, we must gauge the non-anomalous 1-form symmetries of the theory. We find:

$$N_{\text{vac}}[T[L(p, q), su(2), H]] = \begin{cases} \frac{p+1}{4} + 1 & p \text{ even, } H = \mathbb{Z}_2 \\ p+1 & H = 1 \end{cases}$$
Other Seifert manifolds

- We may repeat this for more general \( L(p, q) \), as well as other rational homology spheres. Eg, for \( S^3/\Gamma_{D_n} \) and \( g = u(2) \), we have:

\[
N_{\text{vac}}[T_{\mathcal{N}=2}[S^3/\Gamma_{D_n}, u(2)]] = n + 7
\]

- Here it is not possible to deform the \( \mathcal{N} = 1 \) minimal twist to \( \mathcal{N} = 2 \), so one must compute the Witten index by other means. The result should agree with [Acharya-Vafa].

- For the \( SU(2) \) theory, the theories are labeled by \( H \subset H^2(M_3, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \):

\[
N_{\text{vac}}[T[S^3/D_n, su(2), H]] = \begin{cases} 
2n + 7 & H = 1 \\
\frac{n}{2} + 4 & H = \mathbb{Z}_2 \times 1, \ n \ \text{even} \\
\frac{n+1}{2} + 4 & H = \mathbb{Z}_2 \times 1, \ n \ \text{odd} \\
n + 1 + 4 & H = (\mathbb{Z}_2)_{\text{diag}} \\
\left\lfloor \frac{n}{2} \right\rfloor + 1 & H = \mathbb{Z}_2 \times \mathbb{Z}_2
\end{cases}
\]

- For general \( M_3 \), these results should be computed by a new TQFT invariant appearing in the \( 3d \) \( -3d \) correspondence dictionary, which is labeled by \( M_3 \) and a choice of subgroup \( H \subset H^1(M_3, \mathbb{Z}_2) \).
Summary

and outlook
The $6d \mathcal{N} = (2, 0)$ is a relative QFT, which leads to several subtleties in understanding its compactifications.

The theories $T[M_3, g]$ obtained by compactification of M5 branes on a 3-manifold depend on extra topological data related to the polarization of the $6d$ theory.

This can be naturally phrased in the language of higher form symmetries, and different choices can be related by gauging these symmetries.

This data can be probed by computing the Witten index of these theories, and lead to generalizations of the complex $BF$ and $BFH$ models.
Describe the new entries in the dictionary of the $3d - 3d$ correspondence, and more refined observables in the TQFTs.

Higher form symmetries have led to powerful insights into theories with little or no supersymmetry, and can be used to better understand the $3d \mathcal{N} = 1$ theories obtained by reduction of M5 branes on associative cycles.

They also are important for analyzing defect theories inside higher dimensional quantum field theories (eg, [Gaiotto-Kapustin-Komargodski-Seiberg]), and can play a role in the correspondence between defect theories and calibrated cycles.

This analysis can be generalized to compactifications on four-manifolds [Gadde-Gukov-Putrov,Gukov-Putrov-Pei-Vafa], and to compactifications of general $6d \mathcal{N} = (1, 0)$ theories, which are also typically relative QFTs.