Global aspects of the $3d - 3d$ correspondence

Brian Willett
KITP, UC Santa Barbara

September 9, 2019

based on work w/ J. Eckhard, H. Kim, and S. Schäfer-Nameki (*arXiv:1909.xxxxx)*

M-theory and special holonomy manifolds

- Special holonomy manifolds play an important role in string and M-theory, describing compactifications that preserve some supersymmetry.
- In particular, compactification of *M*-theory on special holonomy manifolds, X_{11-D}, with singularities can give rise to supersymmetric quantum field theories (SQFTs) with non-abelian gauge symmetry and matter in D dimensions.
- We may also consider wrapping M5 branes on a calibrated cycle M_{6-d} in a special holonomy manifold.
- This describes a d dimensional SQFT, which can exist inside the D-dimensional theory as a "defect theory."
- This potentially gives a dictionary between calibrated cycles in special holonomy manifolds and defect theories in SQFTs.

Compactification and QFT$_d$'s

- We may also directly start with the low energy description of N M5 branes.
- This is believed to be described by the A_{N-1}-type 6d $\mathcal{N} = (2,0)$ SCFT.
- More generally, 6d $\mathcal{N} = (2,0)$ SCFTs are classified by an ADE Lie algebra, $\mathfrak{g}$, or $\mathfrak{g} = \mathfrak{u}(1)$ (free tensor multiplet). We consider these on a spacetime:

$$M_{6-d} \times \mathbb{R}^{d-1,1} \underset{\text{low energies}}{\rightsquigarrow} T[M_{6-d}, \mathfrak{g}] \rightarrow d\text{-dim'l SQFT}$$

  where we perform a "topological twist" along M_{6-d} to preserve some supersymmetry.
- This perspective also generalizes to 6d $\mathcal{N} = (1,0)$ SCFTs, which have a much richer classification (e.g., [Heckman-Morrison-Vafa]), and can describe M5 branes probing singularities in special holonomy manifolds.

Example: 4d $\mathcal{N} = 4$ super Yang-Mills theory and S-duality

- Compactification gives a useful perspective on many lower dimensional theories
- For example, the compactification of the theory on T^2 gives the celebrated **4d $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory**, eg, arising in Vafa-Witten invariants and geometric Langlands program.
- The coupling strength of the theory is identified with the complex structure parameter of the torus:

$$\tau = \frac{\theta}{2\pi} + \frac{i}{4\pi g^2}$$

- Then Montonen-Olive/electric-magnetic duality corresponds to modular invariance:

$$\tau \to -\frac{1}{\tau}$$

Example: $4d$ $\mathcal{N} = 4$ super Yang-Mills theory and S-duality

- Compactification gives a useful perspective on many lower dimensional theories
- For example, the compactification of the theory on T^2 gives the celebrated **4d $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory**, eg, arising in Vafa-Witten invariants and geometric Langlands program.
- The coupling strength of the theory is identified with the complex structure parameter of the torus:

$$\tau = \frac{\theta}{2\pi} + \frac{i}{4\pi g^2}$$

- Then Montonen-Olive/electric-magnetic duality corresponds to modular invariance:

$$\tau \to -\frac{1}{\tau}$$

- However, the SYM theory is determined by a choice of Lie group, G, not just Lie algebra.
- Moreover, electric-magnetic duality acts on this choice, e.g.:

$$SU(N) \leftrightarrow SU(N)/\mathbb{Z}_N$$

6d theory as a "relative QFT"

- This extra data comes from the fact that the 6d theory is a "relative QFT" [Witten,Freed-Teleman].
- Due to anomalies, it is ill-defined by itself, but requires the specification of a 7d topological QFT, and observables of the 6d theory are valued in this 7d TQFT.
- For example, the partition function on M_6 is an element in the Hilbert space of the 7d theory on M_6; it is only defined after we choose a "polarization:"

$$\Lambda \subset H^3(M_6, \Gamma_{\mathfrak{g}}) \quad \to \quad Z_{M_6}(\omega), \quad \omega \in \Lambda$$

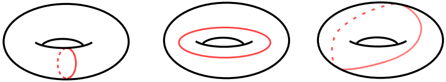
where $\Gamma_{\mathfrak{g}}$ is the center of G_{simp}.

6*d* theory as a "relative QFT"

- This extra data comes from the fact that the 6*d* theory is a "relative QFT" [Witten,Freed-Teleman].
- Due to anomalies, it is ill-defined by itself, but requires the specification of a 7*d* topological QFT, and observables of the 6*d* theory are valued in this 7*d* TQFT.
- For example, the partition function on M_6 is an element in the Hilbert space of the 7*d* theory on M_6; it is only defined after we choose a "polarization:"

$$\Lambda \subset H^3(M_6, \Gamma_{\mathfrak{g}}) \quad \to \quad Z_{M_6}(\omega), \quad \omega \in \Lambda$$

  where $\Gamma_{\mathfrak{g}}$ is the center of G_{simp}.
- For the compactification on T^2, this amounts to a choice of Lagrangian subgroup of $H^1(T^2, \Gamma_{\mathfrak{g}})$. These correspond to the different choices of global form of G.
- Electric-magnetic duality acts on this choice, and maps between different choices.



$SU(N)$ $\quad SU(N)/\mathbb{Z}_N$ $\quad SU(N)/\mathbb{Z}_d$

Compactification and higher form symmetries

- Similar issues arise for compactifications of the 6d $\mathcal{N} = (2,0)$ theory on higher dimensional manifolds, M_{6-d}, where there is a richer dependence on the topology of M_{6-d}.
- This structure can be conveniently summarized in terms of **higher form symmetries** of the theories $T[M_{6-d}, \mathfrak{g}]$.

Compactification and higher form symmetries

- Similar issues arise for compactifications of the $6d$ $\mathcal{N} = (2,0)$ theory on higher dimensional manifolds, M_{6-d}, where there is a richer dependence on the topology of M_{6-d}.
- This structure can be conveniently summarized in terms of **higher form symmetries** of the theories $T[M_{6-d}, \mathfrak{g}]$.
- In this talk we will focus on compactifications on three-manifolds, which lead to the **3d-3d correspondence** of [Dimofte-Gaiotto-Gukov].
- We will see there are additional choices that must be made in defining the theories $T[M_3, \mathfrak{g}]$, which leads to a refinement of the $3d - 3d$ correspondence dictionary.
- We use this structure to gain new insights into these theories, and perform several tests by computing observables which are sensitive to it.

Outline

- $T[M_3, \mathfrak{g}]$ for Seifert manifolds
- $6d$ theory and higher form symmetries
- Witten index and the Bethe-ansatz equations

$T[M_3, \mathfrak{g}]$ for

Seifert manifolds

M5 branes on 3-manifolds

- There are two natural classes of three manifolds on which we can wrap M5 branes:

$$\begin{array}{ccccccc} \text{M5 branes} & \to & M_3 & \times & \mathbb{R}^{2,1} & \to & 3d\ \mathcal{N}=1 \\ & & \cap\ \text{assoc.} & & \cap & & \\ & & X_7 & \times & \mathbb{R}^{3,1} & \to & 4d\ \mathcal{N}=1 \\ & & \uparrow & & & & \\ & & G_2 & & & & \end{array}$$

$$\begin{array}{ccccccc} \text{M5 branes} & \to & M_3 & \times & \mathbb{R}^{2,1} & \to & 3d\ \mathcal{N}=2 \\ & & \cap\ \text{sLag} & & \cap & & \\ & & X_6 & \times & \mathbb{R}^{4,1} & \to & 5d\ \mathcal{N}=1 \\ & & \uparrow & & & & \\ & & CY_3 & & & & \end{array}$$

- Field theoretically, starting from the $\mathcal{N}=(2,0)$ SCFT of type $\mathfrak{g}$, which has a $Sp(4)_R$ R-symmetry, we identify subgroups:

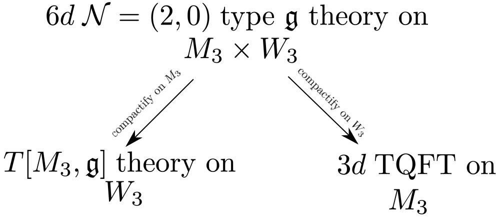
$$\begin{cases} SU(2)_{diag} \subset SU(2)_\ell \times SU(2)_r \subset Sp(4)_R & \text{"DGG twist"} \quad \to \quad \mathcal{N}=2 \\ SU(2)_\ell \subset SU(2)_\ell \times SU(2)_r \subset Sp(4)_R & \text{"minimal twist"} \quad \to \quad \mathcal{N}=1 \end{cases}$$

  which we use to twist the $SU(2)$ symmetry acting on the tangent bundle of M_3 to preserve supersymmetry [Blau-Thompson].
- We denote the resulting $3d$ SQFTs by $T_{\mathcal{N}=2}[M_3,\mathfrak{g}]$ and $T_{\mathcal{N}=1}[M_3,\mathfrak{g}]$.

$3d - 3d$ correspondence

- We expect $T[M_3, \mathfrak{g}]$ to depend topologically on M_3, and physical observables of this theory to compute topological invariants of M_3.

- In particular, we may consider the **supersymmetric partition function** on a spacetime manifold W_3.



W_3	DGG twist	minimal twist
$L(k,1) = S^3/\mathbb{Z}_k$	level k $\mathfrak{g}^{\mathbb{C}}$ CS theory	level k (real) $\mathfrak{g}$ CS-Dirac theory
T^3	complex BF theory	BFH theory

- The complex BF and BFH theories compute the Euler characteristic of moduli spaces, respectively, of flat $\mathfrak{g}^{\mathbb{C}}$ connections and solutions to the generalized Seiberg Witten equations on M_3[Eckhard-Schäfer-Nameki-Wong]:

$$\not{D}\phi^{\alpha\hat{\alpha}} = 0, \quad \epsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^{\alpha}_{\beta}\phi^{\beta\hat{\alpha}}] = 0$$

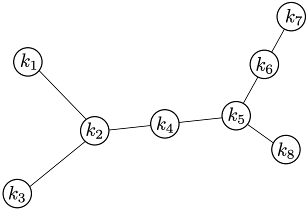
which also arise in the study of associatives in G_2 manifolds [Doan,Walpuski].

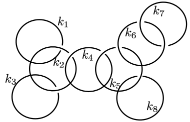
Constructions of $T[M_3, \mathfrak{g}]$

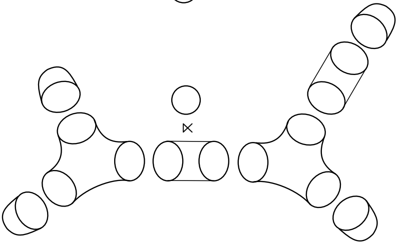
- To compute these observables, we need a Lagrangian description of the $T[M_3]$ theories. There are several approaches
- DGG ($\mathcal{N} = 2$) twist:
  - For M_3 a hyperbolic manifold, we may decompose into ideal hyperbolic tetrahedron and construct $T_{\mathcal{N}=2}[M_3]$ by assigning gluing rules as an abelian Chern-Simons-matter theory. [Dimofte,Gaiotto,Gukov].
  - For M_3 a more general Seifert manifold (or more generally, graph manifold) we can find a quiver gauge theory description by studying boundary conditions of the $4d$ $\mathcal{N} = 4$ theory [Gadde-Gukov-Putrov,Gukov-Putrov-Pei-Vafa,Alday-Bullimore-Genolini-van Loon].
- Minimal ($\mathcal{N} = 1$) twist:
  - For $M_3 = L(k, 1)$, by reduction of the $6d$ theory along the Hopf fiber, one finds a $3d$ $\mathcal{N} = 1$ $\mathfrak{g}$ Chern-Simons theory [Acharya-Vafa,Eckhard-Schäfer-Nameki-Wong]

Seifert and graph manifolds

- A *graph manifold* is can be cut along disjoint embedded tori to form pieces $\Sigma \times S^1$, where Σ is a compact surface with boundary.
- It can also be contstructed as the boundary of a 4-manifold obtained by plumbing with a link of unknots.
- It can be represented by a graph, with vertices labeled by $\Sigma \ltimes S^1$ and edges by S-tranformations.

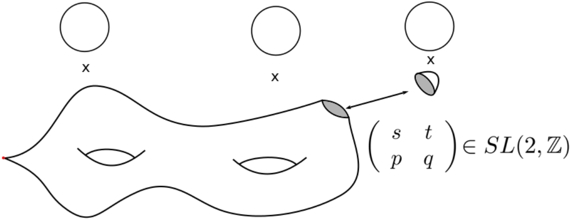






Seifert manifolds

- A special case are *Seifert manifolds*, which are S^1 fibrations over orbifold Riemann surfaces:



- These can be labeled by the underlying genus, g, and a list of special fibers:

$$M_{Seifert} \to \{g,\ (p_1, q_1), \cdots, (p_n, q_n)\}$$

- Examples with $g = 0$:
  - $\{0, (p, q)\} = L(p, q) = S^3/\mathbb{Z}_p$ - lens spaces
  - $\{0, (-1, 2), (1, 2), (1, n)\} = S^3/\Gamma_{D_n}$- prism manifolds
  - $\{0, (-1, 2), (1, 3), (1, n)\} = S^3/\Gamma_{E_n}$ - including Poincaré homology sphere.
  - $\{0, (1, p), (1, q), (1, r)\} = \Sigma(p, q, r)$ - Brieskorn manifolds.
- Many known examples of associative 3-cycles in G_2 manifolds are Seifert manifolds (eg, S^3/Γ).

$T[M_3]$ for graph and Seifert manifolds - abelian case

- The theory $T[M_3, \mathfrak{g}]$ can be directly read off from the graph defining M_3.
- For $T[M_3, \mathfrak{u}(1)]$, the theory is a Chern-Simons theory
- It can be conveniently described by defining the **linking matrix**, L_{ij}, of the graph manifold, which has:

$$L_{ii} = k_i, \quad L_{ij} = -1 \text{ if nodes } i \text{ and } j \text{ connected by } S\text{-gluing}$$

- Then the action is given by:

$$S_{T[M_3,\mathfrak{u}(1)]} = \frac{1}{4\pi} \sum_{i,j} \int L_{ij} A_i \wedge dA_j + \cdots$$

where there are additional matter fields to complete $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supermultiplets [Gadde-Gukov-Putrov,Eckhard-Schäfer-Nameki-Wong].

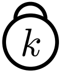
$T[M_3]$ for graph and Seifert manifolds - nonabelian case

- For the non-abelian case, the quiver can be derived by reducing the $4d$ $\mathcal{N} = 4$ theory.
  - To each node, we assign a gauge group with Lie algebra $\mathfrak{g}$ and Chern-Simons level k_i
  $$\frac{k_i}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \cdots$$
  - To each edge, we assign a copy of the duality wall theory $T[\mathfrak{g}]$ of [Gaiotto-Witten], with $\mathfrak{g} \oplus \mathfrak{g}$ symmetry:

- The different twists (minimal vs DGG) correspond to taking $1/2$ or $1/4$ BPS boundary conditions for the $\mathcal{N} = 4$ theory, which we give rise to different vector multiplets ($\mathcal{N} = 1$ or $\mathcal{N} = 2$) when gauging. We mostly focus on the $\mathcal{N} = 2$ twist.

Example: $T[L(p,q), \mathfrak{su}(N)]$

- The theory $T[L(k,1), \mathfrak{su}(N)]$ is given by a degree k fibration over S^2, and leads to a $su(N)$ gauge theory with level k CS term, which we denote:

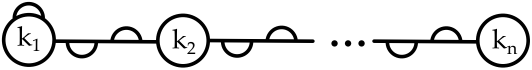


  This agrees with the derivation by first reducing to $5d$.

- For $L(p,q)$, we write:

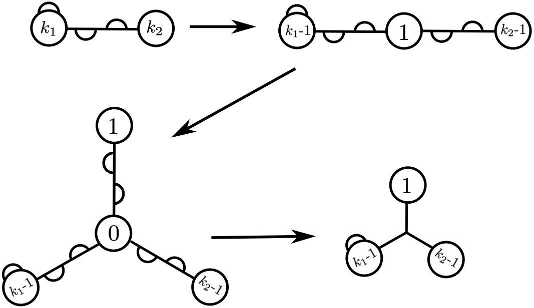
$$\varphi = \begin{pmatrix} s & t \\ p & q \end{pmatrix} = \mathcal{T}^{k_1}\mathcal{S}\mathcal{T}^{k_2}...\mathcal{S}\mathcal{T}^{k_n}, \qquad \frac{p}{q} = k_1 - \frac{1}{k_2 - ...\frac{1}{k_n}}$$

  Then the quiver gauge theory description is:



Symmetries of the quiver

- The description is not unique, due to relations in $SL(2,\mathbb{Z})$ (eg, $(ST)^3 = 1$).
- For example, we find the following dual description of the $T[\mathfrak{g}]$ theory:



where we used the star-shaped quiver duality of [Benini-Tachikawa-Xie].
- For $\mathfrak{g} = \mathfrak{su}(2)$ it gives a useful new description of $T[SU(2)]$ (see also [Teschner-Vartanov]):

$$\mathcal{N} = 4\ U(1) \text{ with two hypermultiplets} \quad \leftrightarrow \quad \mathcal{N} = 2\ SU(2)_{k=1} \text{ with two fundamental flavors}$$

- There should be analogous $3d$ $\mathcal{N} = 1$ dualities related to these relations for the minimal twist.

Global structure of the gauge group

- So far we have not specified the global form of in the various nodes in the quiver.
- This turns out to be intimately related to the relative nature of the 6*d* theory.
- Moreover, there are several versions of $T[M_3, \mathfrak{g}]$, analogous to $SU(N)$ vs $SU(N)/\mathbb{Z}_N$ in 4*d*. These lead to different prescriptions for the global form of the quiver gauge group.

6*d* theory and

higher form symmetries

$6d$ theory on a boundary

- The $6d$ $\mathcal{N} = (2,0)$ theory is a "relative QFT" meaning it only exists as the boundary of a $7d$ topological theory.
- This is given by a certain $7d$ "Wu-Chern-Simons theory" [Witten,Monnier]. Roughly, for $\mathfrak{g} = A_{N-1}$:

$$S_{WCS} = \frac{N}{4\pi} \int C \wedge dC, \qquad C \in \Omega^3(M_7)$$

- Then the partition function of the $6d$ SCFT on M_6 is replaced by a **partition vector:**

$$|Z^{M_6}\rangle \in \mathcal{H}^{WCS}_{M_6}$$

$6d$ theory on a boundary

- The $6d$ $\mathcal{N} = (2,0)$ theory is a "relative QFT" meaning it only exists as the boundary of a $7d$ topological theory.
- This is given by a certain $7d$ "Wu-Chern-Simons theory" [Witten,Monnier]. Roughly, for $\mathfrak{g} = A_{N-1}$:

$$S_{WCS} = \frac{N}{4\pi} \int C \wedge dC, \qquad C \in \Omega^3(M_7)$$

- Then the partition function of the $6d$ SCFT on M_6 is replaced by a **partition vector:**

$$|Z^{M_6}\rangle \in \mathcal{H}^{WCS}_{M_6}$$

- The Hilbert space on M_6 is acted on by Wilson surfaces supported on 3-cycles, but with commutation relation:

$$[\mathcal{O}_\omega, \mathcal{O}_{\omega'}] = e^{\frac{2\pi i}{N} \omega \cap \omega'}, \qquad \omega, \omega' \in H_3(M_6, \Gamma_\mathfrak{g})$$

- A basis of the Hilbert space depends on a choice of "polarization," roughly, a decomposition into "position" vs "momentum" operators:

$$H_3(M_6, \Gamma_\mathfrak{g}) = \Lambda \oplus \tilde{\Lambda} \quad \text{such that} \quad \Lambda \cap \Lambda = \tilde{\Lambda} \cap \tilde{\Lambda} = 0$$

$$\Rightarrow |Z^{M_6}\rangle = \sum_{\lambda \in \Lambda} Z^{M_6}_\lambda |\lambda\rangle$$

- Changing polarization is implemented by a discrete Fourier transform, eg:

$$\tilde{Z}^{M_6}_{\tilde{\lambda}} = \frac{1}{|H_3(M_6, \Gamma_\mathfrak{g})|^{1/2}} \sum_{\lambda \in \Lambda} e^{\frac{2\pi i}{N} \lambda \cap \tilde{\lambda}} Z^{M_6}_\lambda, \qquad \tilde{\lambda} \in \tilde{\Lambda}$$

Compactification to $4d$ $\mathcal{N} = 4$ theory

- For example, suppose $M_6 = W_4 \times T^2$, and assume W_4 is simply connected. Then (for $\mathfrak{g} = \mathfrak{su}(N)$):

$$H_3(M_6, \mathbb{Z}_N) = H^3(M_6, \mathbb{Z}_N) = H^2(W_4, \mathbb{Z}_N) \otimes H^1(T^2, \mathbb{Z}_N) \cong H^2(W_4, \mathbb{Z}_N)_A \oplus H^2(W_4, \mathbb{Z}_N)_B$$

- Then one finds taking $\Lambda = H^2(W_4, \mathbb{Z}_N)_A$, we have:

$$Z_\lambda^{W_4 \times T^2} = \left\{ \begin{array}{c} \text{contribution to path integral of } 4d\ SU(N)/\mathbb{Z}_N \text{ SYM} \\ \text{from bundles } P \text{ with } w_2(P) = \lambda \in H^2(W_4, \mathbb{Z}_N) \end{array} \right\}$$

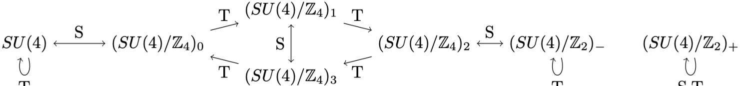
- In particular:

$$Z_0^{W_4 \times T^2} = \text{partition function of } 4d \text{ SYM with } SU(N) \text{ group}$$

while, for the other polarization

$$\tilde{Z}_0^{W_4 \times T^2} \propto \sum_\lambda Z_\lambda^{W_4 \times T^2} = \text{partition function of } 4d \text{ SYM with } SU(N)/\mathbb{Z}_N \text{ group}$$

- More general $SL(2, \mathbb{Z})$ tranformations take us to more complicated choices of polarization, leading to many forms of the $\mathcal{N} = 4$ theory differing by global structure and topological “discrete theta angle” terms [Aharony, Seiberg, Tachikawa],[Tachikawa]



source: Aharony-Seiberg-Tachikawa

Compactification to $3d$

- Now consider $T[M_3, \mathfrak{g}]$ on the spacetime W_3 (assume W_3 torsionless).
- Then we have $M_6 = W_3 \times M_3$, and:

$$H^3(M_6, \mathbb{Z}_N) = (H^2(W_3, \Gamma_{\mathfrak{g}}) \otimes H^1(M_3, \Gamma_{\mathfrak{g}})) \oplus (H^1(W_3, \Gamma_{\mathfrak{g}}) \otimes H^2(M_3, \Gamma_{\mathfrak{g}})) \oplus ...$$

$$\equiv A \oplus B$$

- For simplicity, assume $\Gamma_{\mathfrak{g}}$ is a field (eg, $\mathfrak{g} = A_{N-1}$ for N prime). Then we may use the universal coefficient theorem to write:

$$A = H^2(W_3, \Upsilon), \quad B = H^1(W_3, \Upsilon)$$

  where:

$$\Upsilon = H^1(M_3, \Gamma_{\mathfrak{g}}) \cong H^2(M_3, \Gamma_{\mathfrak{g}})$$

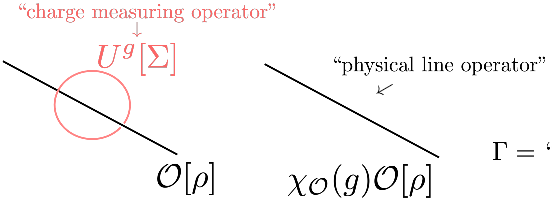
- There two corresponding polarizations:

$$A\text{-polarization:} \quad Z_\lambda, \quad \lambda \in H^2(M_3, \Upsilon), \qquad B\text{-polarization:} \quad \tilde{Z}_{\tilde{\lambda}}, \quad \tilde{\lambda} \in H^1(M_3, \Upsilon)$$

- **Upshot:** the theory $T[M_3, \mathfrak{g}]$ is not uniquely defined, as it depends on a choice of polarization, and even then, its partition function is labeled by a cohomology class.

Higher form symmetries

- This structure can be naturally interpreted in terms of **higher form symmetries** of $T[M_3, \mathfrak{g}]$ [Gaiotto-Kapustin-Seiberg-BW].
- While a global symmetry of a quantum field theory is a group acting on point-like operators, a q-form symmetry group acts on extended q-dimensional operators.
- It can be characterized by codimension $(q+1)$ topological surfaces labled by the group, and can be related to n-group structures acting on the QFT [Kapustin-Thorngren, Cordova-Dumitrescu-Intriligator].
- **E.g.**, in $d = 3$, a 1-form symmetry is both carried and measured by line operators:



$$g \in \Gamma$$
$$\Gamma = \text{"1-form symmetry group"}$$

- Many useful properties of ordinary symmetries hold for higher-form symmetries, such as:
  - Selection rules
  - Spontaneous symmetry breaking
  - ’t Hooft anomalies

Higher form symmetries (cont'd)

- As with ordinary symmetries, we can couple the theory to a classical background $q + 1$-form gauge field. This gives an observable:

$$Z^{W_d}_{\omega}, \quad \omega \in H^{q+1}(W_d, \Gamma)$$

- We may "gauge" this symmetry by summing over gauge fields:

$$\tilde{Z}^{W_d} = \sum_{\omega \in H^{q+1}(W_d,\Gamma)} Z_{W_d}[\omega]$$

- We may naturally interpret the observables in compactifications of the 6d theory as the partition function of a theory with higher form symmetry, and change of polarization as gauging these symmetries!

Higher form symmetries (cont'd)

- As with ordinary symmetries, we can couple the theory to a classical background $q+1$-form gauge field. This gives an observable:

$$Z_{\omega}^{W_d}, \quad \omega \in H^{q+1}(W_d, \Gamma)$$

- We may "gauge" this symmetry by summing over gauge fields:

$$\tilde{Z}^{W_d} = \sum_{\omega \in H^{q+1}(W_d, \Gamma)} Z_{W_d}[\omega]$$

- We may naturally interpret the observables in compactifications of the $6d$ theory as the partition function of a theory with higher form symmetry, and change of polarization as gauging these symmetries!
- **3d − 3d correspondence** - Now we see the two choices of polarization above correspond to theories with either 1-form or 0-form symmetry Υ.
- More generally, there is a polarization for every choice of

$$H \subset \Upsilon = H^1(M_3, \Gamma_{\mathfrak{g}}),$$

which defines a theory we denote:

$$T[M_3, \mathfrak{g}, H]$$

where:

$$A\text{-polarization:} \quad T[M_3, \mathfrak{g}, H = 1], \qquad B\text{-polarization:} \quad T[M_3, \mathfrak{g}, H = \Upsilon]$$

and other choices are obtained by gauging symmetries.

- This is true for both minimal and DGG twists.

Eg, $T[L(k,1),\mathfrak{su}(2)]$

- Recall this is a level k CS theory with gauge algebra $\mathfrak{su}(2)$. More precisely, the theory is specified by:

$$H \subset \Upsilon = H^1(L(k,1),\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k \text{ even} \\ 1 & k \text{ odd} \end{cases}$$

- Then we claim:

$$T[L(k,1),\mathfrak{su}(2),1] \to \quad (\mathcal{N}=1,2) \quad SU(2) \text{ level } k \text{ CS theory}$$
$$T[L(k,1),\mathfrak{su}(2),\mathbb{Z}_2] \to \quad (\mathcal{N}=1,2) \quad SO(3) \text{ level } k \text{ CS theory } \textbf{(only for } k \textbf{ even!)}$$

Eg, $T[L(k,1), \mathfrak{su}(2)]$

- Recall this is a level k CS theory with gauge algebra $\mathfrak{su}(2)$. More precisely, the theory is specified by:

$$H \subset \Upsilon = H^1(L(k,1), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k \text{ even} \\ 1 & k \text{ odd} \end{cases}$$

- Then we claim:

$$T[L(k,1), \mathfrak{su}(2), 1] \to \quad (\mathcal{N} = 1,2) \quad SU(2) \text{ level } k \text{ CS theory}$$
$$T[L(k,1), \mathfrak{su}(2), \mathbb{Z}_2] \to \quad (\mathcal{N} = 1,2) \quad SO(3) \text{ level } k \text{ CS theory } \textbf{(only for } k \textbf{ even!)}$$

- The $SO(3)$ theory is ill-defined when k is odd due to an **'t Hooft anomaly** in the corresponding 1-form symmetry.
- Anomalies represent an obstruction to gauging a symmetry, occuring when the *charge measuring operators*, $U^g[\Sigma]$, are themselves charged under the symmetry.
- For a $3d$ 1-form symmetry, they can be characterized by an "anomaly form" describing two linked charge-measuring operators:

$$\mathcal{A} : \Gamma \times \Gamma \to \mathbb{R}/\mathbb{Z}$$

A subgroup $H \subset \Gamma$ can be gauged iff $\mathcal{A}(H, H) = 0$.

$T[L(p,q),\mathfrak{su}(N)]$

- Next take $M_3 = L(p,q)$ and $\mathfrak{g} = \mathfrak{su}(N)$ (for N prime):

$$\Upsilon = H^1(L(p,q),\mathbb{Z}_N) = \begin{cases} \mathbb{Z}_N & p|N \\ 1 & p \not| N \end{cases}$$

- For our quiver description from before, each $SU(N)$ gauge group has a potential $\mathbb{Z}_N$ 1-form symmetry acting on its Wilson loops, but most of these are anomalous. The anomaly matrix can be computed as:

$$\mathcal{A} = \begin{pmatrix} k_1 & -1 & 0 & \cdots & & 0 \\ -1 & k_2 & -1 & & & \\ 0 & -1 & k_3 & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & k_{n-1} & -1 \\ 0 & & & \cdots & -1 & k_n \end{pmatrix} \mod N\,,$$

- This has $\det \mathcal{A} = p$ and a unique null-vector if and only if $p = 0 \mod N$, corresponding to the generator of the $\mathbb{Z}_N$ 1-form symmetry in this case.
- For general graph M_3, the anomaly matrix is the **linking matrix** of the graph manifold, which has a null-vector for each generator of $H^1(M_3,\mathbb{Z}_N)$.

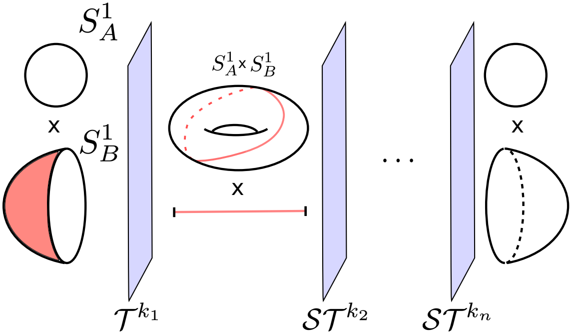
Geometric interpretation

- Explicitly, we can write the null vector as:

$$\delta = (\delta_1, \delta_2, .., \delta_n) \in {\mathbb{Z}_N}^n$$

Then the gauge group determined by δ_j in the jth node is $SU(N)/\mathbb{Z}_d$, $d = N/(\delta_j, N)$.

- Geometrically, δ corresponds to an element of $H^1 \cong H_2$ as follows:



- The cycle of T^2 wrapped in the jth interval then picks a polarization of T^2. This in turn determines the gauge group for this node by the same rule as for the 4d SYM theory.

Summary so far

- So far we have seen that the correct definition of the theory $T[M_3, \mathfrak{g}]$ depends on some extra data:

$$\rightarrow T[M_3, \mathfrak{g}, H], \quad H \subset \Upsilon \equiv H^1(M_3, \Gamma_{\mathfrak{g}})$$

- These theories have higher form symmetries, and can naturally be coupled to higher form gauge fields, leading to more refined observables on a spacetime W_3. eg:

$$Z_\lambda^{W_3}, \quad \lambda \in H^2(W_3, \Upsilon)$$

- Next we compute some explicit observables which depend on this data, and discuss the implications for the $3d - 3d$ correspondence.

Witten index and

the Bethe-ansatz equations

$T[M_3, \mathfrak{g}]$ on a torus

- We can test some of these properties by studying the Hilbert space of $T[M_3, \mathfrak{g}]$ on T^2, and the associated T^3 partition function.
- For supersymmetric theories, we consider the **Witten index**:

$$Z_{T^3} = \mathrm{Tr}(-1)^F = N_{vac}^{bos} - N_{vac}^{ferm}$$

- By the $3d - 3d$ correspondence, this maps to:

$$\begin{cases} \text{DGG twist} \;\rightarrow\; \text{complex } BF \text{ theory} \;\rightarrow\; \text{flat } \mathfrak{g}^{\mathbb{C}} \text{ connections} \\ \text{minimal twist} \;\rightarrow\; BFH \text{ theory} \;\rightarrow\; \text{moduli space of gSW equations} \end{cases}$$

- $3d\ \mathcal{N} = 1$ theories may exhibit wall-crossing where the Witten index jumps. This corresponds to the presence of twisted harmonic spinors on M_3, where the Euler characteristic of the moduli space of gSW equations may jump.
- We focus on the case of **rational homology spheres**, where we do not expect twisted harmonic spinors to arise. Then the BFH theory counts flat $\mathfrak{g}$ connections.

$T[M_3, \mathfrak{u}(1)]$ on a torus

- Recall the $T[M_3, \mathfrak{u}(1)]$ theory is a Chern-Simons theory:

$$S_{T[M_3,\mathfrak{u}(1)]} = \frac{1}{4\pi} \int \sum_{i,j} L_{i,j} A_i \wedge dA_j$$

  where L is the linking matrix of the graph manifold.
- In general, this TQFT has:

$$\dim(\mathcal{H}_{T^2}) = |\det L_{ij}| = |H^1(M_3, \mathbb{Z})|$$

  which agrees with the counting of (real and complex) flat $U(1)$ connections.

Supersymmetric vacua and Bethe equations

- For most other examples, $T[M_3, \mathfrak{g}]$ is an interacting QFT, and it is more difficult to study its Hilbert space on T^2.
- We study the supersymmetric vacua of of the 3d $\mathcal{N} = 2$ $T[M_3]$ theory, using the *effective twisted superpotential*.
- For a 3d G gauge theory with matter in representation R and CS levels k^{ab}, this is:

$$\mathcal{W}(x) = \mathrm{Tr}_R\, \mathrm{Li}_2(x) + \sum_{a,b=1}^{r_G} \frac{1}{2} k^{ab} \log x_a \log x_b, \quad x \in \mathbb{T}_G^{\mathbb{C}} = (\mathbb{C}^*)^{r_G}$$

- The vacua lie at critical points of the twisted superpotential, which are given by *Bethe-ansatz* type equations [Nekrasov-Shatashvili]:

$$\exp\left(x_a \partial_{x_a} \mathcal{W}(u)\right) = \prod_{\rho \in R} (1 - x^\rho)^{\rho_a} {x_b}^{k^{ab}} = 1, \qquad a = 1, \ldots, r_G$$

- The solutions to these equations define SUSY ground states on T^2:

$$\mathcal{H}_{T^2}^{SUSY} = \mathrm{span}\left\{ |x_a\rangle \;\middle|\; x_a \text{ sol'n to Bethe eqn's} \right\}$$

where all states are bosonic.

Example: $T[L(k,1),\mathfrak{u}(N)]$

- **Minimal twist**
  - Here we must first take an $\mathcal{N}=2$ deformation [Bashmakov-Gomis-Komargodski-Sharon]. Then we find:
  $$x_i{}^k = 1$$
  with:
  $$N_{vac} = \binom{k}{N}$$
  which counts the flat $U(N)$ connections.
  - This agrees with the result of [Acharya-Vafa], and counts domain walls in the $4d$ $\mathcal{N}=1$ $SU(k)$ gauge theory.
- **DGG twist**
  - Here the Bethe equations are given, for the DGG twist, by:
  $$x_i{}^k \prod_{j\neq i} \frac{x_i - tx_j}{x_j - tx_i} = 1, \quad i = 1, ..., N$$
  - This gives a system of coupled polynomial equations, with:
  $$N_{vac} = \binom{N+k-1}{N}$$
  - This agrees with the number of flat $GL(N,\mathbb{C})$ connections on $L(k,1)$.

Example: $T[L(k,1),\mathfrak{su}(2)]$

- Recall we have (for the DGG twist)

$$T[L(k,1),\mathfrak{su}(2),1] \to \quad (\mathcal{N}=2) \quad SU(2) \text{ level } k \text{ CS theory}$$
$$T[L(k,1),\mathfrak{su}(2),\mathbb{Z}_2] \to \quad (\mathcal{N}=2) \quad SO(3) \text{ level } k \text{ CS theory} \quad \text{(only for } k \text{ even)}$$

- We may always take the $H=1$, which implies the gauge group is $SU(2)$. The Bethe equations are:

$$x^{2k}\left(\frac{x^2 t-1}{t-x^2}\right)^2 = 1$$

and we find

$$N_{vac}[T[L(k,1),\mathfrak{su}(2),1]] = k+1$$

Example: $T[L(k,1), \mathfrak{su}(2)]$ (cont'd)

- To gauge the $\mathbb{Z}_2$ 1-form symmetry acting on the T^2 Hilbert space, we must project onto the trivial subspace of the charge-measuring operators for each cycle of T^2:

$$U_A^\gamma = 1, \qquad U_B^\gamma = 1$$

- In general, these act on the Bethe vacua as:

$$U_A^\gamma |x_a\rangle = e^{\gamma^a x_a \partial_a \mathcal{W}(x)} |x_a\rangle, \qquad U_B^\gamma |x_a\rangle = |e^{2\pi i \gamma_a} x_a\rangle$$

- In the present case, we find:

$$U_A^\gamma |x\rangle = x^k \left(\frac{x^2 t - 1}{t - x^2} \right) |x\rangle, \qquad U_B^\gamma |x\rangle = |-x\rangle$$

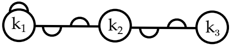
- Note that when k is odd, it is impossible to solve $U_A = U_B = 1$ as they do not commute, reflecting the anomaly.
- The number of solutions in this case is given by:

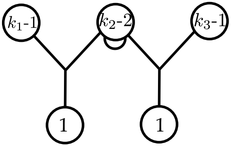
$$N_{vac}[T[L(k,1), \mathfrak{su}(2), \mathbb{Z}_2]] = \left\lfloor \frac{k}{4} \right\rfloor + 1$$

Example: $T[L(p,q),(\mathfrak{s})\mathfrak{u}(2)]$

- For quiver gauge theories, the Bethe equations are more complicated. For example:

$$M_3 = L(p,q), \quad \frac{p}{q} = k_1 - \frac{1}{k_2 - \frac{1}{k_3}}, \qquad p = k_1 k_2 k_3 - k_1 - k_3, \quad q = k_1 k_2 - 1$$





- This leads to a system of 5 coupled equations:

$$\begin{aligned} & {x_1}^{2(k_1-1)} \frac{(x_1 - tx_2x_3)(x_1x_2 - tx_3)(x_1x_3 - tx_2)(x_1x_2x_3 - t)}{(1 - tx_1x_2x_3)(x_2 - tx_1x_3)(x_3 - tx_1x_2)(x_2x_3 - tx_1)} = 1, \\ & {x_2}^{2(k_2-2)} \frac{(x_2 - tx_1x_3)(x_1x_2 - tx_3)(x_2x_3 - tx_1)(x_1x_2x_3 - t)}{(1 - tx_1x_2x_3)(x_1 - tx_2x_3)(x_3 - tx_1x_2)(x_1x_3 - tx_2)} \frac{(x_2 - tx_4x_5)(x_4x_2 - tx_5)(x_2x_5 - tx_4)(x_4x_2x_5 - t)}{(1 - tx_4x_2x_5)(x_4 - tx_2x_5)(x_5 - tx_4x_2)(x_4x_5 - tx_2)} = 1 \\ & \cdots \end{aligned} \tag{1}$$

- These can be solved by finding a **Gröbner basis**, $g_a(x_i)$, for the ideal generated by these polynomial equations. Then:

$$N_{solutions} = \# \left\{ \text{monomials } m = \prod_i {x_i}^{\ell_i} \;\middle|\; \text{lm}(g_a) \not| \; m \right\}$$

- We find Gröbner bases using SINGULAR. In this example we find:

$$N_{solutions} = 4(k_1 k_2 k_3 - k_1 - k_3 + 1) = 4(p+1)$$

Example: $T[L(p,q),(\mathfrak{s})\mathfrak{u}(2)]$ (cont'd)

- To form the $T[L(p,q),\mathfrak{u}(2)]$ theory, we take the tensor product with the $T[M_3,u(1)]$ theory, and then take the quotient corresponding to:

$$U(1)\times SU(2)\to (U(1)\times SU(2))/\mathbb{Z}_2\cong U(2)$$

  at each node, and we find:

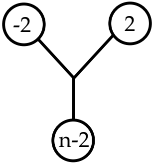
$$N_{vac}[T[L(p,q),\mathfrak{u}(2)]=\frac{p(p+1)}{2}=\binom{p+1}{2}$$

  which agrees with the flat connection counting.
- For the $T[L(p,q),\mathfrak{su}(2)]$ theory, we must gauge the non-anomalous 1-form symmetries of the theory. We find:

$$N_{vac}[T[L(p,q),\mathfrak{su}(2),H]]=\begin{cases}p+1 & H=1\\ \left\lfloor\frac{p}{4}\right\rfloor+1 & p\text{ even}, H=\mathbb{Z}_2\end{cases}$$

Other Seifert manifolds

- We may repeat this for more general $L(p,q)$, as well as other rational homology spheres. Eg, for S^3/Γ_{D_n} and $\mathfrak{g} = \mathfrak{u}(2)$, we have:



$$N_{vac}[T_{\mathcal{N}=2}[S^3/\Gamma_{D_n}, \mathfrak{u}(2)]] = n+7$$

- Here it is not possible to deform the $\mathcal{N}=1$ minimal twist to $\mathcal{N}=2$, so one must compute the Witten index by other means. The result should agree with [Acharya-Vafa].
- For the $SU(2)$ theory, the theories are labeled by $H \subset H^2(M_3, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$$N_{vac}[T[S^3/D_n, \mathfrak{su}(2), H]] = \begin{cases} 2n+7 & H = 1 \\ \frac{n}{2}+4 & H = \mathbb{Z}_2 \times 1, n \text{ even} \\ \frac{n+1}{2}+4 & H = \mathbb{Z}_2 \times 1, n \text{ odd} \\ n+1+4 & H = (\mathbb{Z}_2)_{diag} \\ \lfloor \frac{n}{2} \rfloor + 1 & H = \mathbb{Z}_2 \times \mathbb{Z}_2 \end{cases}$$

- For general M_3, theese results should be computed by a new TQFT invariant appearing in the $3d-3d$ correspondence dictionary, which is labeled by M_3 and a choice of subgroup $H \subset H^1(M_3, \mathbb{Z}_2)$.

Summary

and outlook

Summary

- The 6*d* $\mathcal{N} = (2, 0)$ is a relative QFT, which leads to several subtleties in understanding its compactifications.
- The theories $T[M_3, \mathfrak{g}]$ obtained by compactification of M5 branes on a 3-manifold depend on extra topological data related to the polarization of the 6*d* theory.
- This can be naturally phrased in the language of higher form symmetries, and different choices can be related by gauging these symmetries.
- This data can be probed by computing the Witten index of these theories, and lead to generalizations of the complex *BF* and *BFH* models.

Outlook

- Describe the new entries in the dictionary of the $3d - 3d$ correspondence, and more refined observables in the TQFTs.
- Higher form symmetries have led to powerful insights into theories with little or no supersymmetry, and can be used to better understand the $3d$ $\mathcal{N} = 1$ theories obtained by reducton of M5 branes on associative cycles.
- They also are important for analyzing defect theories inside higher dimensional quantum field theories (eg, [Gaiotto-Kapustin-Komargodski-Seiberg]), and can play a role in the correspondence between defect theories and calibrated cycles.
- This analysis can be generalized to compactifications on four-manifolds [Gadde-Gukov-Putrov,Gukov-Putrov-Pei-Vafa], and to compactifications of general $6d$ $\mathcal{N} = (1, 0)$ theories, which are also typically relative QFTs.