The η invariant under cone-edge degeneration

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Overview

My phd project in short

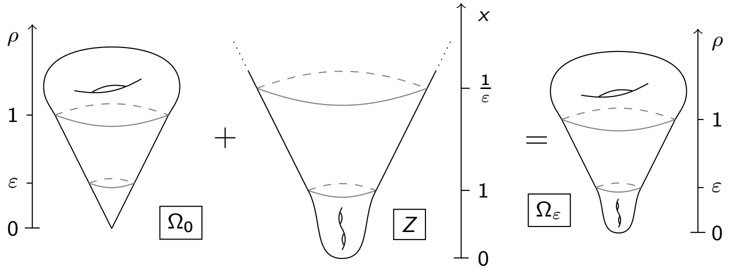
- In [S15], Sher examined the behaviour of the determinant of the Laplacian on a manifold Ω_ε undergoing conic degeneration;
- The determinant of the Laplacian is analytically similar to the η invariant, used in [CGN15] to define the $\bar{\nu}$ invariant for G_2 manifolds;
- A part of Joyce's construction of G_2 manifolds ([J96]) involves blow-ups. The variation of the η invariant under blow-ups can be understood in terms of degeneration processes.

Idea

Apply Sher's techniques to the η invariant and generalise it to the cone-edge case. Try to say something about the $\bar{\nu}$ invariant of Joyce's G_2 manifolds.

Conic degeneration

Use surgery to resolve a conic singularity:



Comments

$\{\rho = 0\} \subseteq \Omega_0$ is the *singular set*. In this case $S = \{pt\}$.
Ω_0 has a *conic singularity* at p iff in a neighbourhood of p it is isometric to the cone $(C(Y), d\rho^2 + \rho^2 h)$, where (Y, h) is a smooth Riemannian manifold.

Conic degeneration

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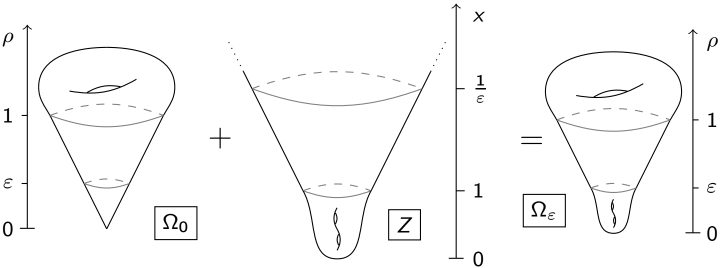
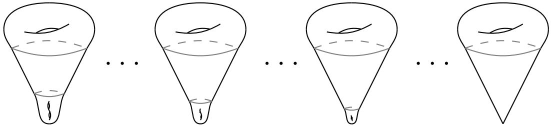


As $\varepsilon \to 0$, Ω_ε degenerates to Ω_0:



Sher's work

Let $\Delta_{\Omega_\varepsilon}$ be the Laplacian acting on functions on Ω_ε, $\lambda^2_{\varepsilon,i}$ its ith eigenvalue, $H^{\Omega_\varepsilon}(t,z,z')$ its heat kernel at time t. The *zeta function* on Ω_ε is:

$$\zeta_{\Omega_\varepsilon}(s) = \sum_{i=1}^{\infty} \left(\lambda^2_{\varepsilon,i}\right)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty (\operatorname{tr} H^{\Omega_\varepsilon}(t) - 1) t^{s-1} dt \tag{1}$$

$\zeta_{\Omega_\varepsilon}(s)$ has a meromorphic continuation to $\mathbb{C}$ without poles at $s = 0$. The determinant of the Laplacian is given by

$$\det \Delta_{\Omega_\varepsilon} = e^{-\zeta'_{\Omega_\varepsilon}(0)} \tag{2}$$

Sher's work

Theorem (Sher '15)

Let Y be a closed Riemannian manifold, let Ω_0 be a compact manifold which has a single isolated conical singularity of cross-section Y, exactly conic in a neighbourhood of 0; let Z be a complete manifold, exactly conic in a neighbourhood of ∞ with cross-section Y. As $\varepsilon \to 0$

$$\begin{aligned}\log\det\Delta_{\Omega_\varepsilon} =& \frac{1}{2}\left(\int_Y u_n(1,y)dy\right)(\log\varepsilon)^2 - 2\log\varepsilon({}^R\zeta_Z(0)) \\ &+ \log\det\Delta_{\Omega_0} + \log^R\det\Delta_Z + o(1)\end{aligned} \tag{3}$$

Here $u_n(1,y)$ is the coefficient of t^0 in the local heat kernel asymptotics for the infinite cone $C(Y)$ at the point $(1,y)$, ${}^R\zeta_Z(0)$ is the term of order 0 in the Laurent expansion of $\zeta_Z(s)$ and $\log^R\det\Delta_Z$ is the term of order 1.

The $\bar{\nu}$ invariant

Let D_ε be either the spin Dirac operator or the signature operator on Ω_ε, $\lambda_{\varepsilon,i}$ its ith eigenvalue, $H^{\Omega_\varepsilon}(t,z,z')$ the heat kernel of D_ε^2 at time t. Then

$$\eta(D_\varepsilon)(s) = \sum_{i=0}^{\infty} \operatorname{sgn}(\lambda)\,|\lambda|^{-s} = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \operatorname{tr} D_\varepsilon H^{\Omega_\varepsilon}(t) t^{\frac{s-1}{2}}\, dt \tag{4}$$

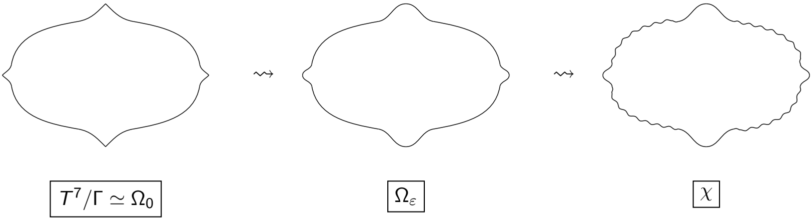
$\eta(D_\varepsilon)(s)$ has a meromorphic continuation to $\mathbb{C}$ without poles at $s = 0$. The η invariant of D_ε is given by $\eta(D_\varepsilon)(0)$.

The $\bar{\nu}$ invariant of a torsion-free G_2 manifold χ is defined as

$$\bar{\nu}(\chi) = 3\eta(B_\varepsilon) - 24\eta(D_\varepsilon) \tag{5}$$

Connection to Joyce's construction of G_2 manifolds

We divide Joyce's construction in two steps:

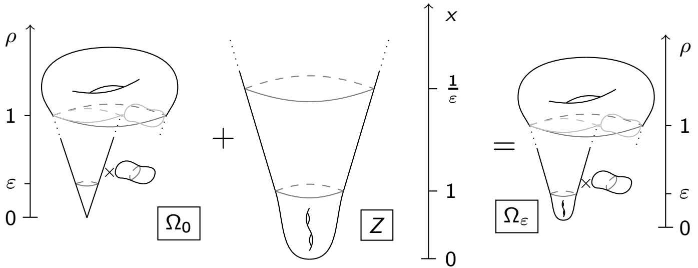


We have a good understanding of η invariants for T^7/Γ in terms of equivariant η invariants. We can use the approach of the previous slides to pass from Ω_0 to Ω_ε, and then hope nothing goes too wrong in the second step.

The singularities of Ω_0 are cone-edge singularities.

(simple) Cone-edge degeneration

Use surgery to resolve a cone-edge singularity:



Comments

$\{\rho = 0\} \subseteq \Omega_0$ is the *singular set*. In this case $S = (A, k)$.
Ω_0 has a *cone-edge singularity* at A iff in a neighbourhood of A it is isometric to $(C(Y) \times A, d\rho^2 + \rho^2 h + k)$, where (Y, h) and (A, k) are smooth Riemannian manifolds. (A, k) is called *edge*.

(simple) Cone-edge degeneration

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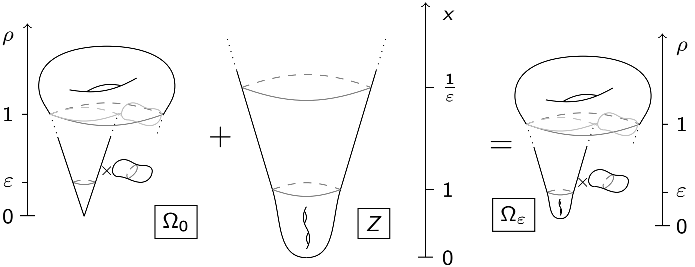

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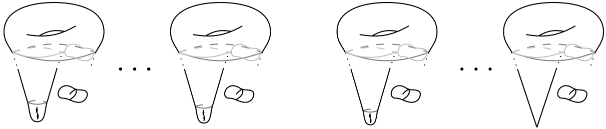


Main result

Definition (Generalised Witt condition)

Let $\dim Z = n$, D_Y the signature/spin Dirac operator of the link Y.

- The Hodge operator is *admissible* if and only if:

$$\begin{cases} \sigma(D_Y^2|_{\Lambda^{n/2-1}}) \cap \{0\} = \sigma(D_Y^2|_{\Lambda^{n/2}}) \cap [0,1] = \emptyset & \text{if } n \text{ is even} \\ \sigma(D_Y^2|_{\Lambda^{(n-1)/2}}) \cap \left[0, \frac{3}{4}\right] = \emptyset & \text{if } n \text{ is odd} \end{cases}$$

- The Dirac operator is *admissible* if and only if:

$$\sigma(D_Y^2) \cap \left[0, \frac{9}{4}\right] = \emptyset$$

Remark

For $Y = S^{n-1}/\Gamma$, $n > 4$, both operators are admissible.

Main result

Theorem (N. '19)

Let Y be a closed space form with $\dim Y > 3$, let Ω_0 be an odd dimensional compact manifold which has a single isolated simple cone-edge singularity of cross-section Y and edge A, exactly conic in a neighbourhood of 0; let Z be a complete manifold, exactly conic in a neighbourhood of ∞ with cross-section Y. Let B_ε be the signature operator on Ω_ε.
As $\varepsilon \to 0$

$$\eta(B_\varepsilon) = \eta(B_0) + \begin{cases} {}^R\eta(B_Z)\sigma(A) & \textit{if } \dim A \textit{ even} \\ \sigma_{L^2}(Z)\eta(B_A) & \textit{if } \dim A \textit{ odd} \end{cases} + o(1) \quad (6)$$

Here ${}^R\eta(B_Z)$ is the term of order 0 in the Laurent expansion of $\eta(B_Z)(s)$ and $\sigma_{L^2}(Z)$ is the L^2 signature of Z.

An application: Example 18 from [J96]

Let $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ with the standard G_2 structure $\hat{\varphi}$, consider the isometries of T^7 defined by

$$\begin{cases} \alpha : & [(x_1, \dots, x_7)] \mapsto [(x_2, x_3, x_7, -x_6, -x_4, x_1, x_5)] \\ \beta : & [(x_1, \dots, x_7)] \mapsto \left[\left(\frac{1}{2} - x_1, \frac{1}{2} - x_2, -x_3, -x_4, \frac{1}{2} + x_5, \frac{1}{2} + x_6, x_7\right)\right] \end{cases}$$

Let Γ be the finite group generated by α and β. The singular set of T^7/Γ consists of one copy of S^1 and it has a neighbourhood isometric to $S^1 \times B^6/\mathbb{Z}_7$. For complex coordinates $(z_1, z_2, z_3) \in \mathbb{C}^3$, the action of $\mathbb{Z}_7$ on B^6 is generated by

$$(z_1, z_2, z_3) \mapsto (e^{2\pi i/7} z_1, e^{4\pi i/7} z_2, e^{8\pi i/7} z_3) \tag{7}$$

This is our Ω_0, and it satisfies the hypothesis of the theorem.

An application: Example 18 from [J96]

To desingularize Ω_0 we require a suitable ALE space with holonomy $SU(3)$. Markushevich, Olshanetsky and Perelomov constructed such a space in [MOP87].
Thus we can use the theorem to obtain, as $\varepsilon \to 0$:

$$3\eta(B_\varepsilon) - 24\eta(D_\varepsilon) = 3\eta(B_0) + 3\sigma_{L^2}(Z)\eta(B_{S^1}) - 24\eta(D_0) - 24\,\mathrm{ind}_{L^2}(D_Z)\eta(D_{S^1}) + o(1)$$

However $\eta(B_{S^1}) = 0 = \eta(D_{S^1})$, so we're left with:

$$3\eta(B_0) - 24\eta(D_0) + o(1) \tag{8}$$

Lastly, Scaduto showed that for Joyce's examples it already holds that $\bar{\nu}(\Omega_\varepsilon) = 3\eta(B_\varepsilon) - 24\eta(D_\varepsilon) \in \mathbb{Z}$, so that:

$$\bar{\nu}(\Omega_\varepsilon) = 3\eta(B_0) - 24\eta(D_0) \tag{9}$$

An application: Example 18 from [J96]

To compute $\bar{\nu}(\Omega_\varepsilon)$ we observe that $\eta(B_0)$ and $\eta(D_0)$ correspond to the equivariant η invariants $\eta^\Gamma(B_{T^7})$ and $\eta^\Gamma(D_{T^7})$ (not true a priori). We don't need to do this explicitly, since the orientation reversing isometry of T^7

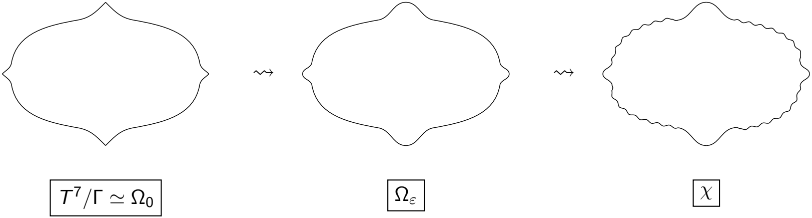
$$\iota : T^7 \to T^7 : [(x_1, \dots, x_7)] \mapsto [(-x_1, \dots, -x_7)] \tag{10}$$

descends to T^7/Γ. Thus:

$$\eta(B_0) = \eta^\Gamma(B_{T^7}) = 0 = \eta^\Gamma(D_{T^7}) = \eta(D_0) \tag{11}$$

so that $\bar{\nu}(\Omega_0) = 0$, hence $\bar{\nu}(\Omega_\varepsilon) = 0$.

An application: Example 18 from [J96]



The last step is to relate $\bar{\nu}(\Omega_\varepsilon)$ to $\bar{\nu}(\chi)$. We have

$$\begin{aligned}\bar{\nu}(\chi) = \bar{\nu}(\Omega_\varepsilon) + 3(\dim\ker B_\chi - \dim\ker B_{\Omega_\varepsilon}) + 6 spf(B_{\Omega_\varepsilon \to \chi}) \\ - 24(\dim\ker D_\chi - \dim\ker D_{\Omega_\varepsilon}) - 48 spf(D_{\Omega_\varepsilon \to \chi})\end{aligned} \tag{12}$$

An application: Example 18 from [J96]

We have

$$\begin{aligned}\bar{\nu}(\chi) =&\bar{\nu}(\Omega_\varepsilon) + 3(\dim\ker B_\chi - \dim\ker B_{\Omega_\varepsilon}) + 6spf(B_{\Omega_\varepsilon\to\chi})\\ &- 24(\dim\ker D_\chi - \dim\ker D_{\Omega_\varepsilon}) - 48spf(D_{\Omega_\varepsilon\to\chi})\\ =&\bar{\nu}(\Omega_\varepsilon) - 24(\dim\ker D_\chi - \dim\ker D_{\Omega_\varepsilon}) - 48spf(D_{\Omega_\varepsilon\to\chi})\\ =&-24 + 24\dim\ker D_{\Omega_\varepsilon} - 48spf(D_{\Omega_\varepsilon\to\chi})\end{aligned}$$

Lemma (N. '19)

For ε small enough
$\dim\ker D_{\Omega_\varepsilon} = \dim\ker D_{\Omega_0} + \dim\ker D_B \dim\ker_{L^2} D_Z$.

Hence

$$\bar{\nu}(\chi) = 24\dim\ker_{L^2} D_Z - 48spf(D_{\Omega_\varepsilon\to\chi}) \quad (13)$$

The general spirit

Two approaches to the analysis of the η invariant:

$$\eta(D)(s) = \begin{cases} \underbrace{\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty \operatorname{tr} DH(t) t^{\frac{s-1}{2}}\, dt}_{\text{Microlocal analysis}} \\ \underbrace{\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty \sum_{i=0}^\infty \lambda_i e^{-t\lambda_i^2} t^{\frac{s-1}{2}}\, dt}_{\text{Spectral theory}} \end{cases} \tag{14}$$

The first works best for short times, the second for large times, so one writes

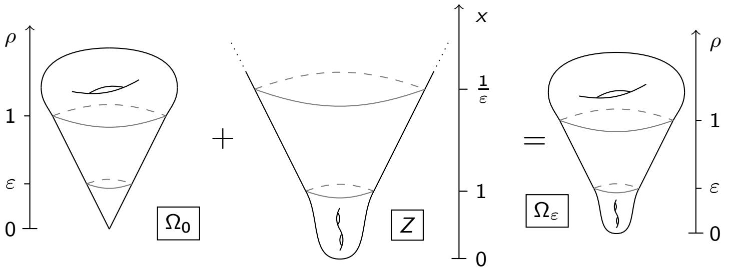
$$\eta(D)(s) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^1 \operatorname{tr} DH(t) t^{\frac{s-1}{2}}\, dt + \frac{1}{\Gamma(\frac{s+1}{2})}\int_1^\infty \sum_{i=0}^\infty \lambda_i e^{-t\lambda_i^2} t^{\frac{s-1}{2}}\, dt \tag{15}$$

The short time component

Split the distributional kernel as:

$$\operatorname{tr} D_\varepsilon H^{\Omega_\varepsilon} = \operatorname{tr} \chi_1 D_0 H^{\Omega_0}(t) + \operatorname{tr} \chi_2 D_{\varepsilon Z} H^{\varepsilon Z}(t) + R(\varepsilon, t) \qquad (16)$$

Here χ_1 is a bump function on Ω_0 supported away from the singularity, and χ_2 on Z is supported away from infinity.



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Here χ_1 is a bump function on Ω_0 supported away from the singularity, and χ_2 on Z is supported away from infinity.

The kernel $\operatorname{tr} \chi_1 D_0 H^{\Omega_0}(t)$ is supported away from the singularity, so we can use the short time expansion for smooth manifolds ([BF86])

$$\chi_2(z) D_0 H^0(t, z, z) = \chi_2(z) b(z) \sqrt{t} + o(t^{\frac{3}{2}}) \text{ as } t \to 0 \qquad (18)$$

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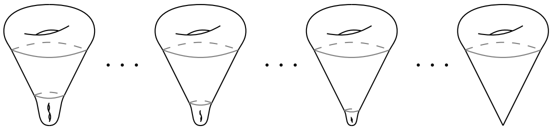
Here χ_1 is a bump function on Ω_0 supported away from the singularity, and χ_2 on Z is supported away from infinity.

As $\varepsilon \to 0$, the space εZ looks more and more like a cone, so we can understand $\operatorname{tr} \chi_2 D_{\varepsilon Z} H^{\varepsilon Z}(t)$ in terms of the same kernel on the exact cone $C(Y)$:



The short time component

Heat kernels behave well under rescaling:

$$H^{\varepsilon Z}(t,z,z') = \varepsilon^{-n} H^Z(\varepsilon^{-2}t, \varepsilon^{-1}z, \varepsilon^{-1}z') \tag{20}$$

This scaling property is useful when analyzing the term tr $\chi_2 D_{\varepsilon Z} H^{\varepsilon Z}(t)$:

$$\begin{aligned}
&\int_0^1 \int_{\varepsilon Z} t^{\frac{s-1}{2}} \chi_2(z) D_{\varepsilon Z} H^{\varepsilon Z}(t,z,z)dtdz = \\
&\quad = \int_0^1 \int_{\varepsilon Z} t^{\frac{s-1}{2}} \chi_1(z) \varepsilon^{-n-1} D_Z H^Z(\varepsilon^{-2}t, \varepsilon^{-1}z, \varepsilon^{-1}z)dzdt \\
&\quad = \varepsilon^{2s-1} \int_0^{\varepsilon^{-2}} \int_Z \tau^{\frac{s-1}{2}} \chi_1(\varepsilon z) D_Z H^Z(\tau,z,z)dzd\tau
\end{aligned} \tag{21}$$

As $\varepsilon \to 0$ this spawns the term ${}^R\eta(D_Z)$.

The long time component

Analysis of the long time component boils down to computing

$$\lim_{\varepsilon \to 0} \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \sum_{i=0}^\infty \lambda_{\varepsilon,i} e^{-t\lambda_{\varepsilon,i}^2} t^{\frac{s-1}{2}} dt \tag{22}$$

To do this, we compute the limit $\lim_{\varepsilon \to 0} \lambda_{\varepsilon,i}$ and then justify passing the limit inside the integral and the series. Spectral convergence results were proven by Anné and Takahashi in [AT15]:

Theorem (Anné-Takahashi, '15)

The spectrum $\sigma(D_\varepsilon^2)$ converges to the union of $\sigma(D_0^2)$ with a set of $\dim \ker_{L^2} D_Z^2$ *zeros.*

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To do this, we compute the limit $\lim_{\varepsilon \to 0} \lambda_{\varepsilon,i}$ and then justify passing the limit inside the integral and the series. To apply Lebesgue's dominated convergence theorem we make use of a Weyl criterion:

Theorem (Sher, '15)

there are constants $C > 0$ and $N_0 \in \mathbb{N}$, both independent of ε, such that for all $\varepsilon \in [0, \varepsilon_0]$, $k \geq N_0$

$$\lambda_{\varepsilon,k}^2 \geq Ck^{2/n} \tag{24}$$

Generalization to the cone-edge case

We follow the same route. For the short time component we have

$$\operatorname{tr} D_\varepsilon H^{\Omega_\varepsilon} = \operatorname{tr} \chi_1 D_0 H^{\Omega_0}(t) + \operatorname{tr} \chi_2 D_{\varepsilon Z \times A} H^{\varepsilon Z \times A}(t) + R(\varepsilon, t) \quad (25)$$

The first summand is analyzed as before, for the second we observe

$$\operatorname{tr} D_{\varepsilon Z \times A} H^{\varepsilon Z \times A}(t) = \operatorname{tr} D_{\varepsilon Z} H^{\varepsilon Z} \operatorname{tr} H^A(t) + \operatorname{tr} H^{\varepsilon Z} \operatorname{tr} D_A H^A(t)$$

These summands spawn the terms ${}^R\eta(B_Z) \operatorname{ind}(D_A)$ and $\operatorname{ind}_{L^2}(Z)\eta(B_A)$.

For the long time component, following Anné and Takahashi we show

Theorem (N, '19)

The spectrum $\sigma(D_\varepsilon^2)$ converges to the union of $\sigma(D_0^2)$ with a set of $\dim \ker_{L^2} D_Z^2$ *copies of $\sigma(D_B^2)$.*

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