Analysis of singular sets in calibrated geometric analysis

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Special Holonomy and Calibrated Geometry
Going beyond Almgren

Regularity results for calibrated integral cycles that improve Almgren et al.

- **Complex cycles in** $\mathbb{C}^n$:
  - [King '71], [Harvey-Schiffman '74], [Siu ’74] techniques do not extend to *almost*-complex.

  - Pseudo-holomorphic integral 2-cycles [Taubes '00], [Rivière-Tian '09]: smooth except possibly at isolated points.

  - Special Lagrangian 3D cones in $\mathbb{R}^6$ [B.-Rivière '13] smooth except possibly for finite number of half-lines.

  - Area-minimizing 3D cones [De Lellis - Spadaro - Spolaor '16] smooth except possibly for finite number of half-lines.
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Ideas from [B.-Rivière]

Special Lagrangian cone $\rightsquigarrow$ slice with $S^5 \rightsquigarrow$ Special Legendrian.

2-D integral cycle in $S^5$, semi-calibrated by a (non-closed) 2-form.

Study Special Legendrian cycles in $S^5$. Key tools to start with:

- Slicing by positive transv. foliations,
- multiple valued graphs.
Tangent cone
Current caught in **yellow**, boundary in **red**
Boundaries don’t cross $\Rightarrow$ Algebraic intersection index conserved
The 3-surfaces intersect the current positively!
Every 3-surface meets the current in $Q = 4$ points.
Get a $Q$-valued graph
unordered $Q$-tuples

\[ D^2 \subset \mathbb{C} \rightarrow \frac{(\mathbb{C} \times \mathbb{R})^Q}{\sim} \]

\[ z \rightarrow \{(\varphi_j(z), \alpha_j(z))\}_{j=1}^Q \]
Make sense of PDE for \( \{(\varphi_j(z), \alpha_j(z))\}_{j=1}^Q \)

Perturbation of Cauchy-Riemann

\[
\begin{align*}
\partial_z \varphi_j &= \nu((\varphi_j, \alpha_j), z) \partial_z \varphi_j + \mu((\varphi_j, \alpha_j), z) \\
\nabla \alpha_j &= h((\varphi_j, \alpha_j), z),
\end{align*}
\]

\( \nu, \mu, h \) small, \( \mathbb{C} \)-valued, 0 at 0

Implement **elliptic PDEs** techniques, e.g. **unique continuation**. How?
Proof by induction on $Q$, say $Q = 4$

Part I: prove that singularities of multiplicity 4 cannot accumulate to 0.

Prove that the average is a $W^{1,2}$ graph (needs uniqueness of tangent at 0) that also solves a perturbation of Cauchy-Riemann

Subtract the average and get a new 4-valued graph that satisfies a perturbation of Cauchy-Riemann

Now the singularities of multiplicity 4 are zeros: implement unique continuation.
Part II: prove that singularities of multiplicity $\leq 3$ cannot accumulate to 0.

Within the induction (on multiplicity), at this stage you know that singularities of multiplicity $\leq 3$ are countable and can only accumulate to 0.

Homological argument: from the calibrating condition, produce a notion of “positive degree” around each isolated singularity and a notion of degree bounded from below on any ball centered at 0. Accumulation of singularities to 0 yields a contradiction.
Positiveness of intersection not to be expected in general.
PDE will depend on the calibration.

“degree” argument: I don’t know...

What I expect to be true (but very hard) is the uniqueness of tangent cones for calibrated integral cycles.
Uniqueness issue for the tangent space

Dilating with factors $r_1, r_2, r_3, ...$ yields ??
Dilating with factors $R_1, R_2, R_3, ...$ yields ??
Uniqueness of tangent cones

Known by “2nd order theory”

[Allard-Almgren ’76]: 1-dimensional integral currents.

[White ’83]: area-minimizing 2-dim. integral currents.

[Simon ’83]: mass minimizers, tangent cone with isolated sing. and multiplicity 1.

Known by “1st order theory”

[Pumberger-Rivière ’10]: 2-dim. calibrated integral cycle.

$$\omega$$ calibration of degree 2, $$\Omega = \frac{\omega^p}{p!}$$ calibration of degree $$2p$$:

[B. ’14] $$2p$$-dim. $$\Omega$$-calibrated integral cycles.
Semi-calibration: form of comass one, not necessarily closed.

<table>
<thead>
<tr>
<th>Theorem (B.)</th>
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$\omega$ non-degenerate $2$-form (possibly $d\omega \neq 0$ even if $d\phi = 0$)

compatible almost complex structure $J$ Riemannian metric $g$

$J(\cdot, \cdot) = \omega(\cdot, J\cdot)$

($\omega$ is a semi-calibration w.r.t. $g$ $J$)

Semi-calibrated $2$-dimensional integral currents have an extra structure: they are pseudo holomorphic.

Semi-calibrated by $\phi$ w.r.t. $g$ $\Rightarrow$ semi-calibrated by $\omega$ w.r.t. $g$ $J$. 


Semi-calibration: form of comass one, not necessarily closed.

Theorem (B.)

\( \phi \) semi-calibration of degree 2 in \((M, g)\). Locally in \(M\) or \(M \times \mathbb{R}\) (whichever is even-dimensional) we can find:

- \( \omega \) non-degenerate 2-form (possibly \( d\omega \neq 0 \) even if \( d\phi = 0 \))
- compatible almost complex structure \( J \)
- Riemannian metric \( g_J(\cdot, \cdot) = \omega(\cdot, J\cdot) \)

(\( \omega \) is a semi-calibration w.r.t. \( g_J \))
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\((\omega \) is a semi-calibration w.r.t. \( g_J \))

s.t. any \( \phi \)-calibrated 2-plane is also \( \omega \)-calibrated.
Semi-calibration: form of comass one, not necessarily closed.

**Theorem (B.)**

ϕ semi-calibration of degree 2 in (M, g). Locally in M or M × R (whichever is even-dimensional) we can find:

- ω non-degenerate 2-form (possibly dω ≠ 0 even if dϕ = 0)
- compatible almost complex structure J
- Riemannian metric gJ(·, ·) = ω(·, J·)

(ω is a semi-calibration w.r.t. gJ)

s.t. any ϕ-calibrated 2-plane is also ω-calibrated.

Semi-calibrated 2-dimensional integral currents have an extra structure: they are pseudo holomorphic.
Semi-calibration: form of comass one, not necessarily closed.

**Theorem (B.)**

\( \phi \) semi-calibration of degree 2 in \((\mathcal{M}, g)\). Locally in \(\mathcal{M}\) or \(\mathcal{M} \times \mathbb{R}\) (whichever is even-dimensional) we can find:

- \(\omega\) non-degenerate 2-form (possibly \(d\omega \neq 0\) even if \(d\phi = 0\))
- compatible almost complex structure \(J\)
- Riemannian metric \(g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)\)

(\(\omega\) is a semi-calibration w.r.t. \(g_J\))

s.t. any \(\phi\)-calibrated 2-plane is also \(\omega\)-calibrated.

Semi-calibrated 2-dimensional integral currents have an **extra structure**: they are pseudo holomorphic.

Semicalibrated by \(\frac{\phi^p}{p!}\) w.r.t. \(g \leadsto\) semicalibrated by \(\frac{\omega^p}{p!}\) w.r.t. \(g_J\).
Make the “waste of mass” visible

Blow-up the origin of $\mathbb{C}^n$ (algebraic/symplectic geometry)
Idea of proof

Implement a *pseudo holomorphic blow up* of a sector

\[ \tilde{\Omega}, \tilde{g}, \tilde{J} \] perturbations of the standard \( \mathbb{CP}^{n-1} \times \mathbb{C} \).

\[ \Omega, g, J \]
Idea of proof

Push-forward a pseudo holomorphic current via singular map

\[ D \subset \mathbb{C} \]

\[ U \subset \mathbb{CP}^{n-1} \]
Idea of proof

Push-forward a pseudo holomorphic current via singular map

\[ \text{Diagram with symbols: } \mathcal{D} \subset \mathbb{C} \]

\[ \text{Diagram with symbols: } U \subset \mathbb{C}P^{n-1} \]
Push-forward well-defined in the limit as a $\tilde{J}$-holomorphic cycle
Idea of proof

Push-forward well-defined in the limit as a $\tilde{J}$-holomorphic cycle

Semi-Calibrated cycle on the right!
Further regularity problems in calibrated geometry

Gauge theory on $G2$ and Spin(7)-manifolds [Walpuski '17]

Very closely related: *triholomorphic maps*

$$\mathbb{H} = \text{span}\{1, i, j, k\} \equiv \mathbb{R}^4,$$ with $i^2 = j^2 = k^2 = ijk = -1$

$$f : \mathbb{R}^4 \equiv \mathbb{H} \to \mathbb{H} \text{ satisfying } \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + j \frac{\partial}{\partial x_3} + k \frac{\partial}{\partial x_4} \right) f = 0 \implies f \text{ is harmonic } \Delta f = 0$$

HyperKähler mfd : tangent model is $\mathbb{H}^m$

$$f \text{ between Compact HyperKähler mfllds satisfying analog. 1st order PDE } \implies f \text{ is a harmonic map}$$
Triholomorphic maps

\( u : \mathcal{M}^{4m} \rightarrow \mathcal{N}^{4n} \)

\( i, j, k \) on domain, \( I, J, K \) on target (with quaternionic rule).

\( u \in W^{1,2}(\mathcal{M}, \mathcal{N}) \)

\[
d u = I \, d u \, i + J \, d u \, j + K \, d u \, k
\]

\( \mathcal{N} \) hyperKähler

\( \mathcal{M} \) almost hyperHermitian (\( \omega_i, \omega_j, \omega_k \) not closed, \( i, j, k \) not integrable).

\[
d (u^* \Omega) = 0 \text{ if } \Omega \text{ closed 2-form.}
\]

\[
|\nabla u|^2 = -C_m \left( \omega_i^{4m-2} \wedge u^* \Omega_I + \omega_j^{4m-2} \wedge u^* \Omega_J + \omega_k^{4m-2} \wedge u^* \Omega_K \right)
\]

\( u \) is (almost) stationary harmonic
Compactness for triholomorphic maps

\{ u_\ell \}_{\ell \in \mathbb{N}} \text{ triholomorphic with equibounded Dirichlet energy.}

**Problem:** Analyse bubbles, bubbling set \( \Sigma \) (dim. \( 4m - 2 \)), limiting map \( u \) (weakly harmonic, not known if stationary harmonic).

\[ |\nabla u_\ell|^2 d\text{vol}_M \rightharpoonup |\nabla u|^2 d\text{vol}_M + \Theta(x) \mathcal{H}^{4m-2} \subseteq \Sigma \]

**Theorem (B. - Tian ’19)**

*Energy identity:* for \( \mathcal{H}^{4m-2} \)-a.e. \( x \in \Sigma \)

\[ \Theta(x) = \sum_{s=1}^{N_x} \int_{S^2} |\nabla \phi_s|^2, \]

where \( \phi_s : S^2 \rightarrow \mathcal{N} \) are holomorphic bubbles.

*(holomorphic for a complex structure depending on \( x \))*

“Usual 2D-bubbling picture in \( T_x \Sigma \perp \)”

Energy identity not known in general for stationary harmonic.
Indication of more rigid behaviour than stationary harmonic maps:

**Theorem (B. - Tian ’19)**

*If* $u$ *does not develop singularity in* $B = B_{R}^{4m} \subset \mathcal{M}$ *and* $\Sigma \cap B$ *is contained in a Lipschitz graph,*

*then*

- the bubbles at points $x \in \Sigma \cap B$ are holomorphic for a complex structure independent of $x$;
- $\Sigma \cap B$ is (pseudo)holomorphic sbmfld for a fixed almost complex structure (with $(\overline{\Sigma} \setminus \Sigma) \cap B = \emptyset$).
Thanks for your attention!