Analysis of singular sets in calibrated geometric analysis

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Special Holonomy and Calibrated Geometry

Going beyond Almgren

Regularity results for calibrated integral cycles that improve Almgren et al.

- *Complex* cycles in $\mathbb{C}^n$:
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- *Special Lagrangian* $3D$ *cones in* $\mathbb{R}^6$ [B.-Rivière '13] Smooth except possibly for finite number of half-lines.
- *Area-minimizing* $3D$ *cones* [De Lellis - Spadaro - Spolaor '16] smooth except possibly for finite number of half-lines.

Ideas from [B.-Rivière]

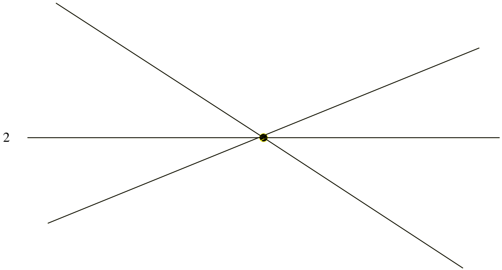
Special Lagrangian cone $\rightsquigarrow$ slice with S^5 $\rightsquigarrow$ *Special Legendrian*.

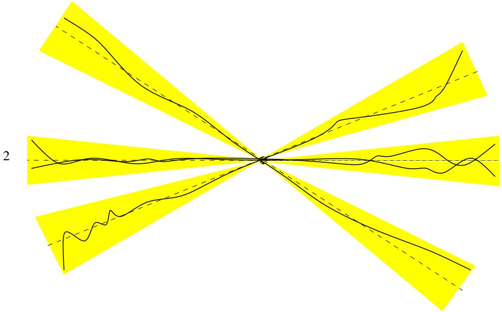
2-D integral cycle in S^5, semi-calibrated by a (non-closed) 2-form.

Study Special Legendrian cycles in S^5. Key tools to start with:

- Slicing by positive transv. foliations,
- multiple valued graphs.

Tangent cone

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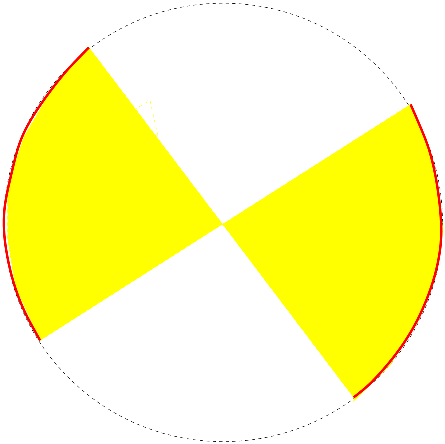
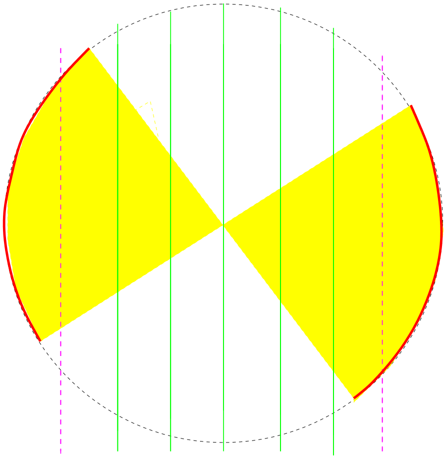
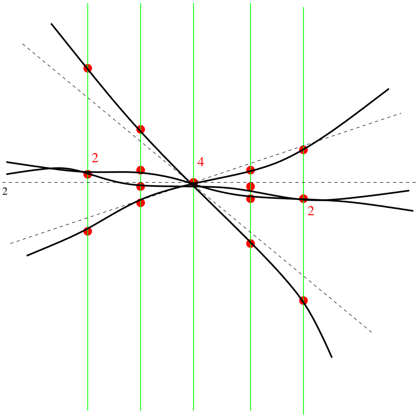


Current caught in yellow, boundary in red

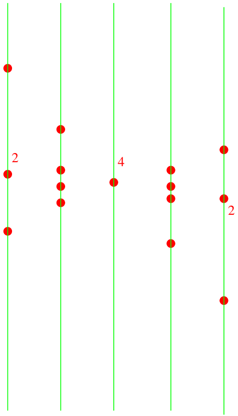
Slicing



Boundaries don't cross $\Rightarrow$ Algebraic intersection index conserved

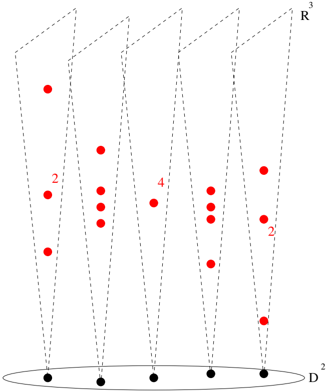


The 3-surfaces intersect the current **positively**!

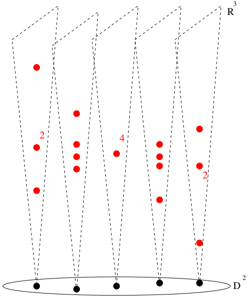


Every 3-surface meets the current in $Q = 4$ points

Current $\rightarrow$ Multiple Valued Graph



Get a Q-valued graph



unordered Q-tuples

$$D^2 \subset \mathbb{C} \quad \to \quad \frac{(\mathbb{C}\times\mathbb{R})^Q}{\sim}$$

$$z \quad \to \quad \{(\varphi_j(z), \alpha_j(z))\}_{j=1}^Q$$

Make sense of PDE for $\{(\varphi_j(z), \alpha_j(z))\}_{j=1}^{Q}$

Perturbation of Cauchy-Riemann

$$\begin{cases} \partial_{\bar{z}}\varphi_j = \nu((\varphi_j, \alpha_j), z)\ \partial_z \varphi_j + \mu((\varphi_j, \alpha_j), z) \\ \\ \nabla \alpha_j = h((\varphi_j, \alpha_j), z), \end{cases}$$

$$\nu, \mu, h \text{ small, } \mathbb{C}\text{-valued}, 0 \text{ at } 0$$

Implement **elliptic PDEs** techniques, e.g. **unique continuation**. How?

Proof by induction on Q, say $Q = 4$

Part I: prove that singularities of multiplicity 4 *cannot accumulate to* 0*.*

Prove that the average is a $W^{1,2}$ graph (needs uniqueness of tangent at 0) that also solves a perturbation of Cauchy-Riemann

Subtract the average and get a new 4-valued graph that satisfies a perturbation of Cauchy-Riemann

Now the singularities of multiplicity 4 are zeros: implement unique continuation.

Part II: prove that singularities of multiplicity ≤ 3 *cannot accumulate to* 0*.*

Within the induction (on multiplicity), at this stage you know that singularities of multiplicity ≤ 3 are countable and can only accumulate to 0.

Homological argument: from the calibrating condition, produce a notion of "positive degree" around each isolated singularity and a notion of degree bounded from below on any ball centered at 0.

Accumulation of singularities to 0 yields a contradiction.

Any hope for more general calibrated integral cycles?

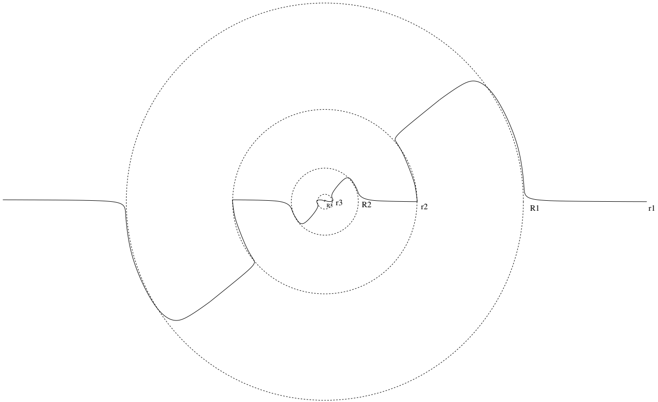
Positiveness of intersection not to be expected in general.

PDE will depend on the calibration.

"degree" argument: I don't know...

What I expect to be true (but very hard) is the uniqueness of tangent cones for calibrated integral cycles.

Uniqueness issue for the tangent space



Dilating with factors $r_1, r_2, r_3, \ldots$ yields ??
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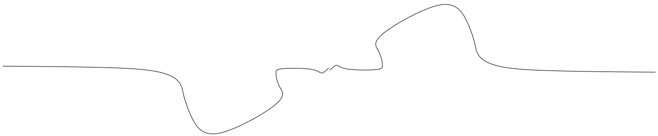

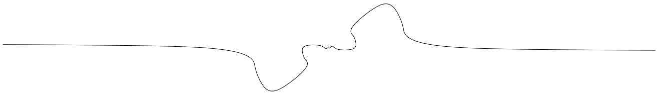

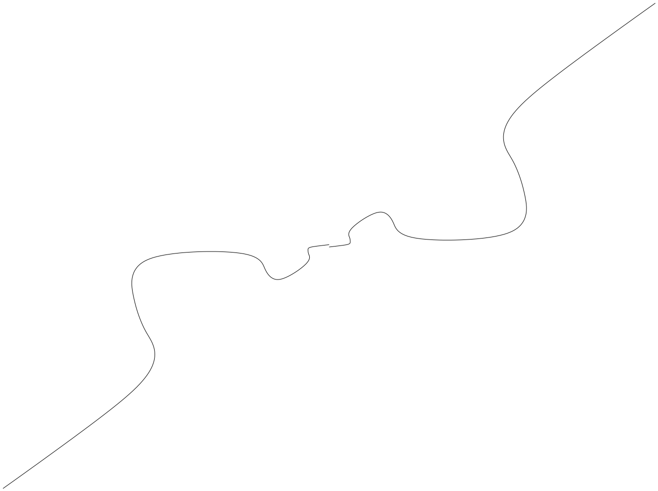

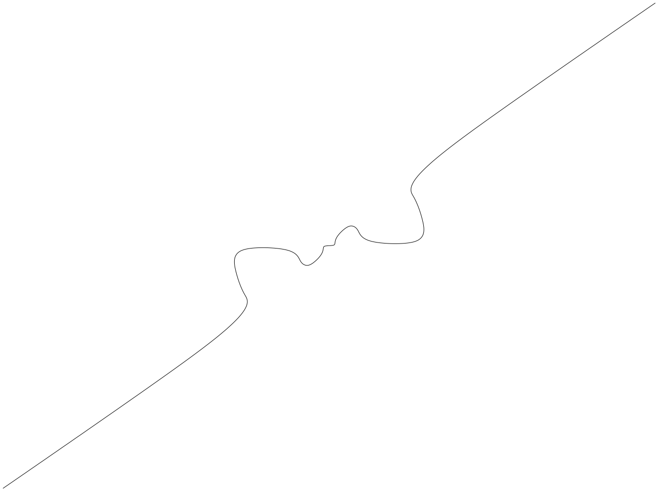

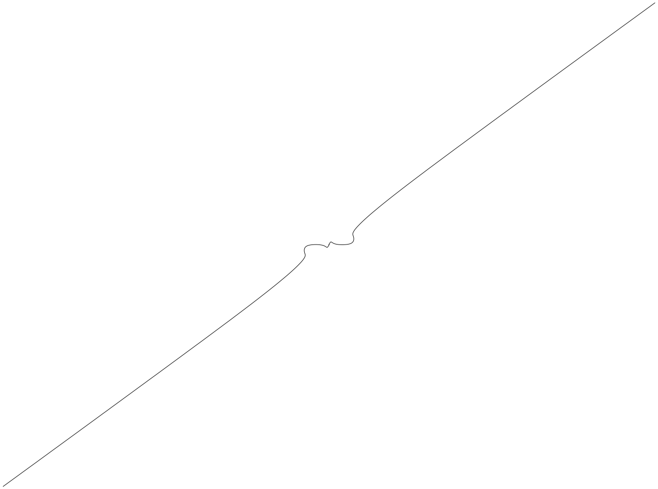

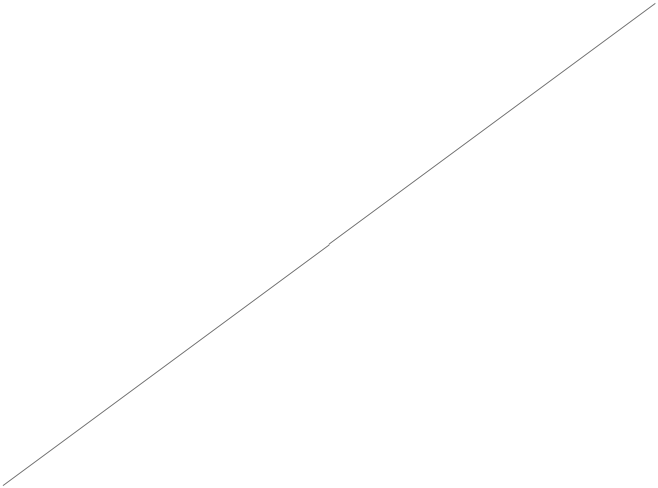

Uniqueness of tangent cones

Known by "2nd order theory"

[Allard-Almgren '76]: 1-dimensional integral currents.

[White '83]: area-minimizing 2-dim. integral currents.

[Simon '83]: mass minimizers, tangent cone with isolated sing. and multiplicity 1.

Known by "1st order theory"

[Pumberger-Rivière '10]: 2-dim. calibrated integral cycle.

ω calibration of degree 2, $\Omega = \frac{\omega^p}{p!}$ calibration of degree $2p$:
[B. '14] $2p$-dim. Ω-calibrated integral cycles.

Semi-calibration: form of comass one, not necessarily closed.

Theorem (B.)

ϕ semi-calibration of degree 2 in $(\mathcal{M}, g)$. Locally in $\mathcal{M}$ or $\mathcal{M} \times \mathbb{R}$ (whichever is even-dimensional) we can find:

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- *ω non-degenerate 2-form (possibly $d\omega \neq 0$ even if $d\phi = 0$)*
- *compatible almost complex structure J*
- *Riemannian metric $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$*

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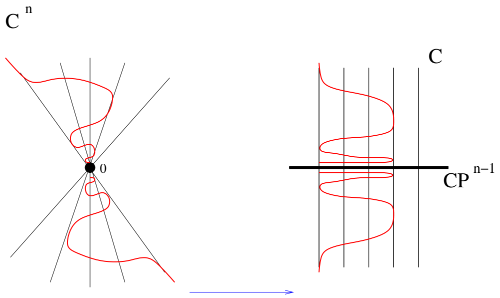
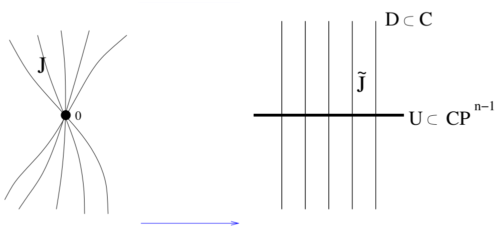
Semicalibrated by $\frac{\phi^p}{p!}$ w.r.t. $g \rightsquigarrow$ semicalibrated by $\frac{\omega^p}{p!}$ w.r.t. g_J.

Blow-up the origin of $\mathbb{C}^n$ (algebraic/symplectic geometry)



Idea of proof

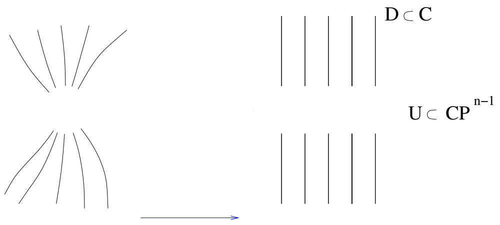
Implement a *pseudo holomorphic blow up* of a sector



$\tilde{\Omega}, \tilde{g}, \tilde{J}$ perturbations of the standard $\mathbb{CP}^{n-1} \times \mathbb{C}$.

Idea of proof

Push-forward a pseudo holomorphic current via singular map



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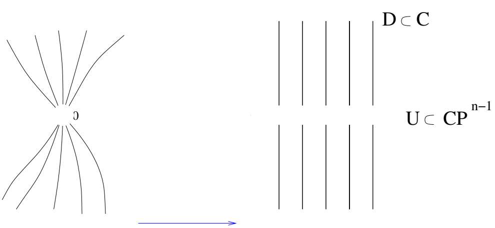
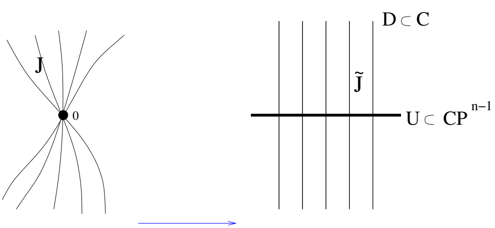


Push-forward well-defined in the limit as a $\tilde{J}$-holomorphic cycle



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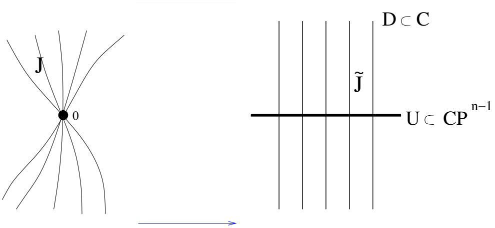

Semi-Calibrated cycle on the right!

Gauge theory on $G2$ and Spin(7)-manifolds [Walpuski '17]

Very closely related: *triholomorphic maps*

$\mathbb{H} = \mathrm{span}\{1, i, j, k\} \equiv \mathbb{R}^4$, with $i^2 = j^2 = k^2 = ijk = -1$

$$\left[\begin{array}{c} f : \mathbb{R}^4 \equiv \mathbb{H} \to \mathbb{H} \text{ satisfying} \\ \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + j \frac{\partial}{\partial x_3} + k \frac{\partial}{\partial x_4} \right) f = 0 \end{array} \right] \Rightarrow \left[\begin{array}{c} f \text{ is harmonic} \\ \Delta f = 0 \end{array} \right]$$

HyperKähler mfld : tangent model is $\mathbb{H}^m$

$$\left[\begin{array}{c} f \text{ between Compact HyperKähler mflds} \\ \text{satisfying analog. 1st order PDE} \end{array} \right] \Rightarrow \left[\begin{array}{c} f \text{ is a} \\ \text{harmonic map} \end{array} \right]$$

$u : \mathcal{M}^{4m} \to \mathcal{N}^{4n}$
i, j, k on domain, I, J, K on target (with quaternionic rule).

$u \in W^{1,2}(\mathcal{M}, \mathcal{N})$

$$du = I\, du\, i + J\, du\, j + K\, du\, k$$

$\mathcal{N}$ hyperKähler
$\mathcal{M}$ almost hyperHermitian ($\omega_i, \omega_j, \omega_k$ not closed, i, j, k not integrable).

$$d\,(u^*\Omega) = 0 \text{ if } \Omega \text{ closed 2-form.}$$

$$|\nabla u|^2 = -C_m \left(\omega_i^{4m-2} \wedge u^*\Omega_I + \omega_j^{4m-2} \wedge u^*\Omega_J + \omega_k^{4m-2} \wedge u^*\Omega_K \right)$$

u is (almost) stationary harmonic

Compactness for triholomorphic maps

$\{u_\ell\}_{\ell\in\mathbb{N}}$ triholomorphic with equibounded Dirichlet energy.

Problem: Analyse bubbles, bubbling set Σ (dim. $4m-2$), limiting map u (weakly harmonic, not known if stationary harmonic).

$|\nabla u_\ell|^2 d\mathrm{vol}_{\mathcal{M}} \rightharpoonup |\nabla u|^2 d\mathrm{vol}_{\mathcal{M}} + \Theta(x)\mathcal{H}^{4m-2} \llcorner \Sigma$

Theorem (B. - Tian '19)

Energy identity: for $\mathcal{H}^{4m-2}$-a.e. $x \in \Sigma$

$$\Theta(x) = \sum_{s=1}^{N_x} \int_{S^2} |\nabla \phi_s|^2,$$

where $\phi_s : S^2 \to \mathcal{N}$ are holomorphic bubbles.
(holomorphic for a complex structure depending on x)

"Usual $2D$-bubbling picture in $T_x\Sigma^\perp$"
Energy identity not known in general for stationary harmonic.

Indication of more rigid behaviour than stationary harmonic maps:

Theorem (B. - Tian '19)

If u does not develop singularity in $B = B_R^{4m} \subset \mathcal{M}$ and $\Sigma \cap B$ is contained in a Lipschitz graph,

then

• *the bubbles at points $x \in \Sigma \cap B$ are holomorphic for a complex structure independent of x;*
• *$\Sigma \cap B$ is (pseudo)holomorphic sbmfld for a fixed almost complex structure (with $(\overline{\Sigma} \setminus \Sigma) \cap B = \emptyset$).*

Thanks for your attention!