Analysis of singular sets in (calibrated) geometric analysis

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Special Holonomy and Calibrated Geometry

Seiberg-Witten and Gromov invariants

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$SW = Gr$ in a compact 4-D symplectic manifold with $b_2^+ > 1$.

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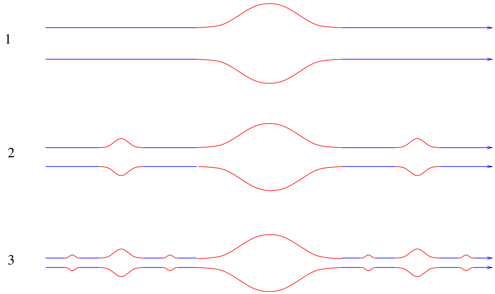
$\uparrow$ **?**

Limit of “submanifolds”



Blue part → Cantor with $\mathcal{H}^1(\text{Cantor}) = \frac{1}{2}$.

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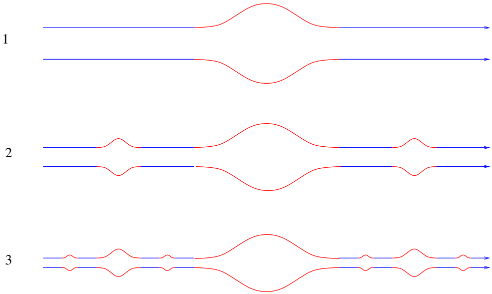
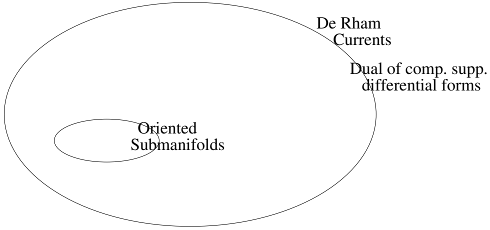


Blue part $\to$ Cantor with $\mathcal{H}^1(\text{Cantor}) = \frac{1}{2}$.

The limiting "object"

- has Cantor "counted twice" $\rightsquigarrow$ "multiplicity" θ,
- DOES NOT "look like a curve" on Cantor $\rightsquigarrow$ SINGULAR POINTS

Need for currents



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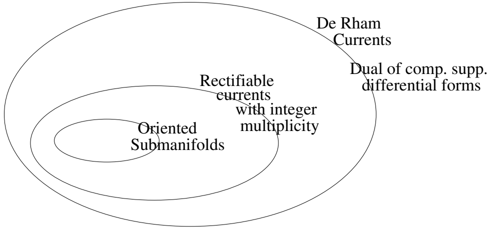


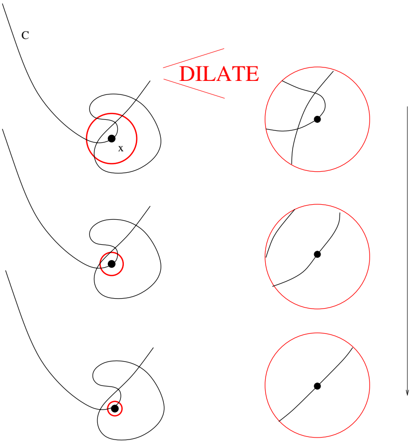
Compactness theorem from GMT: the limit is a

rectifiable current with integer multiplicity

integration on a
"oriented singular submanifold with integer multiplicity"

Rectifiable k-currents with integer multiplicity

"Singular submanifold" = Rectifiable set



limit as measures

$\mathcal{H}^k \llcorner \text{dilated}(\mathcal{C})$

$\mathcal{H}^k$-a.e. conv. $\to \mathcal{H}^k \llcorner C_x$

for a k-dim plane C_x

C_x THE **approximate tangent** at x

Boundaries of currents

C k-dim. current ($C \in$ dual space of k-forms)

Definition (boundary ∂C of C)

∂C is the $(k-1)$-dim. current

$$\partial C(\alpha) := C(d\alpha).$$

$\partial C = 0$ ⇢ **cycles**.

Integral k-cycle:
k-rectifiable set R endowed with multiplicity $\theta \in L^1_{\text{loc}}(R, \mathbb{N})$ (defines a current by integration), boundaryless in the above sense.

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Regularity Theorem [. . .] $\Rightarrow$ it is a pseudo-holomorphic curve.

Regularity Theorem

[Taubes '00]: direct proof for pseudo-holomorphic integral 2-cycles in $4D$ almost complex manifold. Singular set made of isolated points.
(also [Rivière-Tian '04])

Optimal result, e.g. $\{(z,w) : z^2 = w^3\}$.

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Re-done by [De Lellis-Spadaro '14-'16] [De Lellis-Spadaro-Spolaor '16-'18], same approach, more streamlined.

Digression: codimension-1

[De Giorgi-Federer-Fleming-Almgren-Simons '60s '70s]
Area-minimizers in codimension 1 are smooth away from a set of codimension ≥ 7.
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Key difference in codimension 1: De Giorgi and generalizations
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Fails for $\{(z, w) : \varepsilon\, z^3 = w^2\}$ in $\mathbb{R}^4$.

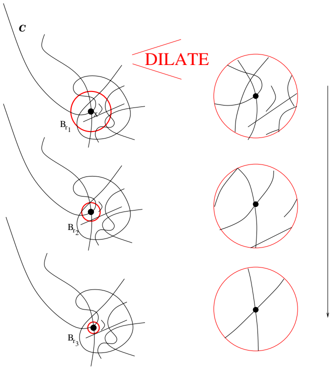
Tangents and Federer-Almgren stratification

C integral k-dim cycle. $\mathcal{H}^k$-a.e.: tangent = k-dim. plane (Q times)

For minimizers, what about other points?

Monotonicity formula $\Rightarrow$ tangent cones exist at all points and are mass-minimizers.

Tangent cone $\approx$ 1st-order model of C in a neighbourhood.



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Fundamental issue: cone may depend on sequence of rescalings.

[Federer-Almgren] $\Rightarrow$ Except for a set of dimension $k-2$, we find a tangent cone that is a plane with multiplicity.
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Codimension ≥ 2: very hard work begins.

De Giorgi's idea in a nutshell

A minimal graph $\{(x, f(x)) : x \in \Omega\}$ is close to a harmonic graph if $|Df|$ is small. Indeed the area is
$\int_\Omega \sqrt{1 + |Df|^2 + \text{squares of minors}} = \int_\Omega \left(1 + \frac{|Df|^2}{2} + O(|Df|^4)\right)$.

Harmonic functions have strong decay properties for integral norms (typically L^2).

"Approximate" the minimizing current with harmonic graph $\rightsquigarrow$ decay properties pass to the current $\rightsquigarrow$ geometric improvement of "flatness" at smaller scales $\rightsquigarrow$ $C^{1,\alpha}$ regularity and higher.

Flatness: L^2-distance to the best approximating plane.

De Giorgi's idea in high codimension

If the tangent plane has multiplicity 1 then it works still!
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Tangent plane with multiplicity ≥ 2:

$\{(z, w) : z^3 = w^2\}$: conclusion fails.
Can't approximate with a graph (nor with 2 graphs).

$\{(z, w) : z^3 = \varepsilon\, w^2\}$:
at scale 1 close to $\{z = 0\}$, tangent at 0 is $\{w = 0\}$.
Can't hope for improvement of flatness as in codimension 1.

Step 1: Q-valued Dir-minimizing functions (linear theory)

Q-valued function f:
$f : \Omega \subset \mathbb{R}^k \to \mathcal{A}_Q(\mathbb{R}^m)$
$\mathcal{A}_Q(\mathbb{R}^m)$: unordered Q-tuples of points in $\mathbb{R}^m$.

Almgren's idea: use these to approximate a k-dim. mass minimizing cycle of codimension m that admits a tangent plane with multiplicity Q.

Linear theory:
• $W^{1,2}$-framework and existence of Dirichlet-minimizers subject to suitable boundary conditions;
• regularity of Dir-minimizing Q-valued functions: smooth except for a set of dimension $\leq k-2$ (away from which f is locally a sum of Q harmonic sheets).

Optimal dimension: $w = \pm z^{3/2}$ is 2-valued Dir-minimizing.

Step 2: Q-valued Lipschitz approximation

Given k-dim. mass minimizing cycle of codimension m that admits a tangent plane with multiplicity Q at a point,

approximate it efficiently with a Q-valued Lipschitz function that is close to being Dir-minimizing.

The Q-valued Lipschitz function should *inherit* the singular set, so that a large singular set for the harmonic Q-valued graph would contradict the linear theory.

The need for "centering"

$\{(z, w) : (w - z^2)^3 = \varepsilon\, z^{1000000}\}$
is a 3-valued graph over $w = z^2$.
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Capturing this separation is crucial for regularity: allows to lower multiplicity except for the singular point under study (the proof proceeds by induction on multiplicity).

Frequency function captures the first term.

Frequency function

If f is a Q-valued harmonic function

$$I(r) := \frac{r \int_{B_r} |Df|^2}{\int_{\partial B_r} |u|^2}$$

satisfies $r \to I(r)$ non-decreasing.

Monotonicity of frequency $\Rightarrow \lim_{r \to 0} I(r) =: I(0)$ exists (finite).

$I(0)$ is, for standard harmonic functions, the degree of the first homogeneous harmonic polynomial in the Taylor expansion of f.

For Q-valued, the first term may correspond to a multiple-valued graph. A multiple-valued "tangent map".
($I(0)$ order of contact, order of vanishing)
(Homogeneity of "tangent map" follows from monotonicity)

Step 3: center manifold

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$\text{area}(\mathcal{N}) = \text{area}(\mathcal{M}) + \frac{1}{2}\int_{\mathcal{M}} |DN|^2 + Q\int_{\mathcal{M}} \vec{H} \cdot \text{average}(N) + \int_{\mathcal{M}} B(N,N) + \text{higher order terms}$

B quadratic form depending on II fund. form of $\mathcal{M}$. Flatness of current $\Rightarrow$ C^3-approximate average is small.
$\vec{H}$ mean curvature of $\mathcal{M}$.
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Leading term is $\int_{\mathcal{M}} |DN|^2$ so N is almost Dir-minimizing.

Step 3: center manifold construction

Average is constructed using the (first) Q-valued approximation over the best approximating plane.

This is done on annuli because of the lack of uniqueness of tangent plane.

Averages on annuli are smoothed and pasted together.

This gives $\mathcal{M}$.

Then produce the second Q-valued approximation N.

Step 4: final blow up argument

Show that N is almost Dir-minimizing and admits a frequency function that is almost monotone.

Produce first order term in the Taylor expansion: non-collapsed thanks to "centering".

Now implement the idea that this multiple valued map inherits the singular set, reducing to linear theory.

Chang's $2D$ result (Almgren++)

2-dimensional integral cycle, area-minimizing
[Almgren] $\Rightarrow$ singular set of dimension 0.

[Chang], [De Lellis–Spadaro-Spolaor] $\Rightarrow$ singular set finite.

Builds on Almgren's theory, producing a "branched center manifold"...

Key role played by
• uniqueness of tangent cones for $2D$ area-minimizers;
• uniqueness of tangent maps for Q-valued Dir-minimizing functions from a $2D$-domain.

Regularity problems in calibrated geometry

[Donaldson-Thomas '98] high-dimensional gauge theory
Yang-Mills fields in dim. > 4 and relations to geometric invariants.

Key aspect: understand compactness properties for sequences of smooth instantons with uniformly bounded energies. Lack of smooth convergence arises on an $(n-4)$-dim set (bubbling set).

[Tian '00]: anti-self-dual instantons. Bubbling set naturally has the structure of a calibrated integral current of dim. $n-4$ (e.g. 4-SLG in CY 4-fold).

Dream statement: the bubbling set is an $(n-4)$-cycle and its singular set is a "subvariety" of dimension $n-6$, possibly stratified.

Can we go beyond Almgren's theory when we have special structures?

Thanks for your attention!