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Zero Modes of Higgs bundles on Spin(7) manifolds

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Motivation/Goals

- Motivation: M-theory/String Theory in four & three space-time dimensions with minimal (N=1) supersymmetry [D=4 N=1& D=3 N=1]
  → compactification on G_2 & Spin(7) holonomy spaces

- Focus: gauge degrees within G_2 & Spin(7) compactifications associated w/ co-dimension four (ADE) singularities, governed by Higgs bundles

- Goal: background solutions & zero modes (charged matter)

  - Higgs bundles on G_2 localized modes w/ gauge flux ("T-branes") [Barbosa, M.C., Heckman, Lawrie, Torres, Zoccarato 1906.02212]

  - Higgs bundles on Spin(7) - geometric unification w/ other bundles [M.C., Heckman, Rochais, Torres, Zoccarato 2003.13682]

    - zero modes: classical & quantum [M.C., Heckman, Torres, Zoccarato 2107.00025]

Previous work: [Becker '00], [M.C., Gibbons, Lü, Pope '01], [Gukov, Sparks '01]…

Outline

- Overview of Higgs bundles on local CY_3, G_2 and Spin(7) spaces

  M-theory/Type IIA perspective → D=5,4,3 N=1
  [Also F-theory/IIB on elliptically fibered CY_4 → D=4 N=1]

- Spin(7) Higgs bundles and relation to G_2 and CY_4 Higgs bundles
  → geometric unification

- Zero modes of Spin(7) Higgs bundles – classical results

  Quantum results c.f., J. Heckman's talk

- Conclusions/Outlook

Overview of Higgs bundles

Within M-theory (11D) compactification: localized gauge degrees live on co-dimension four ADE singular subspace of a special holonomy spaces:

ALE fibration

$$\begin{array}{ccc} \mathbb{R}^4/\Gamma_{ADE} & \rightarrow & X_6 \\ & & \downarrow \\ & & \Sigma_2 \end{array}$$

Local Calabi-Yau threefold

2D Riemann surface Σ_2

→ D=5 N=1

$$\begin{array}{ccc} \mathbb{R}^4/\Gamma_{ADE} & \rightarrow & X_7 \\ & & \downarrow \\ & & Q_3 \end{array}$$

Local G_2 holonomy space

3D subspace Q_3.

→ D=4 N=1

$$\begin{array}{ccc} \mathbb{R}^4/\Gamma_{ADE} & \rightarrow & X_8 \\ & & \downarrow \\ & & M_4 \end{array}$$

Local Spin(7) holonomy space

4D subspace M_4

→ D=3 N=1

ALE subspaces as gauge sectors localized on D6-brane

Singular codimension four subspaces can be interpreted in Type IIA string theory as D6 branes wrapping respective singular subspaces.

[via duality: M-theory on S^1 ~ Type IIA string theory]

- Start with 10D Super Yang-Mills (SYM) action
  w/adjoint valued gauge fields A_μ & fermions ψ^α (Majorana-Weyl)

$$S = \int d^{10}x\, Tr(-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + i\psi^\alpha\gamma^\mu_{\alpha\beta}D_\mu\psi^\beta)$$

$$D_\mu = \partial_\mu + gA_\mu$$
$$F_{\mu\nu} = [D_\mu, D_\nu]$$

& supersymmetry transformations

ALE subspaces as gauge sectors localized on D6-brane

- Truncation to 7D: D6-brane worldvolume action
  w/ gauge fields A_μ, 3 adjoint rep. scalars ϕ_i (3 "transverse dims")
  & pair of fermions ψ_I (w/symplectic Majorana constraint)

$$S = \int dt\, d^6x\, \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}D_\mu\phi_a D^\mu\phi_a + \frac{1}{4}[\phi_a,\phi_b][\phi_a,\phi_b] - \frac{i}{2}\overline{\Psi}^I \not{D}\Psi_I - \frac{i}{2}\sigma^a_{IJ}\overline{\Psi}^I[\phi_a,\Psi^J]\right]$$

& supersymmetry transformations

R-symmetry
(3 transverse dims)

Global symmetry: Spin(1,9) → *Spin*(1,6)⊗*Spin*$(3)_R$

- Compactify D6-brane worldvolume action on Σ_2, Q_3, M_4 spaces → Higgs bundles

D6-brane on Σ_2, Q_3, M_4 spaces → Higgs bundles

Global symmetry & Topological twist

on Σ_2

$$Spin(1,6)\otimes Spin(3)_R \rightarrow Spin(1,4)\otimes \underbrace{Spin(2)}_{\simeq U(1)}\otimes Spin(3)_R$$

$$U(1)_{\mathrm{twist}}$$

Diagonal U(1) → twist

preserves D=5 N=1 SUSY

on Q_3

$$Spin(1,6)\otimes Spin(3)_R \rightarrow Spin(1,3)\otimes Spin(3)\otimes Spin(3)_R$$

$$SU(2)_{\mathrm{twist}}$$

Diagonal SU(2) → twist

preserves D=4 N=1 SUSY

on M_4

$$Spin(1,6)\otimes Spin(3)_R \rightarrow Spin(1,2)\otimes \underbrace{Spin(4)}_{\simeq SU(2)_l\otimes SU(2)_r}\otimes Spin(3)_R$$

$$SU(2)_{\mathrm{twist}}$$

Diagonal SU(2)→ twist

preserves D=3 N=1 SUSY

[CY threefold; D=5 N=1]

Compactification on Σ_2

- Higgs bundle: 2D SUSY gauge theory on Riemann surface Σ_2
  w/ one-form Higgs field ϕ_{Hit} & vector bundle V connection A
  (in adjoint reps. of ADE groups)

N. Hitchin, Lon. Math. Soc. 55 (1987)

- Hitchin equations:
  (BPS eqs.)

$$\overline{\partial}_A \phi_{\text{Hit}} = 0$$

$$F_A + \frac{i}{2}[\phi^\dagger_{\text{Hit}}, \phi_{\text{Hit}}] = 0$$

Unitary frame [$\dagger$ - hermitian conjugate]

Solutions specified modulo gauge transformations

- ϕ_{Hit} characterize local deformations in the shape of the ambient Calabi-Yau threefold
  A parameterizes gauge field deformations (flux), sourced by D-brane

Hitchin System

[G_2; D=4 N=1]

Compactification on Q_3

[Pantev, Wijnholt '09] [Acharya] [Barbosa '19]

- Higgs bundle: 3D gauge SUSY gauge theory on Q_3
  adjoint valued one-form Higgs field ϕ, vector bundle connection A
  (after twisting)

- BPS equations: $D_A\phi = 0 \qquad D_A \star \phi = 0$
  $F = [\phi, \phi]$

Complexified connection: $\mathscr{A} = A + i\phi \quad \rightarrow$

BPS equations: $\mathscr{F} = 0 \qquad D_A \star \phi = 0$

[$D_A\phi = 0$ redundant]

Pantev Wijnholt (PW) System

Compactification on Q_3

[Pantev, Wijnholt '09] [Acharya] [Barbosa '19]

- Chiral matter localized at the center of solitonic configuration of ϕ, with vanishing value there (co-dimension seven singularities).

[Braun, Cizel, Hübner, Schäfer-Nameki '18]
[Hübner'20]

- Subsequent studies developed and extended to co-dimension six singularities (non-chiral matter). Appealing feature: they could possibly connect to building compact G_2 manifolds via twisted connected sums (TCS).

[Kovalev '03] [Corti, Haskins, Nordström, Pacini '13-'15]
[Braun, Schäfer-Nameki '18]

- Most prior analyses of localized matter have assumed one-form ϕ is diagonal and no *A* in Q_3 ➡

[Barbosa, M.C., Heckman, Lawrie, Torres, Zoccarato '19]
c.f., M.C.'s talk at the Santa Barbara Collaboration Meeting '19

PW system generalized non-Abelian case & to include
non-diagonal one-form ϕ & non-zero flux *A*

$$\Phi \sim \begin{bmatrix} \lambda & 1 & 0 \\ z & \lambda & 0 \\ 0 & 0 & -2\lambda \end{bmatrix}$$

- One-form ϕ components will not commute
  - ``T-brane type configuration''
  (fits in the broader scheme of T-brane like phenomena)

- Local model: three-manifold Q_3 as a Riemann surface Σ_2 fibered over an interval S^1: the gauge theory on Σ_2 is *a mild deformation* of a Hitchin system on Σ_2

- As Hitchin system on Σ_2 describes a local Calabi-Yau geometry → obtain a local deformation of a TCS-like construction → Interpreted as building up a local G_2 background

- Localized modes (in co-dim seven)
  → explicit solutions; their existence reduces to a linear algebraic problem
  [à la localized zero modes of T-branes F-theory on Calabi-Yau fourfolds]

[Cecotti, Cheng, Heckman, Vafa'09]
[Cecotti, Cordova, Heckman, Vafa'10]

[Spin(7); D=3 N=1]

Compactification on M_4

[Vafa, Witten '94]
[Heckman, Lawrie, Ling, Zoccarato '18]
[M.C., Heckman, Rochais, Torres, Zoccarato '20]

- Higgs bundle: 4D gauge theory on M_4
  Higgs field ϕ - an adjoint valued self-dual two-form &
  gauge field strength F_{SD} - self-dual part of the bundle curvature

- BPS equations: $D_A \phi = 0$

$$F_{SD} + \phi \times \phi = 0$$

Cross product: $(\phi \times \phi)_{ij} = \frac{1}{4}[\phi_{ik}, \phi_{jl}] g^{kl}$

Equations those of N=4 SYM on non-Kähler M_4 w/ topological twist

[Vafa, Witten '94]

Vafa-Witten (VW) System

VW system includes other Higgs bundles as solutions

[M.C., Heckman, Rochais, Torres,Zoccarato ’20]
c.f., G. Zoccarato’s talk at the virtual Collaboration Meeting 20

Geometric unification of Higgs bundle vacua

- PW system related to VW system:

Consider VW system on $M_4 = Q_3 \times S^1$ S^1 parameterized by t

Write the SD forms as: $\phi = \hat{\phi} \wedge dt + \star_3 \hat{\phi}$ $\hat{\phi} \in \Omega^1(Q_3)$

- BPS equations: $F - [\hat{\phi}, \hat{\phi}] + \underline{\star_3 (D_t A - d_3 A_t)} = 0$ $D_A \hat{\phi} + \underline{\star_3 D_t \hat{\phi}} = 0$

$D_A \star_3 \hat{\phi} = 0$

- One recovers PW w/ $A_t = \partial_t A = \partial_t \hat{\phi} = 0$

PW is the dimensional reduction of VW along S^1

Digression: another way to obtain D=4 N=1

Within F-theory (12D) compactification: localized gauge degrees live on co-dimension four ADE singular subspace of elliptically fibered Calabi-Yau fourfold

$$\begin{array}{ccc} \mathbb{R}^4/\Gamma_{ADE} \rightarrow & X_8 & \text{Local elliptically fibered Calabi-Yau fourfold} \\ & \downarrow & \\ & S_4 & \text{Kähler 4D subspace} \end{array}$$

D=4 N=1

- Singular co-dim 4 subspace S_4 can be interpreted as D7 brane wrapping S_4 [duality: F-theory on T^2 → Type IIB string theory]

→ 8D supersymmetric gauge theory of D7-brane worldvolume action, compactified on S_4

⬇

[Beasley, Heckman, Vafa '08]

Beasley Heckman Vafa (BHV) system

[Elliptically fibered CY fourfold; D=4 N=1]

BHV system

[Beasley, Heckman, Vafa '08]

- Higgs bundle on a Kähler 4D manifold S_4

  Higgs field is a complex adjoint valued (2,0)-form (after twisting)

Kähler form

- BPS equations:

$$\bar{\partial}_A \phi = 0 \qquad J \wedge F + \frac{i}{2}[\phi^\dagger, \phi] = 0$$

$$F_{(0,2)} = 0$$

Holomorphic structure makes construction of solutions feasible
(even non-Abelian ones)

Geometric unification of Higgs bundle bacua

- BHV system is also related to VW system

-Take VW system with $M_4 \sim S_4$ to be Kähler manifold
- SD two-forms admit a Hodge decomposition

$$\Omega^2_+(S_4) \simeq H^{(2,0)}(S_4) \oplus H^{(0,2)}(S_4) \oplus H^{(1,1)}_{n.p.}(S_4)$$

non-primitive

- VW equations become:

$$F_{(0,2)} - \underline{\frac{i}{2}\phi_{(1,1)} \times \phi^\dagger_{(0,2)}} = 0 \qquad \bar{\partial}_A \phi_{(2,0)} - \underline{\frac{i}{2}\partial_A \phi_{(1,1)}} = 0$$

$$J \wedge F = \frac{i}{2}[\phi_{(2,0)}, \phi^\dagger_{(0,2)}]$$

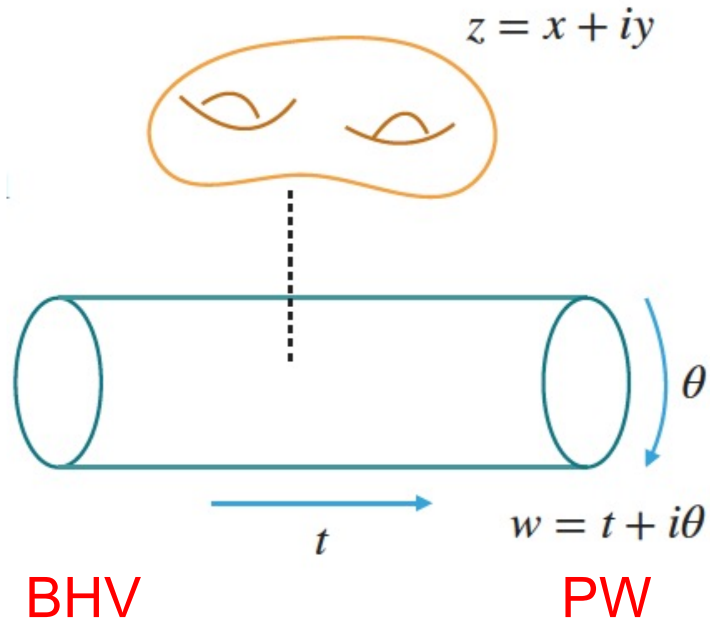
BHV is recovered by setting (1,1)-component set to zero

Geometric unification of Higgs bundle vacua

$$BHV \subset \; VW \; \supset PW$$

- PW to BHV interpolation:

Take M_4 manifold: $$M_4 = \underset{(x,y)}{\Sigma_2} \times \underset{t}{\mathbb{R}} \times \underset{\theta}{S^1}$$



$$|\phi_{(1,1)}| \sim e^{\lambda t} \qquad t \to -\infty$$

$$|\partial_\theta \psi| \sim e^{-\lambda_2 t} \qquad |A_\theta| \sim e^{-\lambda_1 t} \qquad t \to +\infty$$

- For t → - ∞ solution approaches BHV
- For t → +∞ solution approaches PW
- In the middle there is a Spin(7) solution
  (Explicit construction of the full solution)

Higgs bundle version of TCS construction of Spin(7) manifolds

[Braun, Schäfer-Nameki ’18]

- Also examples w/ PW to PW interpolation: $M_4 = Q_3 \times \mathbb{R} \qquad Q_3 = \mathbf{T}^3$

[M.C., Heckman, Torres, Zoccarato ’21]

Zero Modes of VW system – classical

- Zero modes equations – infinitesimal variation around a solution:

$$A = \langle A \rangle + a \qquad \phi = \langle \phi \rangle + \varphi$$

- Zero mode equations for VW:
  [For simplicity: abelian, without flux <A>=0]

$$D_A a + \star D_A a + \phi \times \varphi = 0$$
$$D_A \varphi - [\phi, a] = 0$$

drop < > for < ϕ >

Solutions identified via inf. gauge transformations: $a \simeq a + D_A \xi$
$\varphi \sim \varphi + [\phi, \xi]$

- Where are zero modes localized?
  [loci where ϕ=0 (due to $\phi \to t\,\phi$ rescaling)]

Can we localize zero modes on M_4?

As *BHV⊂ VW ⊃ PW,* one expects that localized VW zero modes will share some similarities with PW and/or BHV ones.

- Comparison to PW: three scalars, generically localize modes at co-dim three subspace of Q_3, i.e. a point.
  VW - three independent components of SD scalars, to localize modes in co-dim three in M_4, i.e. a line
- Comparison to BHV: on a Kähler surface S_4, w/ one (2,0)-form, w/ zeroes localized in complex co-dim one subspace. Gauge flux further localizes to complex co-dim two in S_4, i.e. a point.
  VW – consider a special case of BHV background, to localize in real codimension four on M_4, i.e. a point
- Full-fledged VW: zero modes generically lie in codimension three in M_4 , i.e. a line
  [Pull-back of flux on line does not produce further localization]
  Supported by explicit analysis in a local patch $M_4 = \mathbb{R}^4$
  [Clash of PW and BHV complex structures]

Quantum corrections could lift zero modes

[M.C., Heckman, Torres, Zoccarato ’21]

In spite of lack of the power of holomorphicity (unlike D=4 N=1), there is certain quantum protection in D=3 N=1, due to

- Discrete Symmetry Anomalies
  involving space-time reflection symmetries, inherited from 7D SYM action (D6-brane worldvolume) reduced on M_4

- Examples of gluing local 4D patches into compact M_4
  resulting in D=3 N=1 & counting of quantum protected zero modes

c.f., J. Heckman’s talk

Conclusions

- Higgs bundles in Spin(7) compactifications of M-theory
- This system provides a unification of other known Higgs bundles (contains PW and BHV bundles as special cases)
- Build solutions interpolating between BHV and PW & interfaces between PW solutions
- Localization of zero modes: generically localized at most on codimension three in M_4

Classical results $\rightarrow$ quantum ones   c.f., J. Heckman's talk

Outlook

- Most of the analysis on (ultra-)local patches
  → full-fledged global analysis:
  - Higgs bundles on compact M_4
    Embedding in non-compact Spin(7)
  - Embedding in compact Spin(7) [D=3 N=1 w/ gravity]

Progress   c.f., J. Heckman's talk

- Toward geometric classification of D=3 N=1 dualities

- Type IIA on Spin(7)   D=2 N=(1,1)
  Type IIB on Spin(7)   D=2 N= (0,2) – holomorphicity?

Thank you!