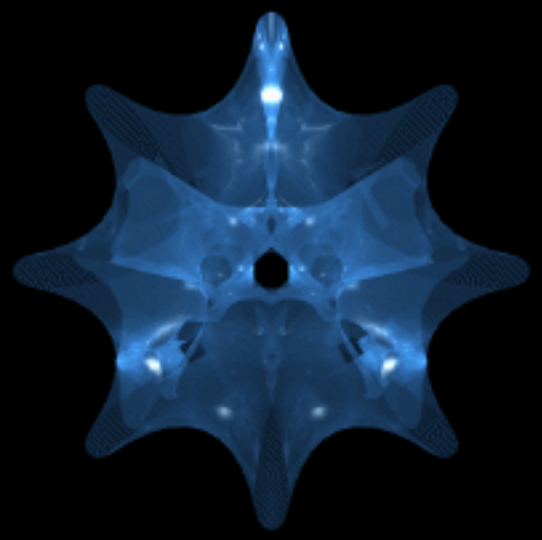
K3 metrics from little string theory



Shamit Kachru (Stanford)

Based on work with Arnav Tripathy and Max Zimet…

arXiv:1810.10540

You will see that the work enjoys connections to earlier or naively distinct approaches:

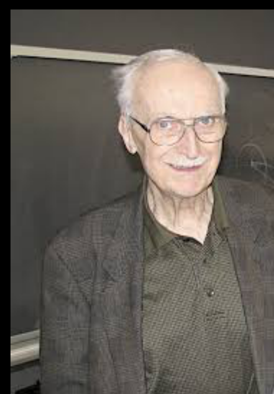
Greene, Shapere, Vafa, Yau
Nucl. Phys. B337 (1990) 1.

Gross, Wilson
J. Diff. Geom. 55 (2000) 475.

+

an ongoing program of research by
Yu-Shen Lin

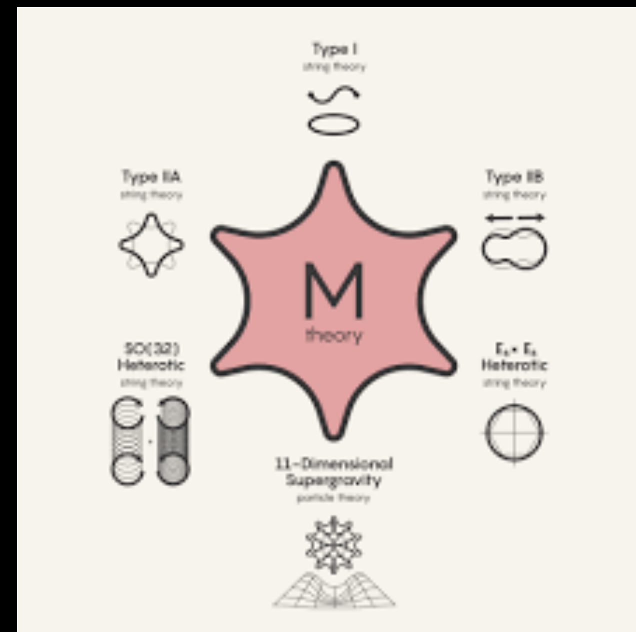
1. Introduction



Calabi conjecture: There is a unique Ricci flat Kahler metric on a Kahler manifold of vanishing first Chern class for each choice of the Kahler form.

Proved by Yau.

The result is famously non-constructive.



String theorists are interested because these spaces solve the vacuum Einstein equations relevant in superstring compactification.

Concretely one studies, for instance, a non-linear sigma model with target a Calabi-Yau M.

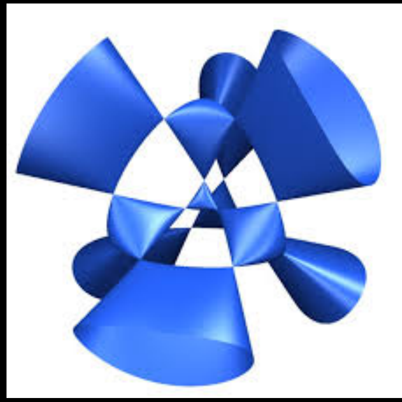
$$S = \int d^2z \; g_{ij}(X)\partial X^i \bar{\partial} X^j + \cdots$$

Then the most basic questions about observables in string compactification on M are decided by facts about the geometry of M.

(Strictly this is only true in a large-volume expansion, but the case we'll study is a bit better.)

Upshot: It would be nice to know the Ricci flat metric on M.

Today, I'll describe a conceptually new approach to this problem; its relations to other ideas; and the kinds of formulas it leads to.



Our approach applies to metrics on the simplest compact CY manifold, K3.

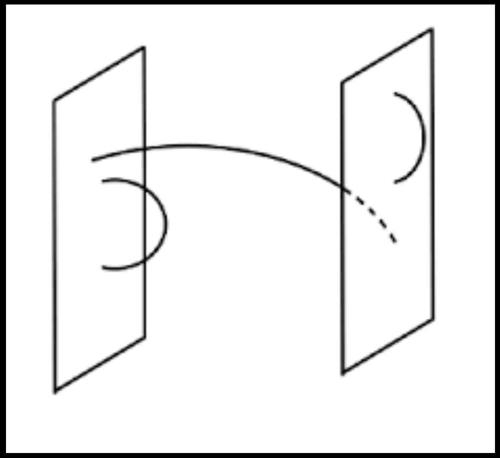
Quick summary: One can determine an exact K3 metric given knowledge of the spectrum of BPS states of an auxiliary supersymmetric theory.

II. Little string theories and their moduli spaces

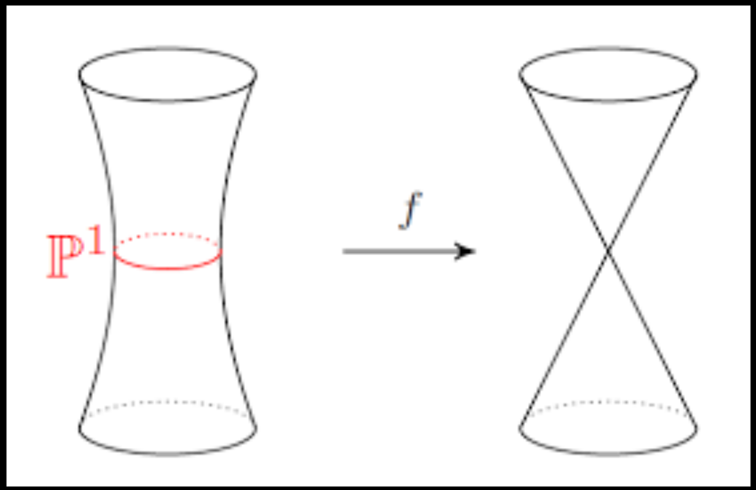
A. Simple new theories without gravity

One of the discoveries of the mid 1990s was a zoo of novel theories not previously expected to exist.

Tools of discovery:

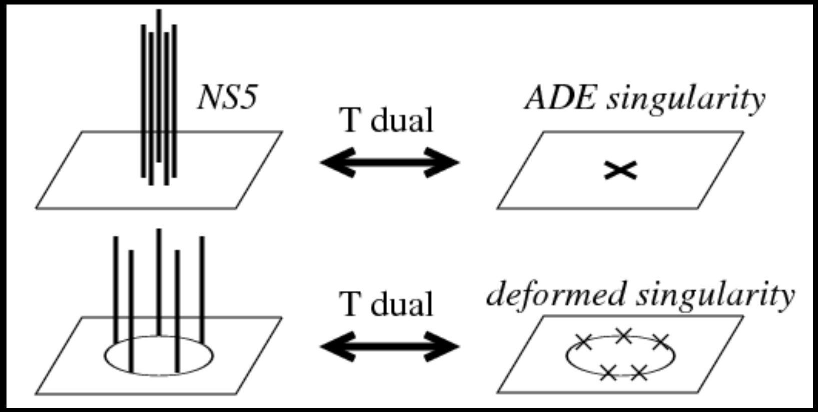


Study the low-energy theories on p-branes embedded in the 10d string theory.



Study a singular Calabi-Yau and focus on "local physics" at the singularity.

A simple Calabi-Yau singularity is an A-D-E singularity of a K3 surface.



It has a dual description in terms of so-called "NS 5 branes."

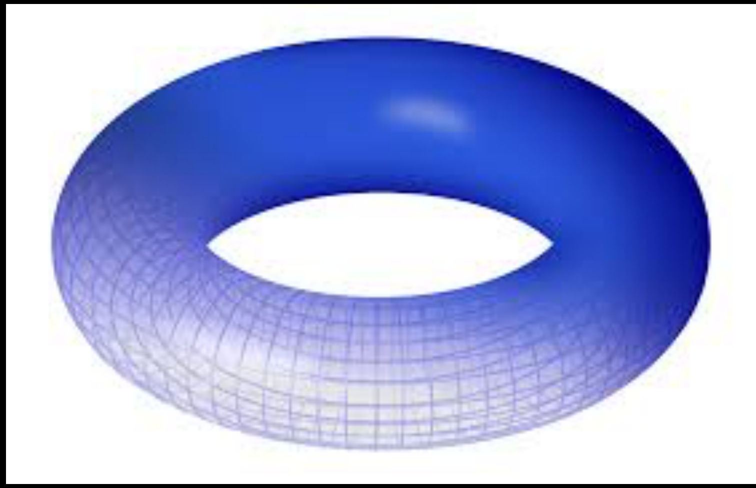
By carefully studying the physics of the singular K3 or on the branes, one finds several new theories.

— Famously, there is a (2,0) supersymmetric field theory in 5+1 dimensions. It can be used to explain S-duality.

— Slightly less famously, there is also a theory which reduces this to low energies but is smaller than the full string theory. It is a so-called "little string theory" living on the NS 5-branes.

The latter is not a quantum field theory, and comes with a scale: the scale of its strings.

The simplest (2,0) NS 5 brane theory has many interesting properties.



— Compactified on a 2-torus, it gives rise to a theory which looks - at low energy - like a U(1) gauge theory.

— The (Coulomb branch) moduli space of this theory is the (dual of the) mirror torus.

Compactifying on an additional circle, one gets two additional scalars:

— The abelian gauge field present on the Coulomb branch gives a Wilson line on the circle, which is periodic.

— This leaves a 3d photon. But in 3d, via

$$f_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho \phi$$

one can dualize this to a scalar.

From the M theory picture, you see this theory has a moduli space given by a four-torus.

We review this because we will use an analogous, but more complicated, system where there is a little string theory whose moduli space is K3.

B. A theory with K3 as its moduli space

The theory we were talking about has 16 supercharges, or the equivalent of 4d N=4 supersymmetry.

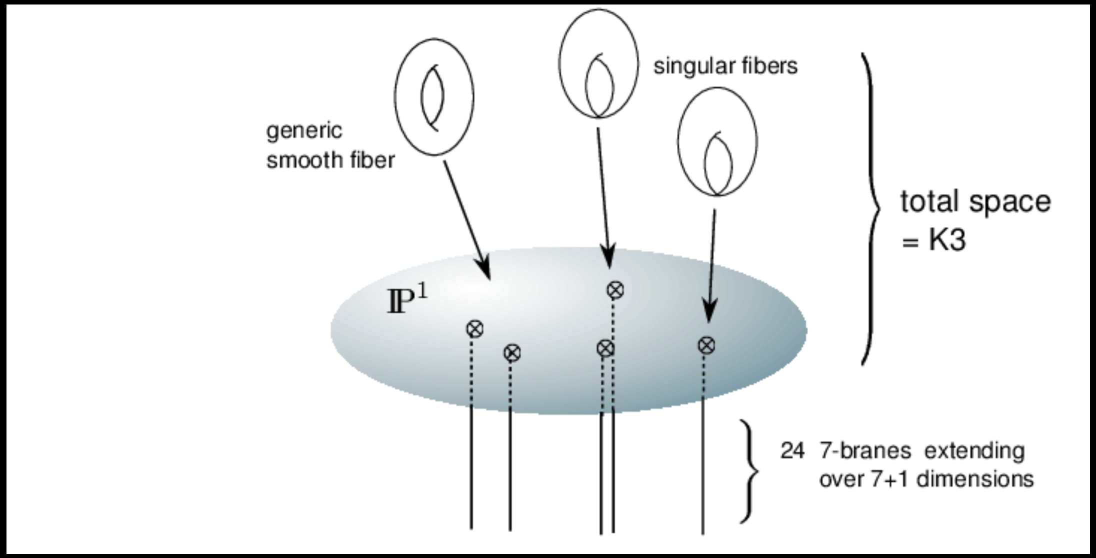
The one we're interested in is a cousin — the first one discovered — that has 4d N=2 supersymmetry.

We will review how to "geometrically engineer" it later.
The salient properties are:

— it arises on NS 5-branes in the heterotic string, or on D5-branes in the type I string.

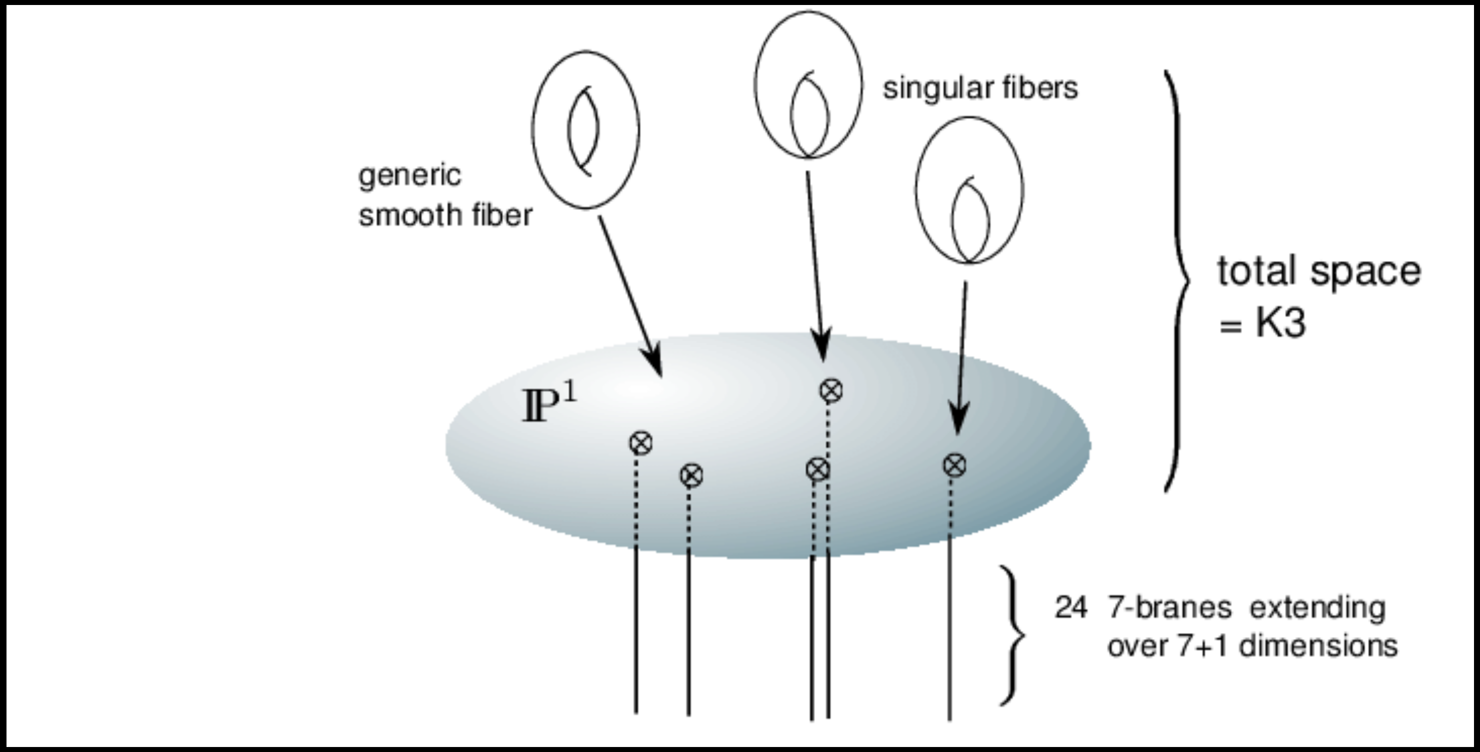
— one description: 6d SU(2) gauge theory with 16 doublet hypermultiplets (SO(32) global symmetry).

— when compactified on a two-torus, it is (at low energies) a rank 1 N=2 field theory; but its Coulomb branch is a sphere.



This sphere has 24 marked points on it; in a dual picture, think of D3-branes moving on base of elliptic K3.

Now, when one compactifies this theory on an additional circle, one gets a 3d theory whose moduli space is a K3 surface!



The Wilson line of the gauge field, together with the 3d dual photon, give the elliptic fiber above each point on the base sphere.

III. Hyperkahler metrics from N=2 theories

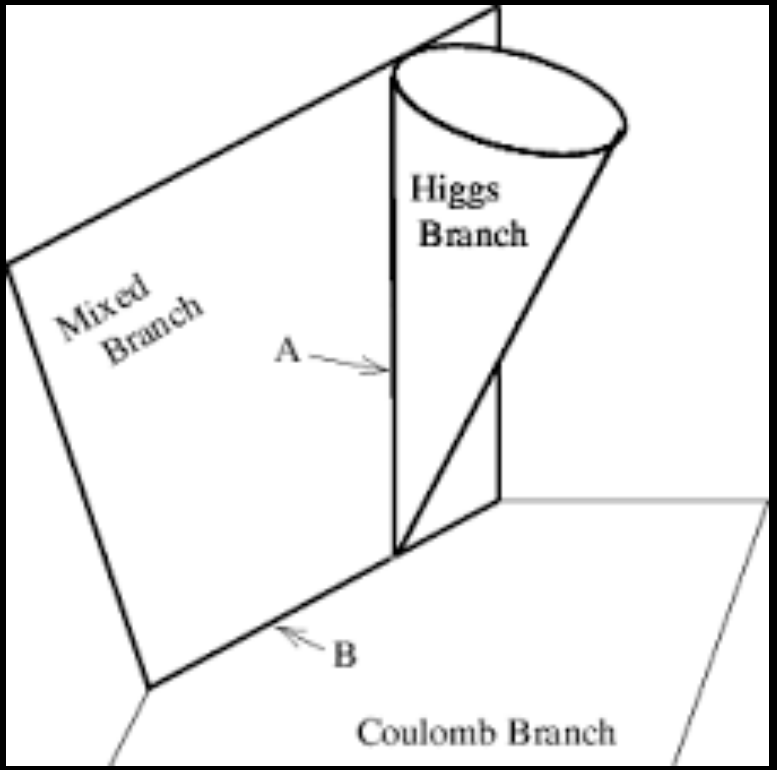
Let us leave this story for a moment, and discuss some generalities about 4d N=2 field theories.

Multiplets:

Vector: $\phi, \psi, \lambda, A_\mu$

Hyper: $Q, \tilde{Q}, \psi_Q, \tilde{\psi}_{\tilde{Q}}$

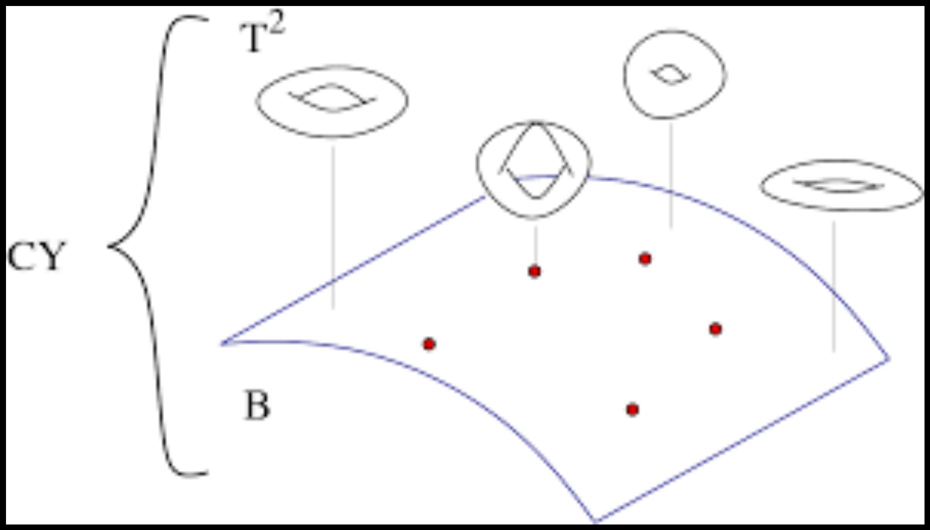
Moduli space of vacua:



Coulomb branch: special Kahler
Higgs branch: hyperKahler

On compactification to 3d, the Coulomb branch becomes hyperKahler too.

e.g. for a one-dimensional (complex) 4d Coulomb branch:



* fibers arise from Wilson line and dual gauge field in 3d

* For F-theory D3 probe, fibers degenerate at D7 branes due to "light electrons"

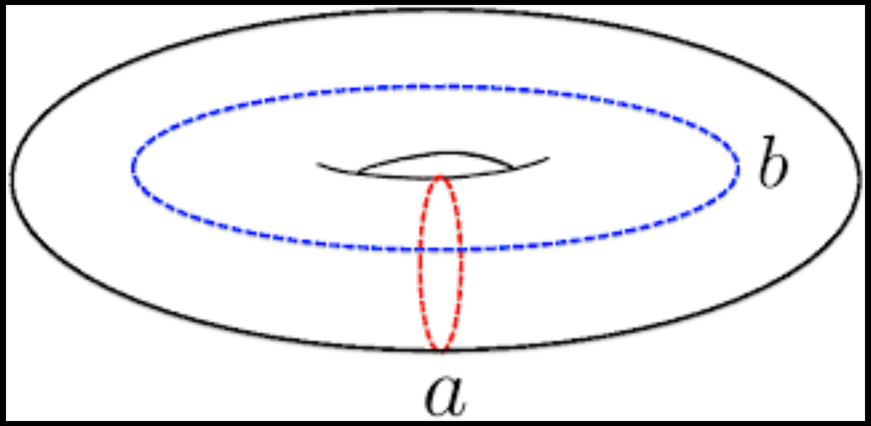
How does one find the metric on the 4d and 3d Coulomb branch?

A. The 4d story

Calling the Coulomb branch modulus “a,” for each point on the “a”-plane one has an auxiliary structure: a Seiberg-Witten curve.

In our case of interest, it is of genus 1.

The periods of a preferred meromorphic differential on this curve then determine the geometry:



$$\int_a \lambda = a$$

$$\int_b \lambda = \frac{\partial \mathcal{F}}{\partial a}$$

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

The central object is the prepotential F.

In terms of F, the Kahler potential is:

$$K = \mathrm{Im}\left(\frac{\partial F}{\partial a}\bar{a}\right)$$

All states satisfy a BPS bound:

$$M \geq |Z(a; q_e, q_m)|$$

States which saturate the bound are in “short” multiplets and are exactly stable.

B. The 3d story

Our 4d theory has an action:

$$L = \int d^4x \left(\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{i\theta}{32\pi^2} F_{\mu\nu} \widetilde{F}^{\mu\nu} \right)$$

At a given point in moduli space, there is an auxiliary torus in the story with

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{e^2}.$$

Reducing to 3d, the torus becomes physical.

The Wilson line for the gauge field and the dualized 3d gauge field have action

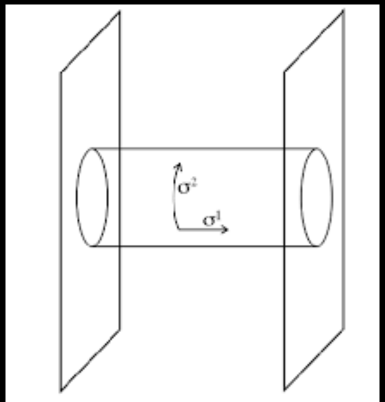
$$\widetilde{L} = \int d^3x \left(\frac{1}{\pi R e^2} |db|^2 + \frac{e^2}{\pi R (8\pi)^2} \left| d\sigma - \frac{\theta}{\pi} db \right|^2 \right)$$

This gives an elliptic fiber E over the old Coulomb branch, with area controlled by

$$V_E(R) = \frac{1}{16\pi R}.$$

The leading large R metric is then given by combining the kinetic terms for the new scalars with those of the 4d Coulomb branch fields.

There are three sources of physical corrections to a naive classical picture:



— one loop corrections in 4d.

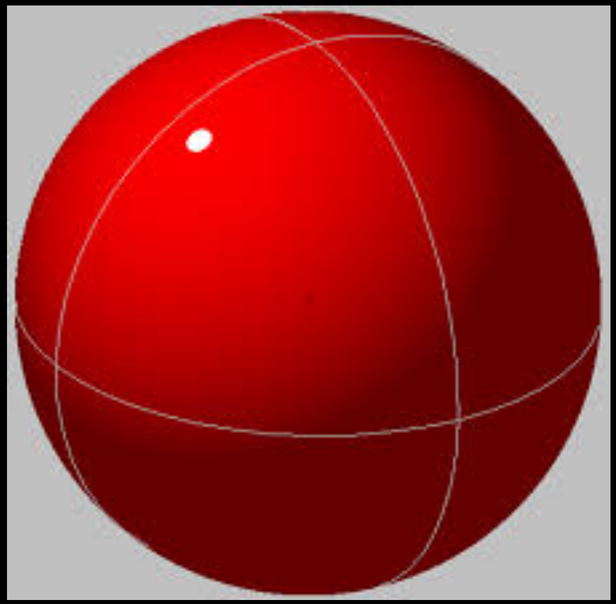
— corrections from 4d instantons.

— instanton corrections from BPS particles in the 4d theory running around the circle.

Gaiotto-Moore-Neitzke have developed a beautiful story for how to compute the hyperKahler metric on a Coulomb branch given the indices defined in the presence of certain massive probe particles:

$$X_\gamma(u,\theta;\zeta) = \mathrm{Tr}(-1)^F e^{-2\pi R H} e^{i\theta_Q}$$

The trace is taken in the presence of a massive probe particle of charge γ .



Here ζ parametrizes the choice of complex structure on the hyperKahler Coulomb branch; there is a sphere of such possible choices.

Now, define the triplet of two forms

$$\varpi(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_- \ ,$$

$$\omega_\pm = \omega_1 \pm i\omega_2 \ .$$

“holomorphic symplectic form”

The definition uses the three Kahler forms at the north, front, and right poles of the sphere.

It follows from standard results in hyperKahler geometry that

$$g = \omega_3 J_3 = -\omega_3 \omega_1^{-1} \omega_2 \ .$$

And it is a result of GMN that

$$\varpi(\zeta) = \frac{1}{8\pi^2 R} \epsilon_{ij} d'\mathcal{Y}^i(\zeta) \wedge d'\mathcal{Y}^j(\zeta) \ ,$$

Darboux coordinates

where

$$\mathcal{Y}_\gamma = \log \mathcal{X}_\gamma \ .$$

Upshot: if we can compute the X_γ for a basis of the charge lattice, we are in business.

Now lets see how this all fits together in some formulae.

IV. Toy formulae

A. The large radius limit

We should expect from the physics that at large radius of the circle, our formulae simplify. (This is a semiclassical limit from the 3d point of view.)

Recall that to compute a line operator index in charge sector γ, we need to evaluate

$$\mathrm{Tr}_{\mathcal{H}_{u;\gamma,\zeta}}(-1)^F e^{-2\pi R H} e^{i\theta Q} \ .$$

In the large radius limit, semiclassical reasoning gives:

$$-2\pi R H = \frac{\pi R}{\zeta} Z_\gamma + \pi R \zeta \overline{Z_\gamma} \ ,$$

For the problem at hand, using the 4d formulae for the Coulomb branch geometry gives the known “semi-flat” metric.

That is, there are 24 points with a light electron (in some duality frame), and one finds:

$$R\,ds^2 = \left[e^{\phi} - \frac{(z-\bar{z})^2 \partial\tau \bar{\partial}\bar{\tau}}{4\tau_2^3}\right] du d\bar{u} + \frac{dz d\bar{z}}{\tau_2} - \left[\frac{(z-\bar{z})\bar{\partial}\bar{\tau}}{2i\tau_2^2} dz d\bar{u} + \text{h.c.}\right] \ .$$

Here:

Greene, Shapere, Vafa, Yau

$$e^{\phi} = \tau_2 \left| \eta^2 \prod_{a=1}^{24} (u - u_a)^{-1/12} \right|^2$$

u are heuristically coordinates on the 4d Coulomb branch, and z coordinates on the elliptic fibers:

$$z \sim z + 1 \sim z + \tau$$

B. Including the instantons

To get a more complete answer, one needs to include the instantons.

We can only state the answers in terms of some as yet unknown BPS numbers. They are morally more "topological" information than a metric.

Heuristically, one expects:

$$\mathcal{X}_\gamma = \mathcal{X}_\gamma^{\mathrm{sf}}(1 + \mathcal{O}(e^{-const \cdot R}))$$

Formally, the Xs solve a Riemann-Hilbert problem written down by GMN. One can show that they solve an integral equation

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{\rm sf}_\gamma(\zeta)\exp\left[-\frac{1}{4\pi i}\sum_{\gamma'\in\Gamma_u}\Omega(\gamma';u)\,\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}(u)}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_{\gamma'}(\zeta')\right)\right]$$

where

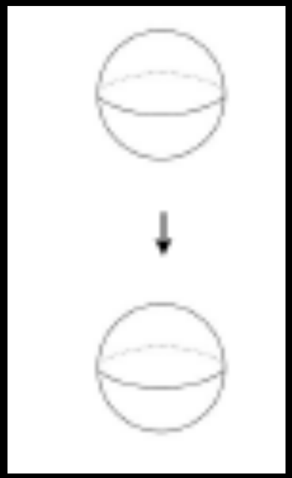
$$\Omega(\gamma;u) = \text{BPS count in charge sector } \gamma$$

This can be solved by an iterative technique, given the BPS counts.

Something rather similar to the Gross-Wilson approximation to the metric will emerge if one works on a smooth elliptic K3, and includes only the instanton corrections to the U(1) gauge theory at each “electron point.”

The most important question is: how do we determine the BPS invariants?

There are models of the little string in question that suggest how to attack this question. For instance, one can “geometrically engineer” it in F-theory on a certain Calabi-Yau threefold.

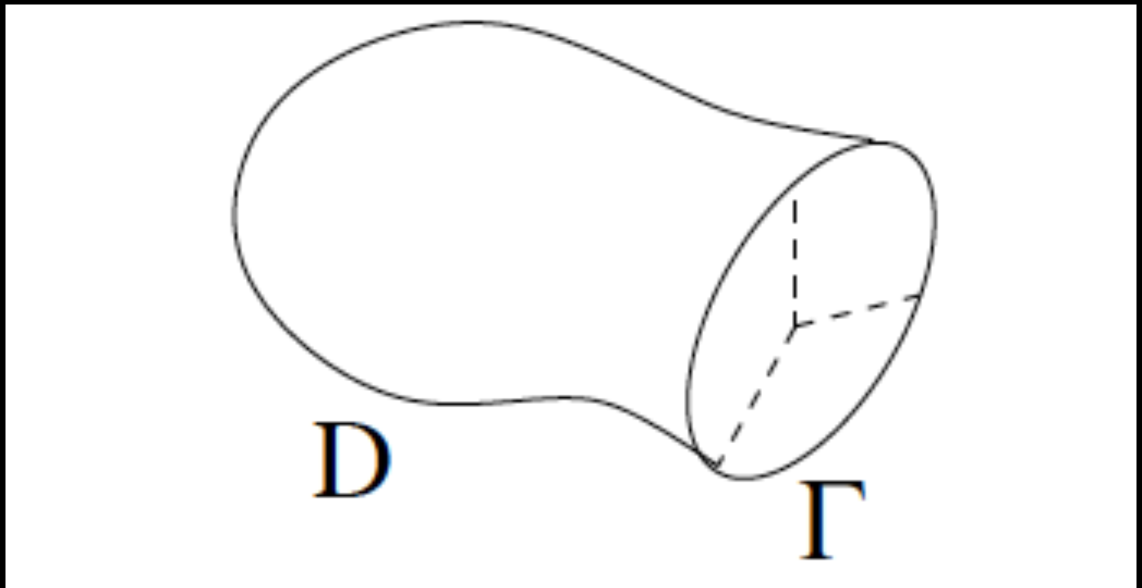


For instance, we can consider an elliptic fibration over $\mathbb{P}^1 \times \mathbb{P}^1$

Putting an A_1 singularity at a point on one of the spheres, and a D_{16} at a point on the other, we engineer the 6d low-energy theory we started with.

Going down to IIA string theory gives us the 4d N=2 theory whose BPS spectrum we need. Computing the BPS invariants of this kind of theory is a subject of intense research. The subset we need may be within reach.

Another approach could proceed via IIA duality frame:
D4-brane wrapping elliptic fibers of an elliptic K3.
(This relates to open Gromov-Witten theory.)



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The instantons become disc instantons with boundary on a 1-cycle of the elliptic curve the D4 wraps, and ending on a degenerate fiber.