Superconformal algebras for twisted connected sums and G_2 mirror symmetry

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Marc-Antoine Fiset

University of Oxford

KITP, Santa Barbara, 12 April 2019

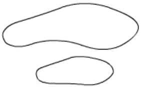
Introduction

$$(\mathcal{M}^d, g)$$

$\mathcal{M}^d \times \mathbb{R}^{1,9-d}$

Not M-theory!

Introduction



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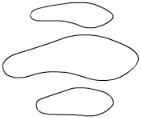

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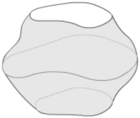

2-dim Σ^2 (complex curve; coords $z, \bar{z}$)

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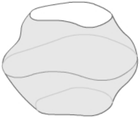

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$$\sigma : \Sigma^2 \to \mathcal{M}^d$$

In coords: $x^i(z, \bar{z}), \quad i = 1, \dots d$ (**Field**)

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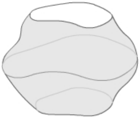

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(Polyakov action / Non-linear sigma model)

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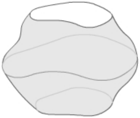

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(Canonical / WZW / α'-expansion)

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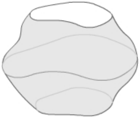

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2-dim QFT

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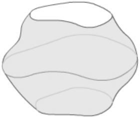

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RNS **Super**strings
(IIA, IIB, heterotic):
Add fermions

Quantization

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2-dim $\mathcal{N} = 1$ SQFT

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$\boxed{(\mathcal{M}^d, g) \mid \text{Ricci}(g) \approx 0}$

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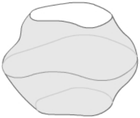

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$\boxed{\text{2-dim } \mathcal{N} = 1 \text{ SCFT}}$

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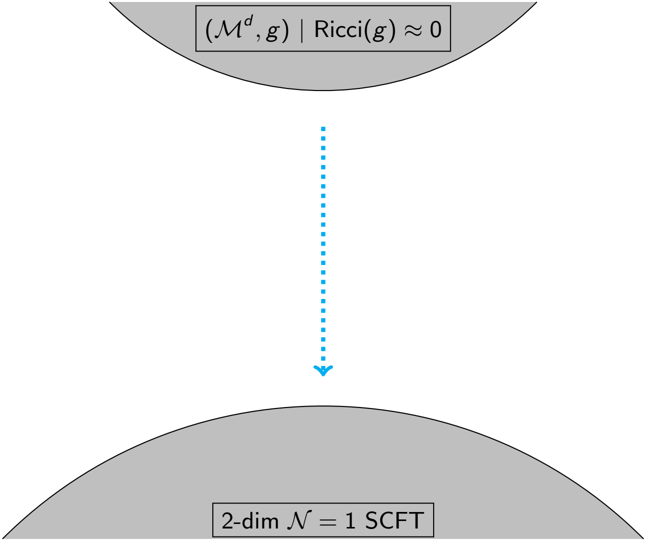

Introduction

$$\boxed{(\mathcal{M}^d, g) \mid \text{Ricci}(g) \approx 0}$$

- Not surjective:

$\exists$ other constructions of SCFTs

Landau-Ginzburg,
Gepner,
etc.

$$\boxed{\text{2-dim } \mathcal{N} = 1 \text{ SCFT}}$$

Introduction

$\boxed{(\mathcal{M}^d, g) \mid \mathrm{Ricci}(g) \approx 0}$

- Not surjective:

$\exists$ other constructions of SCFTs

- Not injective:

Mirror symmetry, T-duality

Mirror symmetry for TCS G_2?

[Braun, Del Zotto '17; '18]

Landau-Ginzburg,
Gepner,
etc.

$\boxed{\text{2-dim } \mathcal{N} = 1 \text{ SCFT}}$

Contents

Operator algebras: formalism

$\boxed{(\mathcal{M}^d, g) \mid \mathrm{Ricci}(g) \approx 0}$ $\dashrightarrow$ $\boxed{\text{2-dim } \mathcal{N} = 1 \text{ SCFT}}$

Virasoro $\boldsymbol{\mathcal{N} = 1}$ acts on the Hilbert space.

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∞-dimensional $\mathbb{Z}_2$-graded Lie algebra

$$\left\langle \{G_n\,,\ T_n\}_{n\in\mathbb{Z}}\ , c \,\middle|\, \text{Lie bracket} \right\rangle$$

c : **central charge**

$$[T_m, T_n] = (m-n)T_{m+n} + \frac{1}{12}(m^3 - m)c$$

$$[T_m, G_n] = \dots$$

$$[G_m, G_n] = \dots$$

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<ins>"**Operator algebra**"</ins>

$$\overset{\mathcal{N}=1}{\text{Vir}}{}_c := \left\langle G_{\left(\frac{3}{2}\right)}(z)\,,\ T_{(2)}(z) \;\middle|\; \textbf{OPEs} \right\rangle$$

$$T(z)T(z') \sim \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial'_z T(z')}{z - z'}$$

$$T(z)G(z') \sim \ldots$$

$$G(z)G(z') \sim \ldots$$

Generators$_{\textbf{(weights)}}$

Operator algebras: formalism

Assemble your favorite name! (closely related concepts):

Chiral (Super-)Conformal Vertex $\mathcal{W}$ Lie conformal / Λ-bracket	Operator Field Current	Algebra

- Vertex Operator Algebra

[Borcherds '86; Frenkel, Lepowsky, Meurman '88; Kac '98; Frenkel, Ben-Zvi '01]

Operator Algebra [Thielemans '95 hep-th/9506159] Mathematica package

1. Vector space $\mathcal{V}$ of *operators* denoted $\{\mathbb{1}, A, B, \ldots\}$.
   We think of elements of $\mathcal{V}$ as operators acting on the Hilbert space of the QFT.
2. $\mathbb{Z}_2$-grading $\mathcal{V} = \mathcal{V}^b \oplus \mathcal{V}^f$ and $\mathbb{1} \in \mathcal{V}^b$.
3. Even linear map $\partial : \mathcal{V} \longrightarrow \mathcal{V}$ (e.g. ∂_z).
4. Sequence of bilinear pairings $\boxed{\cdot\ \cdot}_n : \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{V}$, $n \in \mathbb{Z}$, compatible with the grading and s.t. (terms in OPEs)
   (0) $\forall\ A, B \in \mathcal{V}$, $\exists\ n(A,B) \in \mathbb{Z}$ such that $\boxed{AB}_{n \geq n(A,B)} = 0\,.$
   (1) (*unity*):
   $$\boxed{\mathbb{1} A}_n = \delta_{0,n} A \qquad \forall A \in \mathcal{V}$$
   (2) (*commutativity*):
   $$\boxed{BA}_n = (-1)^{|A||B|} \sum_{m \geq n} \frac{(-1)^m}{(m-n)!} \partial^{(m-n)} \boxed{AB}_m \qquad \forall n \in \mathbb{Z}$$
   (3) (*associativity*):
   $$\boxed{A \boxed{BC}_m}_n = (-1)^{|A||B|} \boxed{B \boxed{AC}_n}_m + \sum_{l \geq 1} \binom{n-1}{l-1} \boxed{\boxed{AB}_l C}_{m+n-l}$$

Normal ordering: $:\!AB\!: := \boxed{AB}_0$

Calabi-Yau: Virasoro $\mathcal{N} = 2$

$$\boxed{(\mathcal{M}^d, g) \mid \text{Ricci}(g) \approx 0} \dashrightarrow \boxed{\text{2-dim } \mathcal{N} = 1 \text{ SCFT}}$$

$$\overset{\mathcal{N}=1}{\text{Vir}}{}_c := \left\langle G_{\left(\frac{3}{2}\right)} \, , \, T_{(2)} \mid \text{OPEs} \right\rangle$$

$$\cap$$

$$\overset{\mathcal{N}=2}{\text{Vir}}{}_c := \left\langle \left(G_{\left(\frac{3}{2}\right)} \, , \, T_{(2)}\right) , \left(J_{(1)} \, , \, \tilde{G}_{\left(\frac{3}{2}\right)}\right) \mid \text{OPEs} \right\rangle$$

Properties / applications:

- Spacetime **supersymmetry**
- **Spectral flow**: Isomorphism NS / R
- **Chiral ring** of special NS fields (chiral primaries) related to Ramond ground states and **Dolbeault cohomology**
- **Topological twists**
- etc.

[Lerche, Vafa, Warner '89; Banks, Dixon, Friedan, Martinec '88, etc.]

Calabi-Yau: Odake^{2n}

Calabi-Yau manifolds have **enhanced** operator algebra. [Odake '89]
Complex dimension $n \in \mathbb{N}_{\geq 0} \Rightarrow$ discrete sequence of such algebras.

$$\begin{array}{l} \overset{\mathcal{N}=2}{\mathrm{Vir}}_{c} \\ \quad \cap \\ \mathrm{Odake}^{2n}_{c=3n} := \left\langle \left(G_{\left(\frac{3}{2}\right)}, T_{(2)}\right), \left(J_{(1)}, \tilde{G}_{\left(\frac{3}{2}\right)}\right), \left(A_{\left(\frac{n}{2}\right)} + iB_{\left(\frac{n}{2}\right)}, C_{\left(\frac{n+1}{2}\right)} + iD_{\left(\frac{n+1}{2}\right)}\right) \right\rangle \end{array}$$

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- e.g. $n = 2$ (K3): $\mathbf{Odake^4_{c=6}} = \overset{\mathcal{N}=4}{\mathrm{Vir}}_{c=6}$ (little)

[Eguchi, Ooguri, Taormina, Yang '89; Aspinwall, Morrison '94, etc.]

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- e.g. $n = 3$: $\mathbf{Odake^6_{c=9}}$

Parallel tensors and algebra generators

$$\{(\ \mathrm{CY}^{2n}\ ;\ g\ ,\ \omega_{(2)}\ ,\ \Omega_{(n)}\ )\}$$

$$\mathrm{Odake}^{2n}_{c=3n} := \left\langle \overbrace{\left(G_{\left(\frac{3}{2}\right)}\ ,\ T_{(2)}\right)}, \overbrace{\left(J_{(1)}\ ,\ \tilde{G}_{\left(\frac{3}{2}\right)}\right)}, \overbrace{\left(A_{\left(\frac{n}{2}\right)} + iB_{\left(\frac{n}{2}\right)}\ ,\ C_{\left(\frac{n+1}{2}\right)} + iD_{\left(\frac{n+1}{2}\right)}\right)} \right\rangle$$

Metric g, $\nabla_i g = 0$	$\dashrightarrow$	Generators $\left(G_{\left(\frac{3}{2}\right)}, T_{(2)}\right)$ of $\overset{\mathcal{N}=1}{\mathrm{Vir}}$
p-form ϕ, $\nabla_i \phi = 0$	$\dashrightarrow$	Generators $\left(\Phi_{\left(\frac{p}{2}\right)}\ ,\ \Phi'_{\left(\frac{p+1}{2}\right)}\right)$

(Symmetries leading to these currents understood for general $(1,0)$ non-linear sigma-models. [Howe, Papadopoulos '93; de la Ossa, Fiset '18])

Free

$$\{(\ \mathbb{R}^1 \text{ or } \mathbb{S}^1\ ;\ g = \text{flat}\ ,\ \mathrm{d}x\ )\}$$

$$\mathbf{Free}^1 := \left\langle \overbrace{(\psi_{(1/2)}\ ,\ j_{(1)})} \quad \middle| \quad \psi(z)\psi(w) \sim \tfrac{1}{z-w}\ , \quad j(z)j(w) \sim \tfrac{1}{(z-w)^2} \right\rangle$$

$$\bigcup$$

$$\overset{\mathcal{N}=1}{\mathrm{Vir}}{}_{c=3/2} = \left\langle \underbrace{:j\psi:}_{G^{\mathrm{Free}}}\ ,\ \underbrace{\frac{1}{2}:\psi\psi: + \frac{1}{2}:jj:}_{T^{\mathrm{Free}}} \right\rangle$$

Shatashvili-Vafa algebra (for G_2)

Operator algebras associated to G_2 & Spin(7) holonomy [Shatashvili, Vafa '95]
(c.f. talk by S. Shatashvili 12 Sep 2017 @ Simons Center)

$$\left\{\left(\ G_2\text{-manifold } \mathcal{M}^7 \ ;\ g\ ,\ \varphi_{(3)}\ ,\ *\varphi_{(4)}\ \right)\right\}$$

$$\mathbf{SV}^{G_2}_{c=21/2} := \left\langle \left(\overbrace{G_{\left(\frac{3}{2}\right)}\ ,\ T_{(2)}}\right), \left(\overbrace{\Phi_{\left(\frac{3}{2}\right)}\ ,\ K_{(2)}}\right), \left(\overbrace{X_{(2)}\ ,\ M_{\left(\frac{5}{2}\right)}}\right) \ \middle|\ \text{OPEs} \right\rangle$$

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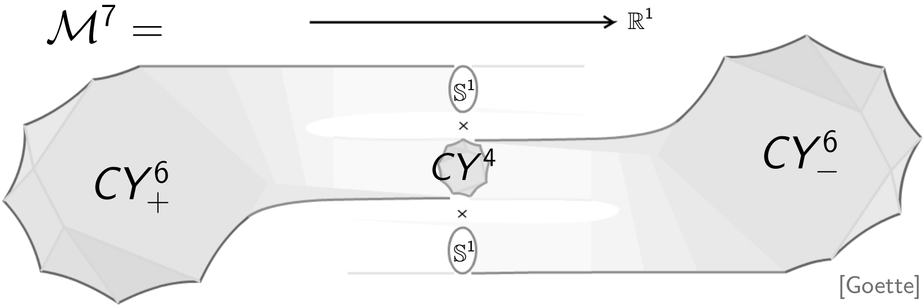
$$\bigcup$$

$$\overset{\mathcal{N}=1}{\text{Vir}}{}_{c=7/10} = \Big\langle \underbrace{\frac{i\Phi}{\sqrt{15}}}_{G^{\text{3-Ising}}}, \ \underbrace{-\frac{X}{5}}_{T^{\text{3-Ising}}} \Big\rangle \qquad \text{Tricritical Ising sector}$$

Properties / applications (some conjectural):

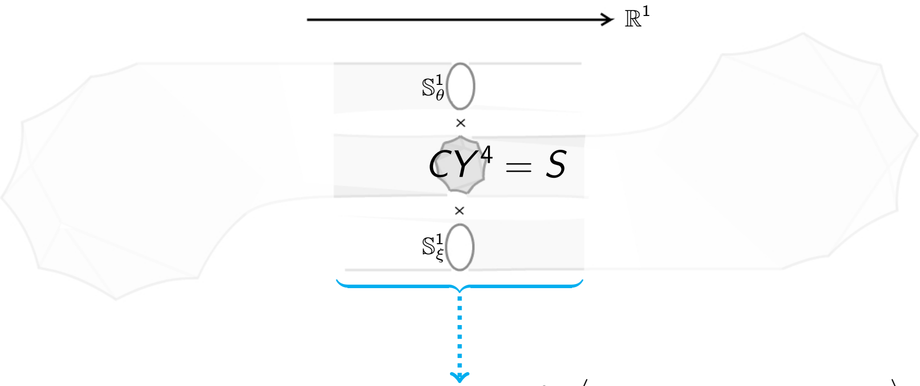
- Spacetime **supersymmetry**
- **Spectral flow**: Isomorphism NS / R
- **Chiral ring** of special NS fields (~~chiral primaries~~) related to Ramond ground states and ~~Dolbeault~~**Check cohomology** [Fernandez, Ugarte '98]
- **Topological twists**, etc??? [Shatashvili, Vafa '95; de Boer, Naqvi, Shomer '05]

Algebra for TCS [Kovalev '03; Corti, Haskins, Nordström, Pacini '13, '15]



$$
\begin{tikzcd}[column sep=small]
\mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3 \arrow[rr] \arrow[d, phantom, "\cup"] & & \mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3 \arrow[d, phantom, "\cup"] \\
\mathrm{Odake}^6_{c=9} \times \mathrm{Free} \arrow[dr, phantom, "\cup"] & & \mathrm{Odake}^6_{c=9} \times \mathrm{Free} \arrow[dl, phantom, "\cup"] \\
& \mathrm{SV}^{G_2} &
\end{tikzcd}
$$

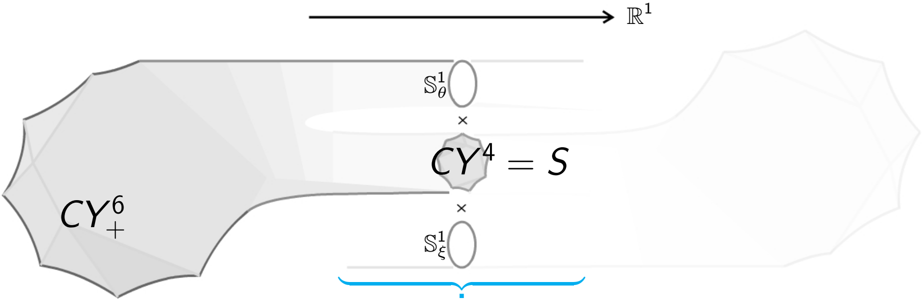
Algebra for TCS



$$\mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3 = \Big\langle G_S \,,\; J_S \,,\; A_S + iB_S \,,\; \psi_t \,,\; \psi_\theta \,,\; \psi_\xi \Big\rangle$$

& partners

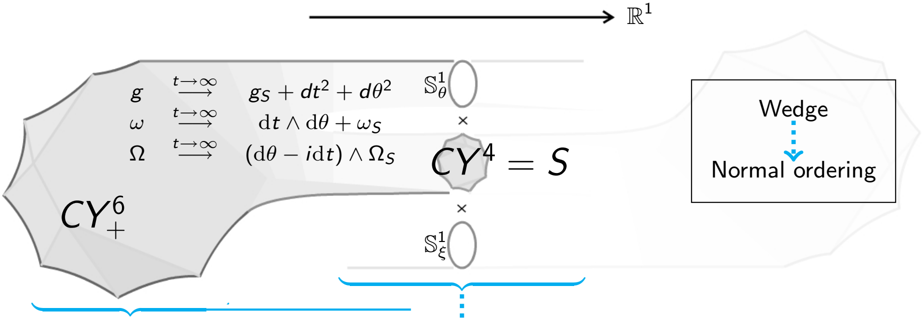
Algebra for TCS



$$\mathrm{Odake}^6_{c=9} \times \mathrm{Free} = \langle G, J, A + iB, \psi_\xi \rangle$$

$$\mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3 = \left\langle G_S, J_S, A_S + iB_S, \psi_t, \psi_\theta, \psi_\xi \ \text{\& partners} \right\rangle$$

Algebra for TCS



Wedge

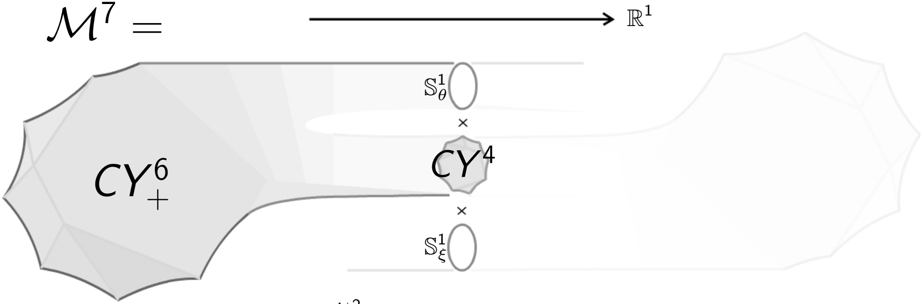
Normal ordering

$$\mathrm{Odake}^6_{c=9} \times \mathrm{Free} \quad \subset \quad \mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3 = \left\langle G_S\,,\ J_S\,,\ A_S + iB_S\,,\ \psi_t\,,\ \psi_\theta\,,\ \psi_\xi \ \ \&\text{ partners} \right\rangle$$

$$\begin{array}{cccccc} \| & & & & \\ \widehat{G\,,} & \longrightarrow & G_S + :j_t\psi_t: + :j_\theta\psi_\theta: & =: & G_+ \\ J\,, & \longrightarrow & :\psi_t\psi_\theta: + J_S & =: & J_+ \\ A + iB\,, & \longrightarrow & :(\psi_\theta - i\psi_t)(A_S + iB_S): & =: & A_+ + iB_+ \\ \underbrace{\psi_\xi} & & & & \end{array}$$

Algebra for TCS

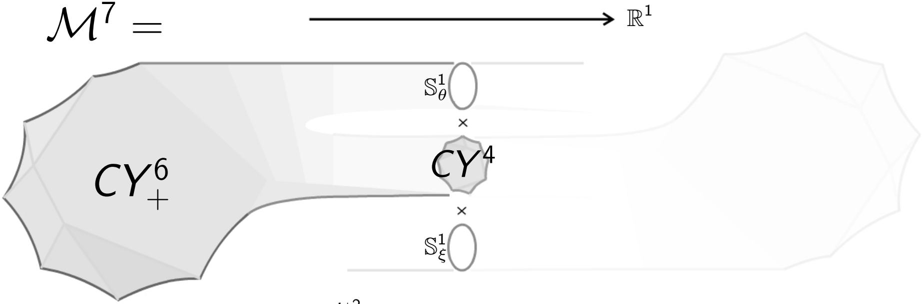
$$\mathcal{M}^7 =$$



$$g_{\mathcal{M}} := g + d\xi^2$$
$$\varphi := \mathrm{d}\xi \wedge \omega + \mathrm{Re}(\Omega)$$
$$\psi := \frac{1}{2}\omega \wedge \omega - \mathrm{d}\xi \wedge \mathrm{Im}(\Omega)$$

Algebra for TCS

$$\mathcal{M}^7 =$$



$$\begin{aligned} g_{\mathcal{M}} &:= g + d\xi^2 \\ \varphi &:= \mathrm{d}\xi \wedge \omega + \mathrm{Re}(\Omega) \\ \psi &:= \tfrac{1}{2}\omega \wedge \omega - \mathrm{d}\xi \wedge \mathrm{Im}(\Omega) \end{aligned}$$

$$\begin{aligned} G_{\mathcal{M}} &:= G + {:}j_\xi \psi_\xi{:} \\ \Phi &:= {:}\psi_\xi J{:} + A \\ X &:= \tfrac{1}{2}{:}JJ{:} - {:}\psi_\xi B{:} \end{aligned}$$

$\mathrm{Odake}^6_{c=9} \times \mathrm{Free}$

[Figueroa O'Farrill '96]

SV^{G_2}

Algebra for TCS

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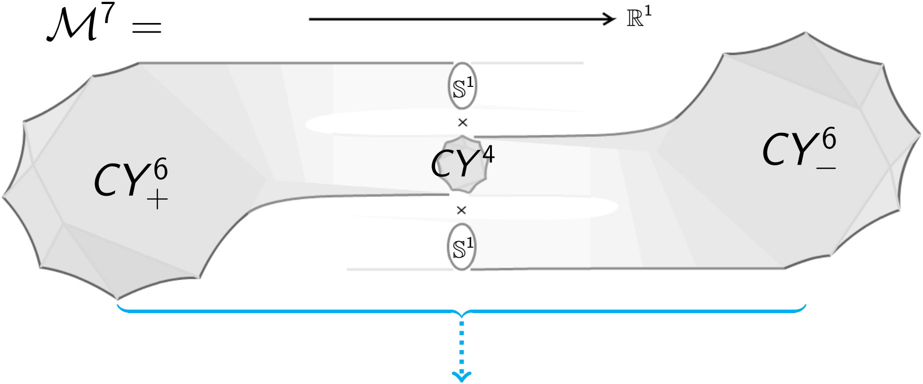


$$\mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3$$

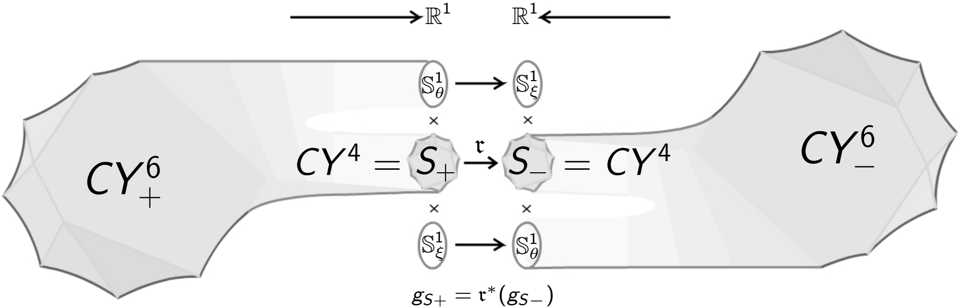
$$\supset$$

$$\mathrm{Odake}^6_{c=9} \times \mathrm{Free}$$

$$\supset$$

$$\mathrm{SV}^{G_2}$$

Algebra for TCS



$$g_{S_+} = \mathfrak{r}^*(g_{S_-})$$

$$\omega_{S_+} = \mathfrak{r}^*\mathrm{Re}(\Omega_{S_-}) \qquad \mathrm{Re}(\Omega_{S_+}) = \mathfrak{r}^*\omega_{S_-} \qquad \mathrm{Im}(\Omega_{S_+}) = -\mathfrak{r}^*\mathrm{Im}(\Omega_{S_-})$$

$$(\theta, \xi, t) \longmapsto (\xi, \theta, -t)$$

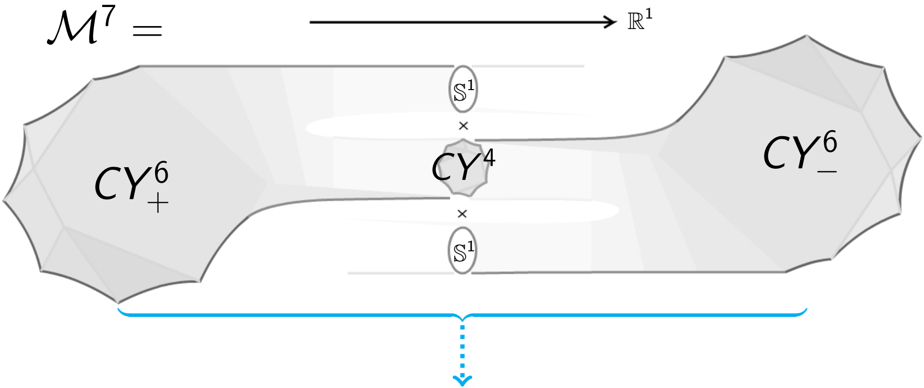
Preserve OPEs ("**Automorphism**")

$$G_S \longmapsto G_S$$

$$J_S \longmapsto A_S \qquad A_S \longmapsto J_S \qquad B_S \longmapsto -B_S$$

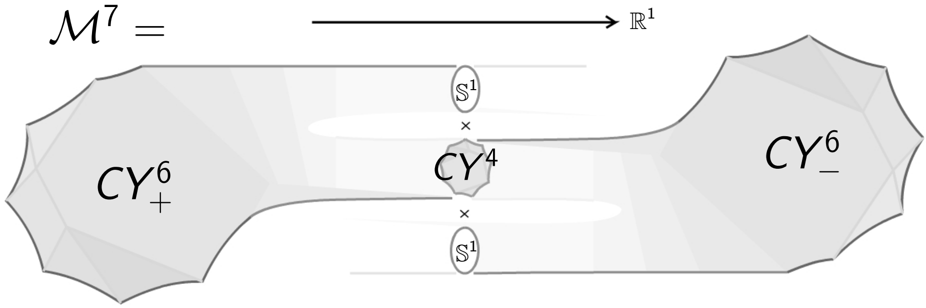
$$(\psi_\theta, \psi_\xi, \psi_t) \longmapsto (\psi_\xi, \psi_\theta, -\psi_t)$$

Algebra for TCS



$$\begin{tikzcd}
\mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3 \arrow[rrrr] \arrow[dr, phantom, sloped, "\subset"] & & & & \mathrm{Odake}^4_{c=6} \times \mathrm{Free}^3 \arrow[dl, phantom, sloped, "\supset"] \\
& \mathrm{Odake}^6_{c=9} \times \mathrm{Free} \arrow[dr, phantom, sloped, "\subset"] & & \mathrm{Odake}^6_{c=9} \times \mathrm{Free} \arrow[dr, phantom, sloped, "\subset"] & \\
& & \mathrm{SV}^{G_2} & & \mathrm{SV}^{G_2}
\end{tikzcd}$$

Algebra for TCS



$\Phi_+ = {:}\psi_\xi J_S{:} + {:}\psi_\xi\psi_t\psi_\theta{:} + {:}\psi_\theta A_S{:} + {:}\psi_t B_S{:}$

All SV^{G_2} generators map to themselves.

$$\begin{tikzcd}
\mathrm{Odake}^4_{c=6}\times\mathrm{Free}^3 \arrow[rrrr] \arrow[dr, phantom, sloped, "\subset"] & & & & \mathrm{Odake}^4_{c=6}\times\mathrm{Free}^3 \arrow[dl, phantom, sloped, "\supset"] \\
& \mathrm{Odake}^6_{c=9}\times\mathrm{Free} \arrow[dr, phantom, sloped, "\subset"] & & \mathrm{Odake}^6_{c=9}\times\mathrm{Free} \arrow[dl, phantom, sloped, "\supset"] & \\
& & \mathrm{SV}^{G_2} & &
\end{tikzcd}$$

Algebra for TCS: a subtlety

- The OPEs of SV^{G_2} fail to define an associative operator algebra (axiom 3). However associativity is achieved by setting [Figueroa-O'Farrill '97]

$$0 = 4{:}GX{:} - 2{:}\Phi K{:} - 4\partial_z M - \partial_z^2 G$$

(and the ideal it generates).

- Similarly, Odake[6] is only associative up to the ideal generated by [Odake '89]

$$0 = \partial_z A - {:}JB{:} \qquad\qquad 0 = \partial_z B + {:}JA{:}\,.$$

- Inclusions above hold up to these ideals.

Automorphisms

Free:

	j_ξ	ψ_ξ
T-duality $\mathbf{T}_\xi$	$-$	$-$

Odake^{2n}:

	T	G	$\tilde{G}$	J	A	B	C	D
Mirror symmetry **M**	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$
Phase $\mathbf{Ph}^{\phi=\pi}$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

SV^{G_2}:

	T	G	Φ	X	K	M
GK mirror symmetry $\mathcal{M}$	$+$	$+$	$-$	$+$	$-$	$+$

First observed in [Becker, Becker, Morrison, Ooguri, Oz, Yin '96] . Interpreted as mirror symmetry for Joyce orbifolds in [Gaberdiel, Kaste '04] .

Automorphisms

$$
\begin{tikzcd}
\mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \arrow[rr] & & \mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \\
\mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] \\
& SV^{G_2} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

Legend:

T-duality

Mirror symmetry

Phase

GK mirror symmetry

Automorphisms

$$\begin{tikzcd}[column sep=small]
\mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \arrow[rrrr] & & & & \mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \\
& {\color{green}\mathrm{Odake}^6} \times {\color{lime}\mathrm{Free}_\xi} \arrow[ul, hook] & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[ur, hook] & \\
& & \mathrm{SV}^{G_2} \arrow[ul, hook] \arrow[ur, hook] & &
\end{tikzcd}$$

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\end{tikzcd}$$

Legend:
T-duality
Mirror symmetry
Phase
GK mirror symmetry

$\mathcal{T}_4$ [Braun, Del Zotto '17] : T-dualities along $\mathbb{T}^3$ fibres and external $\mathbb{S}^1_\xi$.
Here purely quantum mechanical statement.

Automorphisms

$$\begin{array}{ccc}
\mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t & \longrightarrow & \mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \\
\cup & & \cup \\
\mathrm{Odake}^6 \times \mathrm{Free}_\xi & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \\
& \cup \qquad \cup & \\
& \mathrm{SV}^{G_2} &
\end{array}$$

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& \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[ul, hook] & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[ur, hook] & \\
& & \mathrm{SV}^{G_2} \arrow[ul, hook] \arrow[ur, hook] & &
\end{tikzcd}
$$

T-duality

Mirror symmetry

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GK mirror symmetry

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\begin{tikzcd}
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\end{tikzcd}
$$

T-duality
Mirror symmetry
Phase
GK mirror symmetry

Automorphisms

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\begin{array}{ccccc}
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& & \mathrm{SV}^{G_2} & &
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$$
\begin{tikzcd}
\mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \arrow[rrrr] & & & & \mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \\
& \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[ul, hook] & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[ur, hook] & \\
& & \mathrm{SV}^{G_2} \arrow[ul, hook] \arrow[ur, hook] & &
\end{tikzcd}
$$

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\mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] \\
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\mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] \\
& \mathrm{SV}^{G_2} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

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$\mathcal{T}_3$ [Braun, Del Zotto '18]

Automorphisms

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\begin{tikzcd}
\mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \arrow[rr] & & \mathrm{Odake}^4 \times \mathrm{Free}_\theta \times \mathrm{Free}_\xi \times \mathrm{Free}_t \\
\mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] & & \mathrm{Odake}^6 \times \mathrm{Free}_\xi \arrow[u, hook] \\
& \boxed{\mathrm{SV}^{G_2}} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

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$\mathcal{T}_3$ [Braun, Del Zotto '18]

Conclusion

Summary:

- Operator algebras: Free, Odake^{2n}, SV^{G_2}
- Parallel objects lead to generators
- Network of inclusions (modulo ideals) inspired by TCS
- Automorphisms $\Rightarrow$ mirror symmetry $\mathcal{T}_3$, $\mathcal{T}_4$ on TCS

Conclusion

Summary:

- Operator algebras: Free, Odake^{2n}, SV^{G_2}
- Parallel objects lead to generators
- Network of inclusions (modulo ideals) inspired by TCS
- Automorphisms $\Rightarrow$ mirror symmetry $\mathcal{T}_3$, $\mathcal{T}_4$ on TCS

Outlook:

- Systematic search for automorphisms
- Hilbert space / representation theory $\Rightarrow$ Betti numbers?
- Finer conjecture on the effect of mirror symmetry on b_2 & b_3
- Example of a CFT fitting this mould; Gepner for TCS?
- Spin(7) equivalent for generalised connected sums [Braun, Schäfer-Nameki '18]