Canonical Orientations for the Moduli Space of G_2-instantons

Markus Upmeier
(joint with Dominic Joyce)

Talk based on:

Joyce, Upmeier - *Orientations for moduli spaces of G_2-instantons*. (Soon)

Joyce, Tanaka, Upmeier - *On orientations for gauge-theoretic moduli spaces*
http://people.maths.ox.ac.uk/joyce/JTU.pdf

Outline

Preliminary statement of results

Theorem (Joyce–U. 2018)

Let $(X, \phi^3, \psi^4 = *_\phi \phi)$ *be a closed* G_2*-manifold. A flag structure* $\mathcal{F}$ *on* X *determines, for every principal* $\mathrm{SU}(n)$*-bundle* $E \to X$*, an orientation of the moduli space* $\mathcal{M}_E^{\mathrm{irr}}$ *of* G_2*-instantons*

$$\{A \in \mathcal{A}_E^{\mathrm{irr}} \mid F_A \wedge \psi = 0\} \big/ \operatorname{Aut}(E).$$

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The moduli space of G_2-instantons $\mathcal{M}_E^{\mathrm{irr}}$ is orientable.

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Theorem (Walpuski 2013)

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Theorem (Donaldson 1987)

For ASD-connections on closed oriented Riemannian 4-manifolds, canonical orientations depend on an orientation of $H^1(M) \oplus H^+(M)$.

Deformation complex

For a G_2-instanton A and deformation $a \in \Omega^1(X; \mathfrak{g}_E)$ the G_2-instanton condition becomes

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$$\Omega^0(X; \mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(X; \mathfrak{g}_E) \xrightarrow{d_A \wedge \psi} \Omega^6(X; \mathfrak{g}_E) \xrightarrow{d_A} \Omega^7(X; \mathfrak{g}_E) \quad (1)$$

has been made elliptic by adding the right-most term.

Simplification of problem

More generally, for any connection A, we may roll up the complex and define a self-adjoint elliptic operator

$$L_A = \begin{pmatrix} 0 & d_A^* \\ d_A & *(\psi \wedge d_A) \end{pmatrix} : \Omega^0 \oplus \Omega^1 \to \Omega^0 \oplus \Omega^1.$$

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Hence the line bundle on $\mathcal{M}_E^{\mathrm{irr}}$ we want to orient extends to $\mathcal{A}_E^{\mathrm{irr}} / \operatorname{Aut}(E)$ as the determinant line bundle $\operatorname{Det}\{L_A\}_{A \in \mathcal{A}_E}$.

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The principal symbols of L_A and of the twisted Diracian $\not{D}_A$ agree $\implies$ their orientation problems agree.

Plan

Introduction

Orientations in gauge theory

Flag structures

Main theorem

Outline of proof

Determinant line bundles

Definition

Let $\{P_t\}_{t\in T}$ be a T-family of elliptic operators. The Quillen determinant line bundle is

$$\mathrm{Det}\{P_t\} := \bigcup_{t\in T} \Lambda^{\mathrm{top}}(\mathrm{Ker}\, P_t)^* \otimes \Lambda^{\mathrm{top}}(\mathrm{Coker}\, P_t) \;\searrow\; T.$$

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(limit exists) Categorifies $w_1\left(\mathrm{ind}\{P_t\}_{t\in T} \in KO(T)\right) \in H^1(T;\mathbb{Z}_2)$.

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Proposition

$\mathrm{Or}_{E \oplus F} \cong \mathrm{Or}_E \otimes_{\mathbb{Z}_2} \mathrm{Or}_F$ *canonically.*

Special case of excision

Theorem (Excision)

Let $E \searrow X$, $E' \searrow X'$ be $\mathrm{SU}(n)$-bundles over closed spin manifolds.

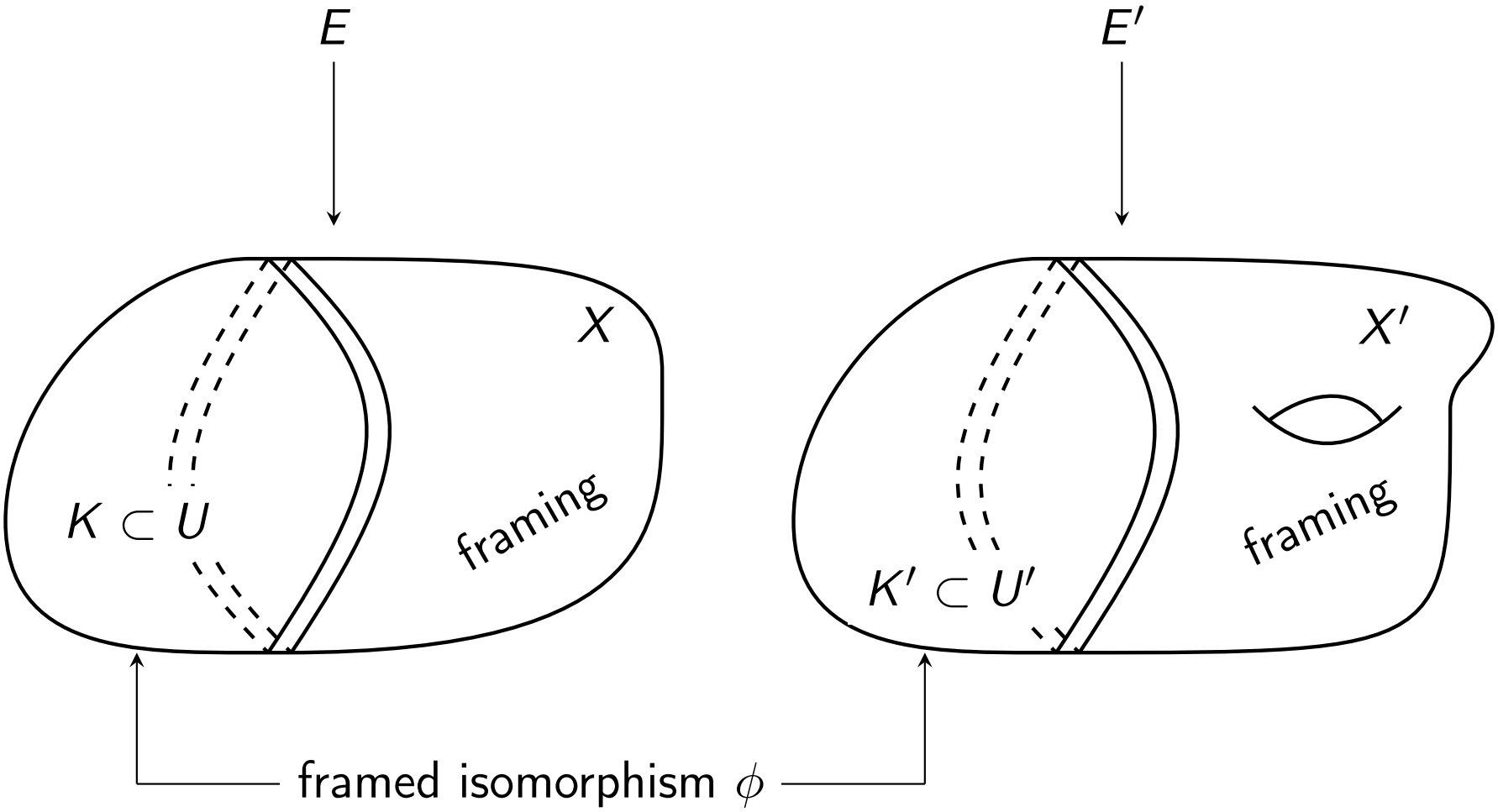
1. *Let ϕ be a spin diffeomorphism of open subsets*

$$X \supset U \xrightarrow{\phi} U' \subset X'.$$

2. *Let s and s' be $\mathrm{SU}(n)$-frames of $E|_{X \setminus K}$ and $E'|_{X' \setminus K'}$ defined outside compact subsets $K \subset U$ and $K' \subset U'$.*
3. *Let $\Phi \colon E|_U \to \phi^* E'|_{U'}$ be an $\mathrm{SU}(n)$-isomorphism with $\Phi(s) = \phi^* s'$.*

Then we get an excision isomorphism

$$\mathrm{Or}_{E \searrow X} \xrightarrow{O(\Phi, s, s')} \mathrm{Or}_{E' \searrow X'} .$$



$$\Longrightarrow \mathrm{Or}_{E \searrow X} \to \mathrm{Or}_{E' \searrow X'} .$$

Families index for real self-adjoint operators

For families $\{P_t\}_{t\in S^1}$ of real self-adjoint operators (Ati-Pat-Si)

$$w_1\left(\operatorname{ind} P_t \in KO^0(S^1)\right) \in \mathbb{Z}_2$$
$$=\text{index of a } \textit{single} \text{ operator } \frac{\partial}{\partial t} + P_t \text{ on the space } X \times S^1$$

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Theorem

Let X be a closed odd-dim. spin manifold, $\Phi\colon E \to E$ an $\mathrm{SU}(n)$-isomorphism over a spin diffeomorphism $\phi\colon X \to X$. Then

$$\mathrm{Or}(\Phi) = (-1)^{\delta(\Phi)}\cdot \mathrm{id}_{\mathrm{Or}_E}, \quad \delta(\Phi) := \int_{X_\phi} \hat{A}(TX_\phi)\left(\mathrm{ch}(E_\Phi^* \otimes E_\Phi) - n^2\right),$$

where $E_\Phi = E \times_{\mathbb{Z}} \mathbb{R} \searrow X_\phi = X \times_{\mathbb{Z}} \mathbb{R}$ are the mapping tori.

Simplification of formula in 7D

In dimension 7 for $\mathrm{Or}(\Phi) = (-1)^{\delta(\Phi)}\colon\ \mathrm{Or}_E \to \mathrm{Or}_E$ we have

$$\delta(\Phi) \equiv \frac{1}{2}\int_{X_\phi} p_1(TX_\phi)c_2(E_\Phi)$$

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Proof.
Add to $\delta(\Phi) = \mathrm{ind}(\not{D}_{\mathfrak{g}_E}) - \mathrm{ind}(\not{D}_{\mathfrak{su}(n)})$ a suitable even multiple of the integer $\mathrm{ind}(\not{D}_E)$ to simplify the integral. $\square$

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- *In particular* $\mathrm{Or}(\Phi) = \mathrm{id}$ *for every gauge transformation* $\phi = \mathrm{id}_X$ (*Walpuski*).

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- *In particular* $\mathrm{Or}(\Phi) = \mathrm{id}$ *for every gauge transformation* $\phi = \mathrm{id}_X$ (*Walpuski*).
- *The second formula is a self-intersection in* X^8_ϕ *of a homology class Poincaré dual to* $c_2(E_\Phi)$.

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A flag structure on X^7 associates signs $\mathcal{F}(Y, s)$ to submanifolds $Y^3 \subset X$ with non-vanishing normal sections s such that

$$F(Y_0, s_0) = (-1)^{D(s_0, s_1)} F(Y_1, s_1) \qquad \forall [Y_0] = [Y_1].$$

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For $S\colon Z \to N_Z$ a transverse extension of s_0, s_1 over a bordism $Z \subset X \times [0,1]$ from Y_0 to Y_1 we have

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A flag structure $\mathcal{F}$ is a notion of parity for (Y, s). When N_Y has an SU(2)-structure, a flag structure reduces choices by helping to pick out a normal SU(2)-framing.

Example

Let $Y := Y_0 = Y_1$ with trivializable normal bundle. For $s_0, s_1 \colon Y \to \mathbb{H}$ unit length write $s_1 = q \cdot s_0$ with $q \colon Y \to S^3$.

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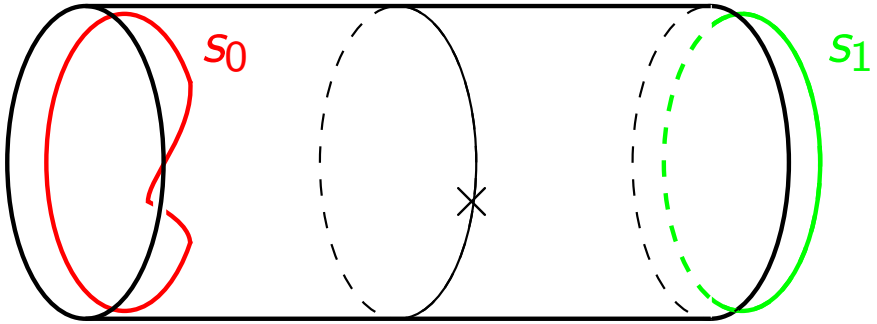
$$S(y,t) = [(1-t) + qt] \cdot s_0(y) = 0 \iff t = \frac{1}{2} \text{ and } q(y) = -1.$$

$$\implies D(s_0, s_1) = \text{degree}(q \colon Y \to S^3).$$



Proposition

Flag structures are a (non-empty) torsor over $H_3(X;\mathbb{Z}_2)$.

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Ratio $\mathcal{F}(Y,s) : \mathcal{F}'(Y,s)$ is independent of s (axiom for $\mathcal{F}$). $\square$

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Let $\phi\colon X\to X$ be an orientation-preserving isometry with $\phi|_Y = \mathrm{id}_Y$. Then $(\mathcal{F}/\phi^*\mathcal{F})[Y] = \mathcal{F}(Y,s):\mathcal{F}(Y,\phi_* s)$ equals the self-intersection of $Y\times S^1$ in the mapping torus $X_\phi = X\times_{\mathbb{Z}}\mathbb{R}$.

Corollary

Every manifold with $H_3(X;\mathbb{Z}_2) = \{0\}$ has a unique flag structure. A set of submanifold generators $[Y_i] \in H_3(X;\mathbb{Z}_2)$ with preferred normal sections s_i determines a unique flag structure with $\mathcal{F}(Y_i, s_i) := 1$.

Example

$X = S^7$ has a unique flag structure. $Y^3 \times S^4$ has a preferred one.

Plan

Theorem (Main theorem)

A flag structure $\mathcal{F}$ on a closed spin 7-manifold X induces uniquely, for every $\mathrm{SU}(n)$-bundle $E \searrow X$, a canonical orientation

$$o^{\mathcal{F}}(E) \in \mathrm{Or}_E$$

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1. (*Normalization*) *For $E = \underline{\mathbb{C}}^k$ trivial, evaluation defines $\mathrm{Or}_E = \mathbb{Z}_2$. Let $o^{\mathrm{triv}}(E) \in \mathrm{Or}_E$ be the image of $1 \in \mathbb{Z}_2$. Then*

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To be continued ...

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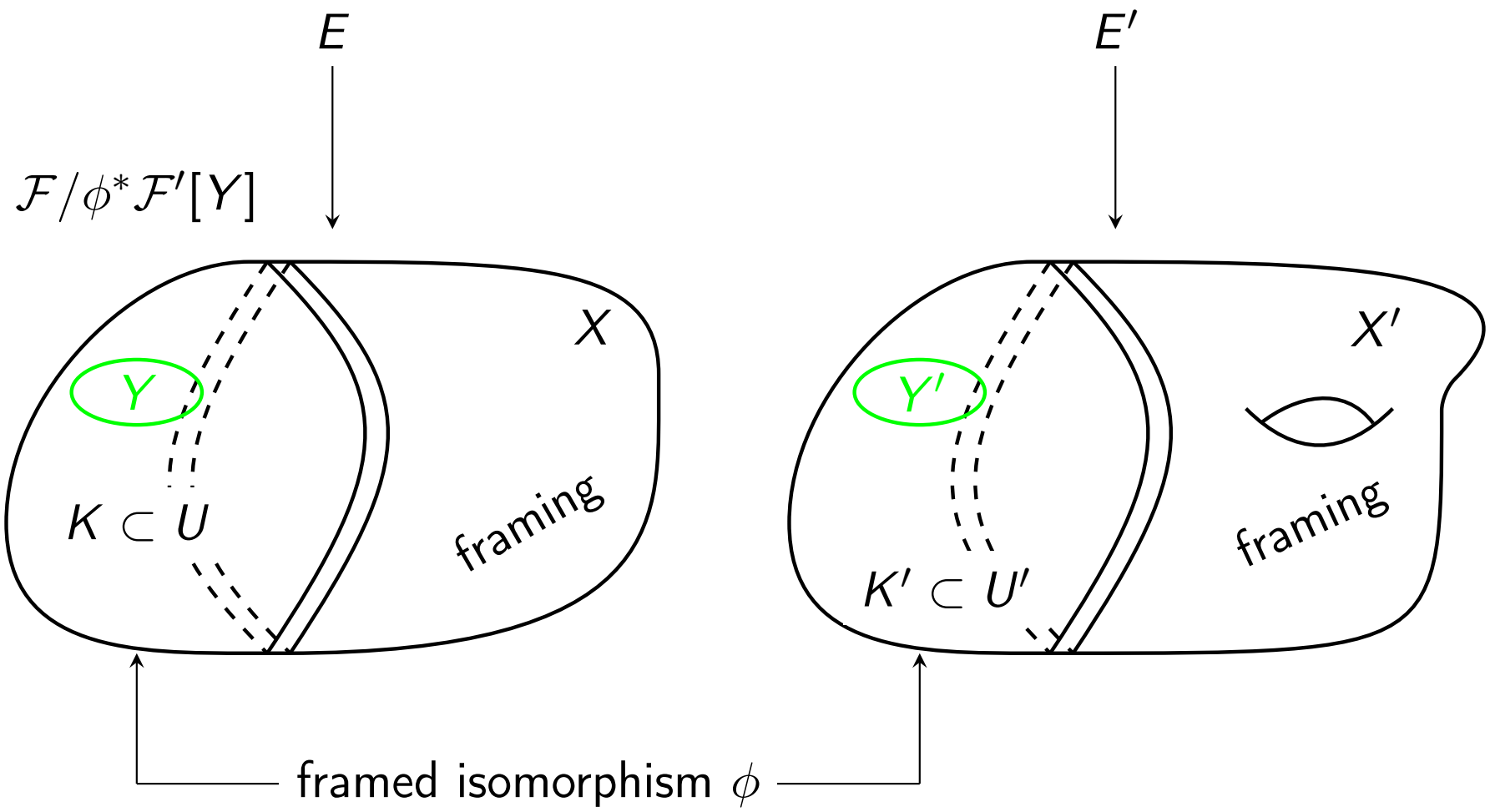
$$\mathrm{Or}(\Phi, s, s')\left(o^{\mathcal{F}}(E \searrow X)\right) = (\mathcal{F}/\phi^*\mathcal{F}')[Y] \cdot o^{\mathcal{F}'}(E' \searrow X'),$$

where $[Y] \in H_3(U;\mathbb{Z})$ *is the homology class Poincaré dual to the relative Chern class* $c_2(E|_U, s) \in H^4_{\mathrm{cpt}}(U;\mathbb{Z})$*.*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Example

Let U, U' be tubular neighborhoods of spin submanifolds Y, Y'. Let $\Phi\colon N_{Y'} \to N_Y$ be a spin isomorphism covering a spin diffeomorphism $\phi_\circ\colon Y' \to Y$. This determines $\phi = \phi(\phi_\circ, \Phi)$.

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By the excision axiom

$$\begin{array}{ccc} \mathrm{Or}_{E' \searrow S^7} & \xrightarrow{\quad \mathrm{Or}(\mathrm{can}, \phi^* s, s) \quad} & \mathrm{Or}_{E \searrow X} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ o^{\mathcal{F}_7}(E') & \longmapsto & (-1)^{(\mathcal{F}_7/\phi^*\mathcal{F})[Y]} \cdot o^{\mathcal{F}}(E) \end{array}$$

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Therefore

$$o^{\mathcal{F}}(E) = \mathrm{Or}(\mathrm{can}, \phi^* s, s) \circ \mathrm{stab}((-1)^{(\mathcal{F}_7/\phi^*\mathcal{F})[Y]} \cdot o^{\mathrm{triv}}) \qquad (*)$$

is uniquely determined by the axioms.

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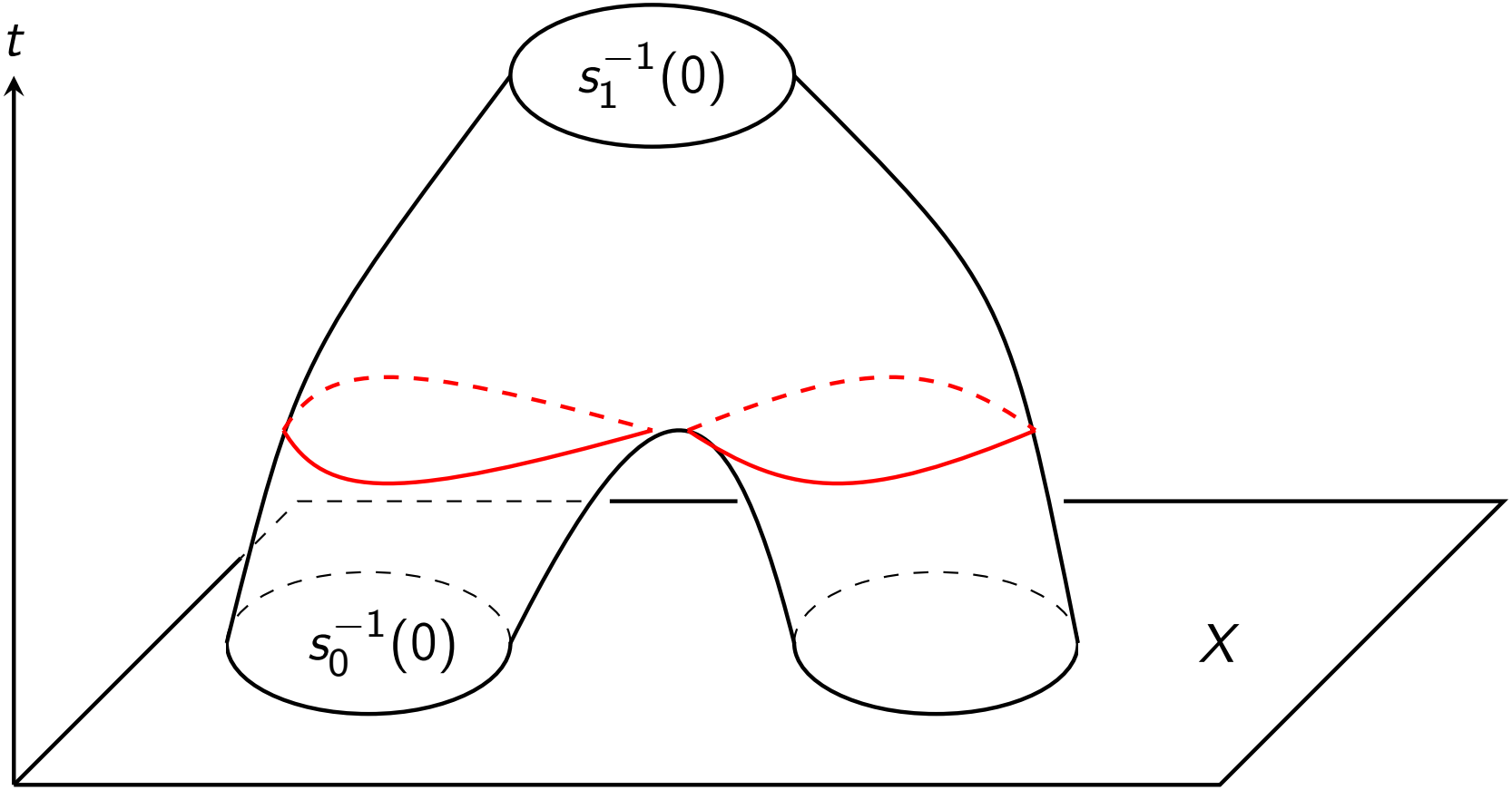
- Let s_0, s_1 be transverse sections of $E \searrow X$. Since the excision isomorphisms $\mathrm{Or}(\mathrm{can}, \phi_0^* s_0, s_0)$, $\mathrm{Or}(\mathrm{can}, \phi_1^* s_1, s_1)$ can be deformed into each other, they are equal by discreteness. The deformation is by excision isomorphisms as in the basic step, but not coming from a submanifold (example).


t
$s_1^{-1}(0)$
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X

Generalization to SU(n)-bundles

- To reconstruct E pick a generic $s\colon \underline{\mathbb{C}}^{n-1} \to E$.
- Let $Y \subset X$ be the points where $\dim \operatorname{Ker} s(y) = 1$.
- The kernel defines a map $\phi\colon Y \to \mathbb{C}P^{n-2}$.
- We get a U(2)-structure ξ on $\pi\colon N_Y \to Y$.

The reconstruction principle gives an isomorphism

$$E|_{\text{near } Y} \cong \left(E_{\text{std}}(N_Y) \oplus \pi^*\phi^* T\mathbb{C}P^{n-2}\right) \otimes \pi^*\phi^*\mathcal{O}(-1)|_{\text{on } N_Y}$$

under which s corresponds to a framing outside Y defined by ξ.

Now proceed similarly as before using Spin^c in place of Spin.