A Strong Stability Condition on Minimal Submanifolds

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Geometric Flows in Special Holonomy

Imperial College London, United Kingdom

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The main purpose of this project is to understand that given a minimal submanifold, under what condition the mean curvature flow takes a nearby submanifold to it.

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Settings/Notations. The ambient Riemannian manifold is denoted by (M^{n+m}, g), and its (minimal) submanifold is denoted by Σ. In this talk, Σ^n is always assumed to be closed and oriented.

§I. First and Second Variational Formula

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where $H = \mathrm{tr}_\Sigma \, \mathrm{II} = \sum_i \nabla^\perp_{e_i} e_i$ is the mean curvature vector.

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A submanifold Σ is said to be a minimal submanifold if its mean curvature vanishes, $H \equiv 0$. In other words, Σ is a critical point of the volume functional.

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$$\delta_V^2 \mathrm{vol}(\Sigma) = \int_\Sigma \Big[|\nabla^\perp V|^2 - \langle \mathcal{R}(V), V \rangle - \langle \mathcal{A}(V), V \rangle \Big] \, \mathrm{dvol}_\Sigma \ .$$

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Here, V is a normal vector field. Note that the first term is a first order differential operator, and the second and third terms are linear (zero-th order) operators.

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Let i, j be indices for Σ, and μ, ν be indices for the normal bundle $N\Sigma$. Write a normal vector as

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The operator $\mathcal{R}$ is the partial Ricci curvature defined by

$$\mathcal{R}(V) = \sum_{i,\mu,\nu} R^{M}_{i\mu\, i\nu} V^{\mu} e_{\nu} \ .$$

The convention here is that $R_{1212} > 0$ on the spheres.

And $\mathcal{A}$ is basically the square of the second fundamental form:

$$\mathcal{A}(V) = \sum_{i,j,\mu,\nu} h_{\mu\, i\, j}\, h_{\nu\, i\, j}\, V^{\mu}\, e_{\nu}$$

where

$$h_{\mu\, i\, j} = \langle \nabla^{M}_{e_i} e_j, e_{\mu} \rangle$$

are the coefficients of the second fundamental form.

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- More precisely, there exists a constant $c > 0$ such that

$$-\sum_{i,\mu,\nu} R^M_{i\mu\, i\nu} V^\mu V^\nu - \sum_{i,j,\mu,\nu} h_{\mu\, ij} h_{\nu\, ij} V^\mu V^\nu \geq c|V|^2$$

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  for any $x \in \Sigma$, $V \in N_x\Sigma$.
- It is clear that strong stability implies strict stability.

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Example (1)

The hypersurface (codimension one) case.

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More more generally, a totally geodesic (not necessary codimension one) in a negatively curved (sectional curvature) space is strongly stable.

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Hence, L is minimal, and stable, $\delta^2 \operatorname{vol}(L) \geq 0$.

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Since $\nabla^M J = 0$,

$$h_{k\,i\,j} = \langle \nabla^M_{e_i} e_j, Je_k \rangle \quad \text{is totally symmetric.}$$

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Hence, we find that a special Lagrangian is strongly stable if the induced metric has positive Ricci curvature. Note that this is purely a condition on the intrinsic geometry of the submanifold.

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Due to the Bochner formula, the positive Ricci implies that $b_1(L) = 0$. In this regards, the strong stability condition is the natural condition governs the triviality of deformation.

Example (3)

Coassociative in a $G2$.

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A $G2$ geometry is characterized by the existence of a parallel, positive 3-form φ.

A four dimensional submanifold C is coassociative if $\varphi|_C$ vanishes.

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In this linear model,

$$\varphi = (\mathrm{d}x^{12} + \mathrm{d}x^{34}) \wedge \mathrm{d}y^1 + (\mathrm{d}x^{13} - \mathrm{d}x^{24}) \wedge \mathrm{d}y^2 + (\mathrm{d}x^{14} + \mathrm{d}x^{23}) \wedge \mathrm{d}y^3 - \mathrm{d}y^{123} \, .$$

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One can easily see that $\varphi|_C = 0$, and see that $V \mapsto (V \lrcorner \varphi)|_C$ identifies NC with $\Lambda^2_+ C$.

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The condition is only on the intrinsic geometry of C. From the Bochner–Weitzenböck formula, it implies that $b^2_+(C) = 0$.

There is also the natural curvature operator in R. McLean's study of deformation:

$$\begin{aligned} &= (\nabla^\perp)^*\nabla^\perp - \mathcal{R} - \mathcal{A} && \text{on } NC \\ (d + d^*)^2 &= (\nabla^C)^*\nabla^C - 2W_+ + \frac{s}{3} && \text{on } \Lambda^2_+ C \end{aligned}$$

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In contrast to the previous two cases, $-(\mathcal{R} + \mathcal{A})$ can be rewritten as the normal bundle curvature contracting with ω_Σ. And the strong stability would imply that $TM/T\Sigma$ has no holomorphic section.

Example (5)

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4. The degree -4 complex line bundle over S^2 with the Atiyah–Hitchin metric [Phys. Lett. A 1985].

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- The rigidity result in the following section holds globally in these manifolds.

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- Except the last one, the Atiyah–Hitchin manifold, the zero sections are totally geodesic.

§III. Strong Stability and Rigidity

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Both stability and strong stability are defined through the second variational formula, which is in the infinitesimal level. A natural question is that whether it implies some uniqueness phenomenon in the geometric/physical submanifold level?

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This local uniqueness theorem not only exclude any possible deformation, integrable or not, but an effective neighborhood can be identified where there is no other closed minimal surface.

Sketch of the proof. Let $\{x_i\}$ be a local coordinate system on Σ, and $\{e_\mu\}$ be a local orthonormal frame for $N\Sigma$. The map

$$(x_i, y_\mu) \mapsto \exp^M_{\{x_i\}}(\sum_\mu y_\mu\, e_\mu)$$

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For simplicity, denote this function by $|y|^2$.

We may also choose an orthonormal frame $\{e_i\}$ for $T\Sigma$. Then parallel transport $\{e_i\} \cup \{e_\mu\}$ along normal radial geodesics to obtain a local orthonormal frame. Use the same notation to denote this frame.

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- $\nabla^2_{e_i,e_j}|y|^2 = -2y^\mu\, h_{\mu\,ij} - 2y^\mu\, y^\nu\, (R^M_{\mu\,i\nu\,j} + h_{\mu\,ik}h_{\nu\,jk}) + \mathcal{O}(|y|^3)$;

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Here, $h_{\mu\, ij}$ and $R^M_{\mu\, i\nu\, j}$ are evaluated at $(x_i, 0) \in \Sigma$.

For an n-dimensional subspace Γ, we may apply the singular value decomposition for find an orthonormal basis:

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$$\begin{aligned}\operatorname{tr}_\Gamma \operatorname{Hess}|y|^2 = &-2\sum_{\mu,i} y^\mu(\cos\theta_i)^2 h_{\mu\,i\,i} \\ &- 2\sum_{\mu,\nu,i} y^\mu y^\nu(\cos\theta_i)^2\big(R^M_{\mu\,i\,\nu\,i} + \sum_j h_{\mu\,i\,j}h_{\nu\,i\,j}\big) \\ &+ 2\sum_i(\sin\theta_i)^2 + \mathcal{O}(|y|^3) + \mathcal{O}(|\sin\theta|\,|y|^2) \ .\end{aligned}$$

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Appealing to the maximum principle finishes the proof of this theorem.

§IV. Stability under the Mean Curvature Flow

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According to the first variational formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\Gamma_t) = -\int_{\Gamma_t} |H_{\Gamma_t}|^2 \mathrm{d}\mathrm{vol}_{\Gamma_t} \ .$$

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We first review some facts about the mean curvature flow as well as the general negative gradient flow.

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Then, it is not hard to prove that for any initial point sufficiently close to the origin, the flow of $-\nabla f$ exists for all time, and converges to the origin as $t \to \infty$.

Long time existence

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Note that the second fundamental form is a quantity in the level of second order derivatives.

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Then, if a solution exists for all time, it must converges to a critical state as $t \to \infty$.

Remark. This may not be true without the analyticity condition.

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Based on the theorem of Huisken, it would be more interesting to have a dynamically stability result with closeness condition weaker than $\mathcal{C}^2$.

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3. The evolution equation for II^{Γ_t}:

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Key. The $\mathcal{C}^0$ equation (from strong stability) can be used to turn the $\mathcal{C}^1$ and $\mathcal{C}^2$ equations into a better shape, for which the maximum principle applies.

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By combining with the works of H.-J. Hein–S. Sun and A. Neves, they constructed an example of Lagrangian mean curvature in a compact Calabi–Yau, which has finite time singularity as smooth flow, but has long time existence as the enhanced Brakke flow. Moreover, the flow converges to a special Lagrangian as $t \to \infty$.

Thank you for your attention.