Categorical Kähler Geometry

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I. Physical Theory = Geometric Space

Le temps et l'espace... Ce n'est pas la nature qui nous les impose, c'est nous qui les imposons à la nature parce que nous les trouvons commodes.

Time and space ... it is not Nature which imposes them upon us, it is we who impose them upon Nature because we find them convenient.

- Henri Poincaré

Idea: View geometric features of a spacetime as “emerging” from observations of scattering processes for strings propogating in that space-time.

60 *P. Candelas et al. / Calabi–Yau manifolds*

TABLE 4
The numbers of rational curves of degree k for $1 \leqslant k \leqslant 10$

k	n_k
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 58888 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

II. Take Symmetry Seriously!

Slogan: Never ask if two entities are equal; instead provide an identification of one with the other.

Symmetries = identifications of an object with itself
Symmetries can have symmetries, and so on ad infinitum!
This is modeled by ∞-groupoids

Grothendieck's Homotopy Hypothesis:

$\pi_{\leqslant\infty}$: Top Spaces $\to$ ∞-groupoids

implements an equivalence of homotopy theories for any good model of ∞-groupoids

Homotopical Mathematics

Classical Entity	Homotopical Analogue
Sets	Spaces
Categories	∞-categories
Groups	Loop spaces
Abelian groups	Spectra
modules / field k	chain complexes / k
Associative rings	$A_\infty/\mathbb{E}_1$-rings
assoc. k-algebras	dg-algebras / k
	$\mathbb{E}_n$-rings
Commutative rings	$\mathbb{E}_\infty$-rings
Topoi	∞-topoi
Algebraic Spaces	n-geometric ∞-stacks
Symplectic structures	0-shifted symplectic structures
	n-shifted symplectic structures
Abelian Categories	Stable ∞-categories

- Algebras of observables in classical and quantum field theories are factorization algebras.
  Simplest case: Observables in TFTs are $\mathbb{E}_d$-algebras.
- Solutions to equations of motion = (-1)-shifted symplectic space
- BV-quantization is a natural construction in derived geometry
- Boundary conditions (branes) can naturally be organized into ∞-categories
- Various moduli spaces in math and physics are naturally derived ∞-stacks
- The philosophy "deformation problems are controlled by dg-Lie algebras" becomes a theorem in the homotopical world
- Derived moduli spaces automatically have the "expected dimension"
- Intersection theory is better behaved

Classification of TFTs

Topological Field Theories (TFTs) of dim d are physical theories that assign invariants to manifolds of dimension $\leqslant d$.

Theorem (Lurie)

A TFT Z is completely determined by Z(pt)

In topological string theory: $d = 2$, and $Z(pt)$ is a k-linear stable ∞-category over $k = \mathbb{C}$.

Definition (Kontsevich)

A derived noncommutative space (nc-space) over k is a k-linear stable ∞-category

Examples: $\mathrm{Fuk}(X, \omega)$, $\mathrm{DCoh}(X, I)$, $\mathrm{DRep}(\mathrm{Q})$

nc-geometry

There are well-developed nc-analogues of various notions from complex algebraic geometry and symplectic geometry:

- Properties, such as smoothness and compactness
- Structures, such as orientations (Calabi-Yau structures)
- Invariants, such as K-theory, Betti and de Rham cohomology
- Hodge theory
- Gromov-Witten theory (curve-counting)
- Donaldson-Thomas theory (counting BPS states)

III. Harmonic representatives

Study isomorphism classes of mathematical objects by finding canonical "good" representatives in each isomorphism class.

Schema:

- E isomorphism class of object
- $\mathrm{Met}(E)$ = space of representatives in the isomorphism class
- Auxillary data: convex function $S : \mathrm{Met}(E) \to \mathbb{R}$

Definition

- Unstable: S is not bounded below
- Semistable: S is bounded below
- Polystable: S attains a (unique) minimum

Fixed point of flow generated by $-\mathrm{grad} S$
= Minimizer of S (harmonic representative)

Linear example: finite dimensions

1. $W \subset V$ finite dimensional vector spaces; $E = [v] \in V/W$

   $\mathrm{Met}(E) = v + W$; an isomorphism $v \to v'$ is an element $w \in W$ such that $v - v' = w$.

   Auxillary structure: inner product on $V \rightsquigarrow S(v) = \|v\|^2$.
   $W^\perp \cap \mathrm{Met}([v]) =$ minimizers of S

   $\rightsquigarrow$ isomorphism $V/W \simeq W^\perp$

2. Hodge theory: infinite dimensional analogue
   $V = \Omega^k_{cl}(X)$, $W = d_{dR}\Omega^{k-1}(X)$

   Riemannian metric on X gives inner product on $\Omega^k(X)$

   $\rightsquigarrow H^k_{dR}(X) \simeq$ Harmonic k-forms $:= \{\alpha | \Delta\alpha = 0\}$ (BPS states)

Nonlinear examples: I

Complex reductive $G \circlearrowright V$ inducing $G \circlearrowright X \subset \mathbb{P}(V)$
Auxillary structure: h hermitian metric on V
$G = K_{\mathbb{C}}$; $K \circlearrowright (V, h)$ preserving X

$E = [x] \in X/G$
$\mathrm{Met(E)} := G$−orbit $\simeq G/K$
$S(x) = \|x\|^2$
Φ, the moment map, is essentially the derivative of S.

$X^{ps}/G \simeq \Phi^{-1}(0)/K$
GIT quotient $\simeq$ symplectic quotient (Kempf-Ness theorem)

Nonlinear examples: II

Infinite dimensional analogue of Kempf-Ness (Donaldson-Uhlenbeck-Yau):

X = space of connections on a smooth complex vector bundle on a complex manifold Y with $F^{(2,0)} = 0$;
K = compact gauge group

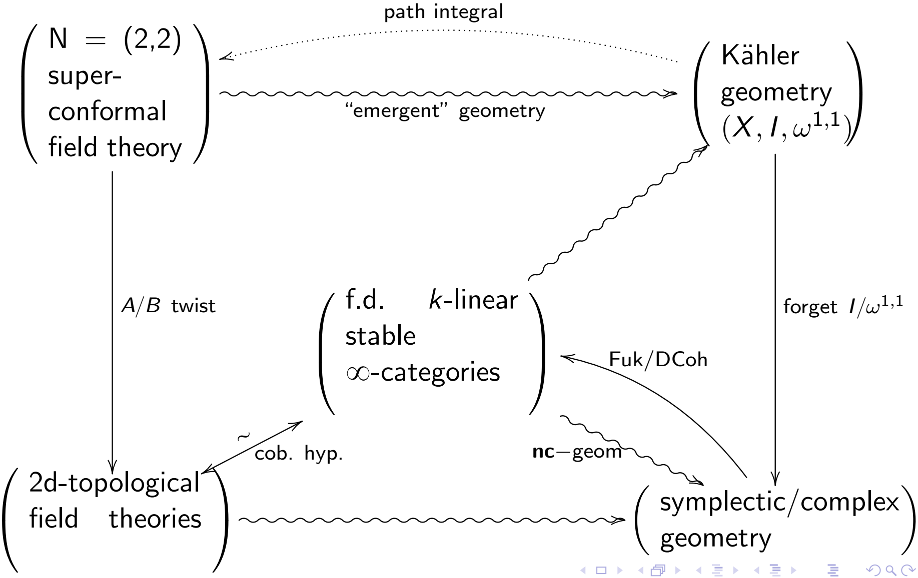
Auxillary structure: Kähler metric on Y
S is given by Bott-Chern secondary characteristic classes
Moment map is given by curvature

(Polystable holomorphic bundles) $\simeq$ (Hermitian-Yang-Mills connections).

RHS = connections satisfying a certain PDE (BPS branes)
Gradient flow for S is Donaldson's heat flow

Problem: Generalize this to complexes of vector bundles

Categorical Kähler geometry



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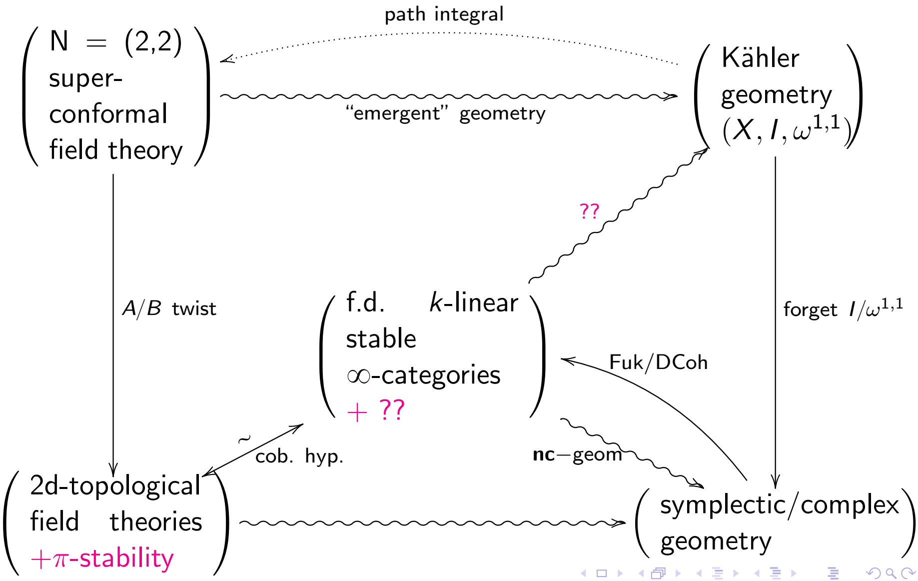

nc-Kähler metrics

Kähler classes : Kähler metrics :: nc-Kähler classes : ??

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Long-term goals of the program:

1. Find a notion of nc-Kähler metric on $\mathcal{C}$ that gives rise to
   - An underlying nc-Kähler class (Bridgeland stability structure) on $\mathcal{C}$
   - A Kähler metric on the moduli of polystable objects of $\mathcal{C}$.
   - A Donaldson-Uhlenbeck-Yau correspondence: $\mathcal{M}_{\mathcal{C}_\theta^{ps}} \simeq \mathcal{M}_{\mathcal{C}}^{\mathrm{harm}}$

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3. Develop local-to-global principles for studying Fukaya categories and stability structures on them, and study applications to mirror symmetry, higher Teichmüller theory, non-abelian Hodge theory, etc.

nc-Kähler classes = Bridgeland stability structures

A Bridgeland stability structure on $\mathcal{C}$ consists of

- A family of full subcats $\{\mathcal{C}_\theta^{ss}\}_{\theta\in\mathbb{R}}$ of semistable objects of phase θ.
- A homomorphism $Z : K_0(\mathcal{C}) \to \mathbb{C}$, the central charge.

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Such that

1. $E \in \mathcal{C}_\theta^{ss}$ then $Z(E) \in \mathbb{R}_{>0} exp(\sqrt{-1}\theta)$
2. $\mathrm{Map}(\mathcal{C}_\theta^{ss}, \mathcal{C}_{\theta'}^{ss}) \simeq 0$ for $\theta > \theta'$.
3. $\mathcal{C}_\theta^{ss}[1] \simeq \mathcal{C}_{\theta+\pi}^{ss}$.

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4. Every $E \in \mathcal{C}$ admits a Harder-Narasimhan "filtration":

$$0 \simeq E_0 \to E_1 \to \cdots \to E_n \simeq E$$

with $\mathrm{gr}_i E \in \mathcal{C}_{\theta_i}^{ss}$ for some

$$\theta_1 > \theta_2 > \cdots > \theta_n$$

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Polystable objects of phase θ: $\mathcal{C}_\theta^{ps} := (\mathcal{C}_\theta^{ss})^{semisimple}$

Category	$\mathrm{Fuk}(X, \omega)$	$\mathrm{Rep}(Q)$
Kähler data	hol vol form Ω	$\{z_v \in \mathbb{H}\}_{v \in \mathrm{Vert}(Q)}$
Object	Lag upto isotopy	$(\{E_v\}_v, \{T_\alpha\}_{\alpha \in \mathrm{Arr}(Q)})$
Metrized object	Lagrangian	(E_v, h_v), h_v hermitian metric
Operator	Ω	$P := \sum z_v pr_v + \sum [T_\alpha^*, T_\alpha]$
Flow $\mathcal{F}$	$\dot{L} = \mathrm{Arg}\Omega_L$	$h^{-1}\dot{h} = \mathrm{Arg} P$
Kähler potential	$dS_{\mathbb{C}}(f) = \int_L \Omega f$	$S_{\mathbb{C}} = \sum \log\det h_v + \sum T_\alpha^* T_\alpha$
Harmonic metric	Fixed points of $\mathcal{F}$ $= \mathrm{Crit}(S_{\mathbb{C}})$ = special Lagrangian	Fixed points of $\mathcal{F}$/rescaling $= \mathrm{Crit}(S_{\mathbb{C}})$
DUY theorem	??	King's theorem

Theorem (King)

There is a stability structure on $\mathrm{DRep}(Q)$ for which the polystable objects are shifts of objects $E \in \mathrm{Rep}(Q)$ that admit a harmonic metric.

Metrized objects: non-archimedean case

- K nonarchimedean field with ring of integers $\mathcal{O}_K$ and residue field k.
- $\mathcal{C}_{\mathrm{met}}$ a $\mathcal{O}_K$-linear stable ∞-category
- $\mathcal{C}_{\mathrm{sp}} := \mathcal{C}_{\mathrm{met}} \otimes_{\mathcal{O}_K} k$ and $\mathcal{C}_{\mathrm{gen}} := \mathcal{C}_{\mathrm{met}} \otimes_{\mathcal{O}_K} K$.
- Stability structure $(\{\mathcal{C}^{ss}_{\mathrm{sp},\theta}\}_{\theta} \in \mathbb{R}, Z_{\mathrm{sp}})$ on the special fiber.

Definition

Let $E \in \mathcal{C}_{\mathrm{gen}}$.

1. A metrization of E is an object $\tilde{E} \in \mathcal{C}_{\mathrm{met}}$ and an equivalence $\alpha : \tilde{E} \otimes_{\mathcal{O}_K} K \to E$.
2. A metrization $(\tilde{E}, \alpha)$ is harmonic of phase θ if $\tilde{E} \otimes_{\mathcal{O}_K} k \in \mathcal{C}^{ps}_{sp,\theta}$.

$\mathrm{Met}(E) :=$ space metrizations of E.

A nonarchimedean categorical DUY theorem

Theorem (Haiden-Katzarkov-Kontsevich-P.)

There is a natural Bridgeland stability structure $\{\mathcal{C}^{ss}_{\mathrm{gen},\theta}\}_{\theta\in\mathbb{R}}, Z_{\mathrm{gen}}$ *on the generic fiber* $\mathcal{C}_{\mathrm{gen}}$*, such that* $E \in \mathcal{C}^{ps}_{\mathrm{gen},\theta}$ *if and only if E admits a harmonic metrization.*

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Key idea: Given $E \in \mathcal{C}_{\mathrm{gen}} + (\tilde{E}, \alpha) \in \mathrm{Met}(E)$,

HN-filtraion of $\tilde{E} \otimes_{\mathcal{O}_K} k$ in $\mathcal{C}_{\mathrm{sp}}$ defines a "tangent vector" to $\mathrm{Met}(E)$

$\rightsquigarrow$ flow on the generalized building Met

The flow converges to a fixed point iff the object is polystable. More generally it converges to the HN-filtration, which is a point in a compactification of $\mathrm{Met}(E)$.