# Gradient flows, iterated logarithms, and semistability

Pranav Pandit

joint work with Fabian Haiden, Ludmil Katzarkov,
and Maxim Kontsevich

University of Vienna

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This talk is based on:

- arXiv:1706.01073
- arXiv:1802.04123

Goals:

- Describe the asymptotic behavior of the flows discussed in the previous talk (in special cases)
- Describe a canonical refinement of the Harder-Narasimhan filtration, and its relation to the asymptotic behavior of the flow

| Category | $\mathrm{Fuk}(X,\omega)$ | $\mathrm{Rep}(Q)$ |
|---|---|---|
| Object | Lag upto isotopy | $(\{E_v\}_v, \{T_\alpha\}_{\alpha\in\mathrm{Arr}(Q)})$ |
| Metrized object | Lagrangian | $(E_v, h_v)$, $h_v$ hermitian metric |
| Kähler data | $\Omega$<br>hol vol form | $P := \sum z_v pr_v + \sum [T_\alpha^*, T_\alpha]$<br>$z_v \in \mathbb{H}$; $v \in Q_0$ |
| Flow $\mathcal{F}$ | $\dot{L} = \mathrm{Arg}\Omega_L$ | $h^{-1}\dot{h} = \mathrm{Arg}P$ |
| Mass $M$ | $M(L) = \int_L \lvert\Omega\rvert$ | $M = \lvert P\rvert$ |
| Central charge | $Z(L) = \int_L \Omega$ | $Z = \sum z_v \chi(E_v)$ |
| Kähler potential | $dS_{\mathbb{C}}(f) = \int_L \Omega f$ | $S_{\mathbb{C}} = \sum \log\det h_v + \sum T_\alpha^* T_\alpha$ |
| Harmonic metric | Fixed points of $\mathcal{F}$<br>$= \mathrm{Crit}(S_{\mathbb{C}})$<br>$=$ special Lagrangian | Fixed points of $\mathcal{F}$/rescaling<br>$= \mathrm{Crit}(S_{\mathbb{C}})$ |
| Amplitude | sup / inf Arg $\Omega_{\lvert L}$ | sup / inf Spec Arg $P$ |
| DUY theorem | ?? | King's theorem |

Good features (some proven): (i) Mass, Amp decrease with flow; (ii) BPS inequality $\lvert Z\rvert \leq M$ and (iii) "properness" of mass.

Recall:

- There is a flow on the space of "metrized objects".
- Fixed points of the flow on metrics upto rescaling are harmonic metrics. The underlying objects should be polystable.
- The "speed" of the rescaling action gives the slope/phase of the polystable object.
- For general objects, the flow should "decompose" the object into its polystable constituents, equipped with harmonic metrics.

The Harder-Narasimhan filtration only decomposes an object into *semistable* constituents.

Basic problem: Describe the decomposition (induced by the flow) of a semistable object into polystable pieces.

$\mathcal{C}$ stable $\infty$-category equipped with Bridgeland stability condition $\{\mathcal{C}_\theta^{ss}\}_{\theta\in\mathbb{R}}, Z : K_0(\mathcal{C}) \to \mathbb{C}$.

For each $\theta \in \mathbb{R}$, $\mathcal{C}_\theta^{ss}$ is an Artinian abelian category equipped with a homomorphism

$$X := exp(-i\theta)Z : K_0(\mathcal{C}_\theta^{ss}) \to \mathbb{R}$$

which is positive on non-zero objects.

# Natural filtrations

$\mathcal{A}$ Artinian abelian category; $E \in \mathcal{E}$
$0 = E_0 \subset E_1 \subset E_2 \subset \ldots E_n = E$ filtration.

1. Socle: $E_k$ is maximal containing $E_{k-1}$ such that $E_k/E_{k-1}$ is semisimple (one extreme).

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3. Let $\mathcal{A}$ be the category of pairs $(V, N)$ consisting of a vector space $V$ and a nilpotent endomorphism $N$. Then there is a unique filtration labelled by half-integers such that

$$N^k : Gr_{k/2}V \to Gr_{-k/2}V$$

is an isomorphism for all $k$. (Balanced; this is the weight/Lefschetz filtration from mixed Hodge theory).

# Balanced filtration

## Theorem (Haiden-Katzarkov-Kontsevich-P.)

*$\mathcal{A}$ Artinian abelian category, $X : K_0(\mathcal{A}) \to \mathbb{R}$, positive on non-zero objects, $E \in \mathcal{A}$. Then there exists a unique $\mathbb{R}$-filtration*

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2. *Balancing condition: $\sum_j \lambda_j X(E_j/E_{j-1}) = 0$*
3. *For any $F_j$ with $E_{j-1} \subset F_j \subset E_j$ such that $F_j/F_k$ is semisimple for $\lambda_j - \lambda_k \leq 1$, we have*

$$\sum_j \lambda_j X(F_j/E_{j-1}) \leq 0$$

# Iterated balanced filtration and asymptotics

The last condition can be formulated as stability of the filtration $F^{\lambda}E$ considered as an object in an auxillary abelian category $\rightsquigarrow$

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- *When $\mathcal{A} = Rep(Q)$ this filtration controls the asymptotic behavior of the gradient flow on the space of metrics*

$$log|h(t)| = \lambda_1 logt + \lambda_2 loglogt + \ldots + \lambda_n log^{(n)} t + O(1)$$

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Meta-principle: The asymptotic dynamics of geometric flows (e.g., mean curvature flow, Yang-Mills flow) can be reduced to the finite dimensional quiver situation using the theory of center manifolds.

# Dynamical systems from quiver representations

$Q = (Q_0, Q_1)$ quiver; $Q_0$ vertices, $Q_1$ arrows.

Quiver representation:
Vertex $i \mapsto E_i$ vector space.
Arrow $\alpha : i \to j \mapsto T_\alpha : E_i \to E_j$ operator.

Metrized quiver representation: hermitian metric $h_i$ on $E_i$
$\rightsquigarrow$ adjoint operator $T_\alpha^* : E_j \to E_i$.

Choosing "masses" $(m_i)_{i \in Q_0} \rightsquigarrow$ flow on the space of metrics.

$$m_i h_i^{-1} \dot{h_i} = \sum_{\alpha : i \to j} h_i^{-1} T_\alpha^* h_j T_\alpha - \sum_{\alpha : k \to i} T_\alpha h_j^{-1} T_\alpha^* h_i$$

This is asymptotic to the previous flow when the central charge takes values in a ray (the positive reals).

$X$ compact Riemann surface, Kähler form $\omega$, $E$ finite dimensional holomorphic vector bundle on $X$, and $h$ a hermitian metric on $E$. Consider the flow:

$$h^{-1}\partial_t h = -2i(\Lambda F - \lambda)$$

**Theorem (Haiden-Katzarkov-Kontsevich-P.)**

*There is a canonical filtration $F^k E =: E_k$ on $E$ labelled by*

$$\beta_k \in \mathbb{R}t \oplus \mathbb{R}\log t \oplus \mathbb{R}\log\log t \oplus \cdots \simeq \mathbb{R}^\infty$$

*such that*

$$||\log h(t)_{|E_k}|| = \beta_k + O(1)$$

$E_k/E_{k-1}$ is a sum of stable bundles of slope $\mu_k$ given by

$$\beta_k = 4\pi \left(\int_X \omega\right)^{-1} (\mu_k - \mu(E))t + \dots$$

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The original system of n differential equations reduces to an identical system of n-1 equations in one less variable upon passing to logarithmic time $x_1 = s := \log t$. .

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$$S(x) := \sum_{\alpha: i \to j} c_\alpha e^{x_j - x_i}$$

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$\rightsquigarrow$ gradient flow of S with respect to metric

$$m_i \dot{x}_i = \sum_{\alpha : i \to j} c_\alpha e^{x_j - x_i} - \sum_{\alpha : k \to i} c_\alpha e^{x_i - x_k}$$

Claim: The asymptotic behavior of these dynamical systems is controlled by iterated logarithms.

# An example

$$m_1 \bullet \xrightarrow{c} \bullet m_2$$

Gradient flow:

$$m_1 \dot{x}_1 = c e^{x_2 - x_1}$$

$$m_2 \dot{x}_2 = -c e^{x_2 - x_1}$$

Solution:

$$x_1(t) = \frac{m_2}{m_1 + m_2} \log t + \log c_1$$

$$x_2(t) = -\frac{m_1}{m_1 + m_2} \log t + \log c_2$$

Where

$$\frac{c_2}{c_1} = \frac{m_1 m_2}{c(m_1 + m_2)}$$

# General case

Ansatz: $x_i = v_i \log t + b_i$;  $v_i, b_i \in \mathbb{R}$.

$$\rightsquigarrow \frac{m_i v_i}{t} = \sum_{\alpha: i \to j} c_\alpha t^{v_j - v_i} e^{b_j - b_i} - \sum_{\alpha: k \to i} c_\alpha t^{v_i - v_k} e^{b_i - b_k}$$

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Interested in asymptotic behavior of $x_i$ upto $O(1)$, so look for solutions $x_i(t)$ of the differential equation correct upto terms in $L^1(\mathbb{R})$.

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$\implies$ $m_i v_i = \sum_{\alpha : i \to j} c_\alpha u_\alpha - \sum_{\alpha : k \to i} c_\alpha u_\alpha$, and $u_\alpha = 0$ if $v_i - v_j > 1$.

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Here $u_\alpha := e^{b_j - b_i} \in \mathbb{R}_{>0}$ for $\alpha : i \to j$.
Key observation: $u_\alpha$'s are Lagrange multipliers for a convex optimization problem.

# Balanced weight grading

$C :=$ set of admissible gradings $\subset \mathbb{R}^{G_0}$ (convex body).
$M : C \to \mathbb{R}_{\geq 0}$; $M((v_i)_i) := \sum_i m_i v_i^2$ convex function.

## Proposition

*TFAE*

1. $(v_i)_i$ *minimizes* $M$.
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3. *(i)* $\sum_i m_i v_i = 0$

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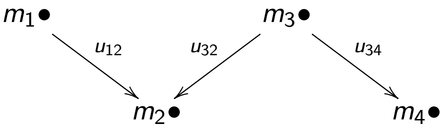
3. *(i)* $\sum_i m_i v_i = 0$ *and (ii)* $\sum_{i \in E} m_i v_i \leq 0$ *for all* $E \subset G_0$ *such that* $i \in G_0$ *and* $\exists \alpha : i \to j \implies j \in E$ *"slope semistability".*

Unique grading satisying (1)-(3) is called the balanced weight grading.

# Iterated balanced weight grading

There are walls in the parameter space of $m_i$'s (recall: $(m_i)_i \in \mathbb{R}^{G_0}_{>0}$ parametrize certain metrics on $\mathbb{R}^{G_0}$) along which the some of the $u_\alpha = 0$ for $\alpha : i \to j$ with $v_i - v_j = 1$.

Simplest example:



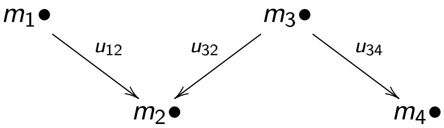

It turns out $u_{32} = \frac{m_2 m_3 - m_1 m_4}{\sum_i m_i}$.

Wall: $m_2 m_3 = m_1 m_4$..

# Iterated balanced weight grading

There are walls in the parameter space of $m_i$'s (recall: $(m_i)_i \in \mathbb{R}^{G_0}_{>0}$ parametrize certain metrics on $\mathbb{R}^{G_0}$) along which the some of the $u_\alpha = 0$ for $\alpha : i \to j$ with $v_i - v_j = 1$.

Simplest example:




It turns out $u_{32} = \frac{m_2 m_3 - m_1 m_4}{\sum_i m_i}$.

Wall: $m_2 m_3 = m_1 m_4$..

$\rightsquigarrow$ Iterate procedure along a certain subgraph $\rightsquigarrow$ asymptotics governed by iterated logarithms along wall.