Gradient flows, iterated logarithms, and semistability

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joint work with Fabian Haiden, Ludmil Katzarkov, and Maxim Kontsevich

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This talk is based on:
- arXiv:1706.01073
- arXiv:1802.04123

Goals:
- Describe the asymptotic behavior of the flows discussed in the previous talk (in special cases)
- Describe a canonical refinement of the Harder-Narasimhan filtration, and its relation to the asymptotic behavior of the flow
<table>
<thead>
<tr>
<th>Category</th>
<th>Fuk($X, \omega$)</th>
<th>Rep($Q$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Object</td>
<td>Lag upto isotopy</td>
<td>(${E_v}<em>v, {T</em>\alpha}_{\alpha \in \text{Arr}(Q)}$)</td>
</tr>
<tr>
<td>Metrized object</td>
<td>Lagrangian</td>
<td>($E_v, h_v$), $h_v$ hermitian metric</td>
</tr>
<tr>
<td>Kähler data</td>
<td>(\Omega)</td>
<td>(P := \sum z_v \text{pr}<em>v + \sum [T^*</em>\alpha, T_\alpha])</td>
</tr>
<tr>
<td></td>
<td>hol vol form</td>
<td>(z_v \in \mathbb{H}; v \in Q_0)</td>
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<tr>
<td>Flow (\mathcal{F})</td>
<td>(\dot{L} = \text{Arg} \Omega_L)</td>
<td>(h^{-1}h = \text{Arg}P)</td>
</tr>
<tr>
<td>Mass (M)</td>
<td>(M(L) = \int_L</td>
<td>\Omega</td>
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<tr>
<td>Central charge</td>
<td>(Z(L) = \int_L \Omega)</td>
<td>(Z = \sum z_v \chi(E_v))</td>
</tr>
<tr>
<td>Kähler potential</td>
<td>(dS_C(f) = \int_L \Omega f)</td>
<td>(S_C = \sum \log \det h_v + \sum T^*<em>\alpha T</em>\alpha)</td>
</tr>
<tr>
<td>Harmonic metric</td>
<td>Fixed points of (\mathcal{F}) = Crit((S_C)) = special Lagrangian</td>
<td>Fixed points of (\mathcal{F}/\text{rescaling}) = Crit((S_C))</td>
</tr>
<tr>
<td>Amplitude</td>
<td>(\sup / \inf \text{Arg} \Omega</td>
<td>_L)</td>
</tr>
<tr>
<td>DUY theorem</td>
<td>??</td>
<td>King’s theorem</td>
</tr>
</tbody>
</table>

Good features (some proven): (i) Mass, Amp decrease with flow; (ii) BPS inequality \(|Z| \leq M\) and (iii) “properness” of mass.
Recall:

- There is a flow on the space of “metrized objects”.
- Fixed points of the flow on metrics upto rescaling are harmonic metrics. The underlying objects should be polystable.
- The “speed” of the rescaling action gives the slope/phase of the polystable object.
- For general objects, the flow should “decompose” the object into its polystable constituents, equipped with harmonic metrics.

The Harder-Narasimhan filtration only decomposes an object into semistable constituents.

**Basic problem:** Describe the decomposition (induced by the flow) of a semistable object into polystable pieces.
$\mathcal{C}$ stable $\infty$-category equipped with Bridgeland stability condition 
$\{C^{ss}_\theta\}_{\theta \in \mathbb{R}}$, $Z : K_0(\mathcal{C}) \to \mathbb{C}$.

For each $\theta \in \mathbb{R}$, $C^{ss}_\theta$ is an Artinian abelian category equipped with a homomorphism

$$X := \exp(-i\theta)Z : K_0(C^{ss}_\theta) \to \mathbb{R}$$

which is positive on non-zero objects.
Natural filtrations

A Artinian abelian category; $E \in \mathcal{E}$
$0 = E_0 \subset E_1 \subset E_2 \subset \ldots E_n = E$ filtration.

1. Socle: $E_k$ is maximal containing $E_{k-1}$ such that $E_k/E_{k-1}$ is semisimple (one extreme).

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2. **Cosocle**: $E_k$ is minimal contained in $E_{k+1}$ such that $E_{k+1}/E_k$ is semisimple (another extreme).
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3. Let \( \mathcal{A} \) be the category of pairs \((V, N)\) consisting of a vector space \( V \) and a nilpotent endomorphism \( N \). Then there is a unique filtration labelled by half-integers such that

\[ N^k : \text{Gr}_{k/2} V \to \text{Gr}_{-k/2} V \]

is an isomorphism for all \( k \). (Balanced; this is the weight/Lefschetz filtration from mixed Hodge theory).
Balanced filtration

Theorem (Haiden-Katzarkov-Kontsevich-P.)

\( \mathcal{A} \) Artinian abelian category, \( X : \mathbb{K}_0(\mathcal{A}) \to \mathbb{R} \), positive on non-zero objects, \( E \in \mathcal{A} \). Then there exists a unique \( \mathbb{R} \)-filtration

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labelled by

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2. **Balancing condition**: $\sum \lambda_j X(E_j / E_{j-1}) = 0$
3. For any $F_j$ with $E_{j-1} \subset F_j \subset E_j$ such that $F_j / F_k$ is semisimple for $\lambda_j - \lambda_k \leq 1$, we have
   $$\sum_{j} \lambda_j X(F_j / E_{j-1}) \leq 0$$
Iterated balanced filtration and asymptotics

The last condition can be formulated as stability of the filtration $F^\lambda E$ considered as an object in an auxiliary abelian category.

**Theorem (Haiden-Katzarkov-Kontsevich-P.)**

- Iterating the construction of the previous theorem gives a canonical filtration of $E$ labelled by $\mathbb{R}^\infty$ equipped with the lexicographic order.

Meta-principle: The asymptotic dynamics of geometric flows (e.g., mean curvature flow, Yang-Mills flow) can be reduced to the finite dimensional quiver situation using the theory of center manifolds.
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\log |h(t)| = \lambda_1 \log t + \lambda_2 \log \log t + \ldots + \lambda_n \log^{(n)} t + O(1)
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**Meta-principle:** The asymptotic dynamics of geometric flows (e.g., mean curvature flow, Yang-Mills flow) can be reduced to the finite dimensional quiver situation using the theory of center manifolds.
Dynamical systems from quiver representations

\( Q = (Q_0, Q_1) \) quiver; \( Q_0 \) vertices, \( Q_1 \) arrows.

Quiver representation:
Vertex \( i \) \( \mapsto \) \( E_i \) vector space.
Arrow \( \alpha : i \to j \mapsto T_\alpha : E_i \to E_j \) operator.

Metrized quiver representation: hermitian metric \( h_i \) on \( E_i \)
\( \mapsto \) adjoint operator \( T_{\alpha}^* : E_j \to E_i \).

Choosing “masses” \( (m_i)_{i \in Q_0} \mapsto \) flow on the space of metrics.

\[
m_i h_i^{-1} \dot{h}_i = \sum_{\alpha : i \to j} h_i^{-1} T_{\alpha}^* h_j T_\alpha - \sum_{\alpha : k \to i} T_\alpha h_j^{-1} T_{\alpha}^* h_i
\]

This is asymptotic to the previous flow when the central charge takes values in a ray (the positive reals).
$X$ compact Riemann surface, Kähler form $\omega$, $E$ finite dimensional holomorphic vector bundle on $X$, and $h$ a hermitian metric on $E$. Consider the flow:

$$h^{-1} \partial_t h = -2i(\Lambda F - \lambda)$$

**Theorem (Haiden-Katzarkov-Kontsevich-P.)**

There is a canonical filtration $F^k E =: E_k$ on $E$ labelled by

$$\beta_k \in \mathbb{R} t \oplus \mathbb{R} \log t \oplus \mathbb{R} \log \log t \oplus \cdots \cong \mathbb{R}^\infty$$

such that

$$\| \log h(t) \|_{E_k} = \beta_k + O(1)$$

$E_k / E_{k-1}$ is a sum of stable bundles of slope $\mu_k$ given by

$$\beta_k = 4\pi \left( \int_X \omega \right)^{-1} (\mu_k - \mu(E)) t + \ldots$$
Iterated Logarithms from a dynamical system

\[ \log^{(1)}(t) := \log t \]
\[ \log^{(n)}(t) := \log(\log^{(n-1)} t) \]
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The original system of n differential equations reduces to an identical system of n-1 equations in one less variable upon passing to logarithmic time \( x_1 = s := \log t \).
1 dim representations

Input:

1. Directed acyclic graph $G = (G_0, G_1)$
1 dim representations

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3. Weights $(c_\alpha)_{\alpha \in G_1} \in \mathbb{R}_{>0}^{G_1} \leadsto$ action functional $S : \mathbb{R}^{G_0} \to \mathbb{R}$

$$S(x) := \sum_{\alpha : i \to j} c_\alpha e^{x_j - x_i}$$
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$\rightsquigarrow$ gradient flow of $S$ with respect to metric

$$m_i \dot{x}_i = \sum_{\alpha:i \to j} c_\alpha e^{x_j - x_i} - \sum_{\alpha:k \to i} c_\alpha e^{x_i - x_k}$$

Claim: The asymptotic behavior of these dynamical systems is controlled by iterated logarithms.
An example

$m_1 \bullet \frac{c}{\bullet} m_2$

Gradient flow:

$m_1 \dot{x}_1 = ce^{x_2 - x_1}$

$m_2 \dot{x}_2 = -ce^{x_2 - x_1}$

Solution:

$x_1(t) = \frac{m_2}{m_1 + m_2} \log t + \log c_1$

$x_2(t) = -\frac{m_1}{m_1 + m_2} \log t + \log c_2$

Where

\[
\frac{c_2}{c_1} = \frac{m_1 m_2}{c(m_1 + m_2)}
\]
General case

Ansatz: \( x_i = v_i \log t + b_i; \quad v_i, b_i \in \mathbb{R}. \)

\[ \sim \sim \frac{m_i v_i}{t} = \sum_{\alpha : i \to j} c_{\alpha} t^{v_j - v_i} e^{b_j - b_i} - \sum_{\alpha : k \to i} c_{\alpha} t^{v_i - v_k} e^{b_i - b_k} \]
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Interested in asymptotic behavior of \( x_i \) upto \( O(1) \), so look for solutions \( x_i(t) \) of the differential equation correct upto terms in \( L^1(\mathbb{R}) \).

\[\Rightarrow \text{ only powers } t^{\leq -1} \text{ in RHS} \]
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\[
\implies v_i - v_j \geq 1 \text{ if } \exists \alpha : i \to j \iff (v_i)_i \text{ is an admissible grading (defn)}
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Comparing coefficients

\[ \implies m_i v_i = \sum_{\alpha : i \to j} c_\alpha u_\alpha - \sum_{\alpha : k \to i} c_\alpha u_\alpha, \text{ and } u_\alpha = 0 \text{ if } v_i - v_j > 1. \]

Here \( u_\alpha := e^{b_j - b_i} \in \mathbb{R}_{>0} \) for \( \alpha : i \to j \).
General case

Ansatz: \( x_i = v_i \log t + b_i; \quad v_i, b_i \in \mathbb{R}. \)

\[
\sum_{\alpha: i \to j} m_i v_i \frac{t^{v_i - v_j} e^{b_j - b_i}}{t} - \sum_{\alpha: k \to i} c_{\alpha} t^{v_i - v_k} e^{b_i - b_k}
\]

Interested in asymptotic behavior of \( x_i \) upto \( O(1) \), so look for solutions \( x_i(t) \) of the differential equation correct upto terms in \( L^1(\mathbb{R}) \).

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\( \implies \) \( v_i - v_j \geq 1 \) if \( \exists \alpha : i \to j \iff (v_i)_i \) is an admissible grading (defn)

comparing coefficients

\( \implies m_i v_i = \sum_{\alpha: i \to j} c_{\alpha} u_{\alpha} - \sum_{\alpha: k \to i} c_{\alpha} u_{\alpha}, \) and \( u_{\alpha} = 0 \) if \( v_i - v_j > 1. \)

Here \( u_{\alpha} := e^{b_j - b_i} \in \mathbb{R}_{>0} \) for \( \alpha : i \to j. \)

Key observation: \( u_{\alpha}'s \) are Lagrange multipliers for a convex optimization problem.
Balanced weight grading

\[ C := \text{set of admissible gradings} \subset \mathbb{R}^{G_0} \text{ (convex body).} \]
\[ M : C \to \mathbb{R}_{\geq 0}; \quad M((v_i)_i) := \sum_i m_i v_i^2 \text{ convex function.} \]

**Proposition**

TFAE

1. \((v_i)_i\) minimizes \(M\).
2. \(\exists u_\alpha \geq 0 \quad \alpha \in G_1 \text{ satisfying} \)

\[ m_i v_i = \sum_{\alpha : i \to j} c_\alpha u_\alpha - \sum_{\alpha : k \to i} c_\alpha u_\alpha \]
Balanced weight grading

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Proposition

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1. $(v_i)_i$ minimizes $M$.
2. $\exists u_\alpha \geq 0 \ \alpha \in G_1$ satisfying

$$m_i v_i = \sum_{\alpha : i \rightarrow j} c_\alpha u_\alpha - \sum_{\alpha : k \rightarrow i} c_\alpha u_\alpha$$

3. (i) $\sum_i m_i v_i = 0$
Balanced weight grading

\[ C := \text{set of admissible gradings} \subset \mathbb{R}^{G_0} \text{ (convex body)}. \]

\[ M : C \rightarrow \mathbb{R}_{\geq 0}; \quad M((v_i)_i) := \sum_i m_i v_i^2 \text{ convex function}. \]

**Proposition**

**TFAE**

1. \((v_i)_i\) minimizes \(M\).
2. \(\exists u_\alpha \geq 0 \ \alpha \in G_1\) satisfying
   \[
   m_i v_i = \sum_{\alpha : i \rightarrow j} c_\alpha u_\alpha - \sum_{\alpha : k \rightarrow i} c_\alpha u_\alpha
   \]
3. (i) \(\sum_i m_i v_i = 0\) and (ii) \(\sum_{i \in E} m_i v_i \leq 0\) for all \(E \subset G_0\) such that \(i \in G_0\) and \(\exists \alpha : i \rightarrow j \implies j \in E\) “slope semistability”.

Unique grading satisfies (1)-(3) is called the balanced weight grading.
Iterated balanced weight grading

There are walls in the parameter space of $m_i$’s (recall: $(m_i)_i \in \mathbb{R}_{>0}^{|G_0|}$ parametrize certain metrics on $\mathbb{R}^{G_0}$) along which the some of the $u_\alpha = 0$ for $\alpha : i \to j$ with $v_i - v_j = 1$.

Simplest example:

\[
\begin{array}{c}
m_1 \bullet \\
\downarrow^{u_{12}} & \downarrow^{u_{32}} \\
m_2 \bullet & m_3 \bullet \\
\uparrow^{u_{34}} & \downarrow^{u_{34}} \\
m_4 \bullet
\end{array}
\]

It turns out $u_{32} = \frac{m_2 m_3 - m_1 m_4}{\sum_i m_i}$.

Wall: $m_2 m_3 = m_1 m_4$.
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→ Iterate procedure along a certain subgraph → asymptotics governed by iterated logarithms along wall.