Strong G_2-structures with torsion

Anna Fino

Universitá di Torino & Florida International University

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Geometries with torsion

Question

Why Geometries with torsion?

Metric connections having totally skew-symmetric torsion and special holonomy have strong relations with

- supersymmetric string theories and supergravity;
- generalized Ricci flow and generalized geometry.

Remark

In particular, the geometry of NS-5 brane solutions of type II supergravity theories is generated by a metric connection with skew-symmetric torsion.

Metric connections

(M^n, g) Riemannian manifold

∇ any connection on M $\hookrightarrow$ we can write

$$\nabla_X Y = \nabla^{LC}_X Y + A(X, Y).$$

Question

If ∇ is a metric connection, what we can say about A?

Remark

∇ is metric $\iff g(A(X,Y),Z) + g(A(X,Z),Y) = 0, \quad \forall X, Y, Z$

$\iff A \in \mathcal{A}^g := \mathbb{R}^n \otimes \Lambda^2 \mathbb{R}^n$

Since a metric connection ∇ is uniquely determined by its torsion

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

we have:

$$A \in \mathcal{A}^g \longleftrightarrow T \in \mathcal{T} \cong \mathcal{A}^g.$$

The space $\mathcal{T} \cong \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)$ has dimension $\frac{n^2(n-1)}{2}$ and for $n \geq 3$ decomposes under the action of $O(n)$ as

$$\mathbb{R}^n \oplus \Lambda^3(\mathbb{R}^n) \oplus \mathcal{T}',$$

where

$$\mathcal{T}' = \{T \in \mathcal{T} | \, \sigma_{X,Y,Z} T(X,Y,Z) = 0, \sum_{i=1}^n T(e_i, e_i, X) = 0, \forall X, Y, Z\}.$$

[E. Cartan, 1925].

Remark

For $n = 2$ the space $\mathbb{R}^2 \otimes \Lambda^2(\mathbb{R}^2) \cong \mathbb{R}^2$ is irreducible.

The skew-symmetric torsion case

Definition

If $A \in \Lambda^3(\mathbb{R}^n)$, ∇ is called connection with skew-symmetric torsion.

Proposition (Cartan, 1926; Agricola, 2006)

A connection ∇ is metric and geodesic preserving $\iff$ T is a 3-form. In this case $2A = T$ and the ∇-Killing vector fields coincide with the Riemannian Killing fields.

$\hookrightarrow \nabla_X Y = \nabla^{LC}_X Y + \frac{1}{2} T(X, Y, \cdot)$.

A key example

G/H reductive homogeneous space

$\hookrightarrow$ $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (as vector space) and $\mathfrak{m}$ is an $Ad(H)$-invariant subspace, i.e. $Ad(H)(\mathfrak{m}) \subseteq \mathfrak{m}$.

A positive definite scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ is naturally reductive if

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0, \quad \forall X, Y, Z \in \mathfrak{m}.$$

$\hookrightarrow$ On a naturally reductive homogeneous space the (canonical) connection ∇^1 with torsion $T^1(X, Y, Z) = -\langle [X, Y]_{\mathfrak{m}}, Z \rangle$ is a metric connection with skew symmetric torsion.

Remark

- $\nabla^1 T^1 = 0$, $\nabla^1 R^1 = 0$.
- $T^1 = 0$ for a symmetric space.
- ∇^1 belongs to a family of metric connections ∇^t with skew-symmetric torsion $T^t(X,Y) = -\langle t[X,Y]_{\mathfrak{m}}, Z\rangle$.

Example

Lie group G with a bi-invariant metric g, i.e.

$$g(ad_X Y, Z) + g(Y, ad_X Z) = 0, \quad \forall X, Y, Z \in \mathfrak{g}.$$

$\hookrightarrow$ $\nabla^{LC}_X Y = \frac{1}{2}[X,Y]$ and the connections $\nabla^\lambda_X Y = \lambda[X,Y]$ are metric and with skew-symmetric torsion

$$T^{\nabla^\lambda}(X,Y) = (2\lambda - 1)[X,Y].$$

Note that $R^{\nabla^\lambda}(X,Y)Z = \lambda(1-\lambda)[Z,[X,Y]]$ $\hookrightarrow$

∇^λ is flat for $\lambda = 0, 1$ ($\pm$ connection) [Cartan, Schouten, 1926].

Curvature of connections with skew-symmetric torsion

(M^n, g, ∇, T) with T a 3-form

$$\hookrightarrow \quad g(\nabla_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} T(X, Y, Z).$$

As a consequence

$$\begin{aligned} dT(X, Y, Z, V) &= \sigma_{X,Y,Z}(\nabla_X T)(Y, Z, V) \\ &\quad -(\nabla_V T)(X, Y, Z) + 2\sigma^T(X, Y, Z, V), \end{aligned}$$

where $\sigma^T(X, Y, Z, V) = \frac{1}{2} \sum_{i=1}^n (i_{e_i} T) \wedge (i_{e_i} T)(X, Y, Z, V)$.

$$
\begin{aligned}
R^{LC}(X,Y,Z,V) &= R^{\nabla}(X,Y,Z,V) - \tfrac{1}{2}(\nabla_X T)(Y,Z,V) \\
&\quad \tfrac{1}{2}(\nabla_Y T)(X,Z,V) - \tfrac{1}{4} g(T(X,Y),T(Z,V)) \\
&\quad -\tfrac{1}{4}\sigma^T(X,Y,Z,V).
\end{aligned}
$$

As a consequence

$$
\mathrm{Ric}^g(X,Y) = \mathrm{Ric}^{\nabla}(X,Y) + \frac{1}{2}\delta_g T(X,Y) - \frac{1}{4}\sum_{i=1}^{n} g(T(e_I,X),T(Y,e_i)),
$$

where $\delta_g(T)$ is the co-differential of T with respect to g.

Remark

In particular, the skew-symmetric part of Ric^{∇} is given by

$$
\mathrm{Ric}^{\nabla}(X,Y) - \mathrm{Ric}^{\nabla}(Y,X) = -\delta_g T(X,Y).
$$

Flat case

Theorem (Cartan, Schouten, 1926; Agricola, Friedrich, 2010)

Let (M, g) be Riemannian manifold admitting a flat metric connection ∇ with torsion $T \in \Omega^3(M)$. Then

$$3dT = 2\sigma_T, \quad \nabla^{1/3} T = 0, \quad \nabla^{1/3}\sigma_T = 0,$$

where $\nabla^{1/3}$ is the metric connection with torsion $\frac{1}{3}T$.

Moreover, $\|T\|$ and $Scal^{\nabla}$ are both constant and the sectional curvature satisfy $K(X, Y) \geq 0$, for every X, Y.

Corollary

If (M, g) is a simply connected, complete and irreducible Riemannian manifold admitting a flat metric connection ∇ with totally skew-symmetric torsion $T \neq 0 \hookrightarrow M$ is a compact irreducible symmetric space and g is Einstein with $Scal^g = \frac{3}{2}\|T\|^2$.

Since σ^T is $\nabla^{\frac{1}{3}}$-parallel there are two cases:

• $\sigma_T \equiv 0 \hookrightarrow (M, g)$ is isometric to a simple compact Lie group;

• $\sigma_T \not\equiv 0$ (i.e. $dT \neq 0$) $\hookrightarrow (M, g)$ is isometric to S^7 [Cartan, Schouten, 1926].

Remark

• The proof by Agricola and Friedrich uses

$$\mathrm{Lie}\{i_X T \mid X \in T_pM\} \subset \mathfrak{so}(T_pM), \quad \forall p \in M.$$

• A different proof has been given by D'Atri, Nickerson and Wolf using the classification of irreducible compact symmetric spaces with vanishing Euler characteristic.

Problem

What we can say about Riemannian manifolds (M, g) admitting a metric connection ∇ with skew-symmetric torsion T satisfying

$$dT = 0 \quad \text{and} \quad Ric^{\nabla} = 0?$$

In general, given (M, g, T) with T a 3-form we can define a "Bismut connection" ∇^B given by

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} T(X, Y, Z).$$

If $dT = 0$, the pair (g, T) is called a generalized metric, since it is associated to generalized metrics on exact Courant algebroids.

Remark

In general $\text{Ric}^B := \text{Ric}(\nabla^B)$ is not symmetric.

Proposition

If $dT = 0$, then

$$\text{Ric}^B = \text{Ric}^g - \frac{1}{4} T_g^2 - \frac{1}{2}\delta_g T, \quad \text{Scal}^B = \text{Scal}^g - \frac{1}{4}|T|^2,$$

where T_g^2 is the non-negative definite symmetric 2-tensor given by

$$T_g^2(X, Y) := g(i_X T, i_Y T).$$

As a consequence

- the symmetric part of Ric^B is $\text{Ric}^g - \frac{1}{4} T_g^2$;
- the skew-symmetric part of Ric^B is $-\frac{1}{2}\delta_g T$.

$\mathrm{Ric}^B = 0$ if and only if $\delta_g T = 0$ (T is harmonic) and $\mathrm{Ric}^g = \frac{1}{4} T^2$.

Definition

The generalized metric (g, T) such that $\mathrm{Ric}^B = 0$ is called Bismut Ricci flat (BRF).

Remark

BRF metrics are also called generalized Einstein and they are the fixed points of the generalized Ricci flow

$$\begin{cases} \partial_t g(t) = -2\mathrm{Ric}^{g(t)} + \frac{1}{2}(T(t))^2_{g(t)}, \\ \partial_t T(t) = -d\delta_{g(t)} T(t). \end{cases}$$

Theorem (Agricola, Friedrich; Garcia-Fernandez, Streets)

The only compact simply connected Riemannian manifolds admitting a flat ∇^B with closed torsion are compact semisimple Lie groups (G, g_b, T_b), where g_b is a bi-invariant metric and $T_b := g_b([\cdot,\cdot],\cdot)$ (the Cartan 3-form).

Question (Generalized Alekseevsky-Kimefield)

M homogeneous Riemannian manifold with an invariant generalized metric (g, T). If $\mathrm{Ric}(\nabla^B) = 0$, is ∇^B flat?

The answer is no!

Example (Podestá, Raffero, 2022)

There exists a family of compact homogeneous spaces $M_{p,q} = SU(2) \times SU(2)/K_{p,q}$, $p \geq q$, $gcd(p,q) = 1$, where

$$K_{p,q} = \{\mathrm{diag}(z^p, z^{-p}), \mathrm{diag}(z^q, z^{-q}) \in SU(2) \times SU(2)\} \cong S^1.$$

admitting an invariant non-flat BRF (g, T).

Remark

- The metric g on $M_{p,q}$ is not Einstein and $\nabla^B T \neq 0$.
- $M_{p,q}$ are principal bundles over $SU(2)/T^2 \cong S^2 \times S^2$ and they are diffeomorphic to $S^3 \times S^2$.
- There exist infinitely many compact homogeneous spaces admitting an invariant non-flat BRF. They are given by $G \times G/K_{diag}$ where G is a compact semisimple Lie group and (G, K) is a symmetric pair of inner type.
- No examples are known in dimension 6 and 7.

Examples (Lauret, Will, 2022)

$M = G/K$ with G a product of two compact simple Lie groups satisfying $b_3(M) > 0$ and some technical hypothesis. The lowest dimensional example is $M^{10} = \frac{SU(2)\times SU(3)}{S^1}$.

In case of an H-structure?

M oriented manifold of dimension n

$\mathcal{F}(M)$ frame bundle

Remark

If the structure group of $\mathcal{F}(M)$ has a reduction to a closed subgroup $H \subset SO(n) \hookrightarrow$ an H-structure $\mathcal{R} \subset \mathcal{F}(M)$.

The existence of an H-connection ∇ with totally skew-symmetric torsion can be characterized in terms of the intrinsic torsion

$$\xi_X Y = \nabla^{LC}_X Y - \nabla_X Y$$

which measures the defect for ∇^{LC} to be an H-connection [Friedrich, Ivanov, 2002].

Remark

In particular, $\xi = 0 \Leftrightarrow \mathrm{hol}(\nabla^{LC}) \subseteq H$.

G_2 case

(M^7, φ) 7-manifold with a G_2-structure

Poinwise: $\varphi = e^{127} + e^{347} + e^{135} - e^{245} - e^{146} - e^{236} + e^{567}$.

Theorem (Friedrich, Ivanov, 2002)

(M^7, φ) has a G_2-connection ∇ with totally skew-symmetric torsion if and only if $d(*\varphi) = \theta \wedge *\varphi$, where θ is a 1-form called the Lee form. Moreover, ∇ is unique.

If there exists such ∇ we will say that φ is a G_2T-structure.

Remark

In general for a G_2-structure φ we have

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \quad d(*\varphi) = 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi.$$

So (M^7, φ) is G_2T if and only if $\tau_2 = 0$.

Remark

Since $g_\varphi(X,Y)dV_\varphi = \frac{1}{6} i_X\varphi \wedge i_Y\varphi \wedge \varphi \hookrightarrow \nabla g_\varphi = 0$, so ∇ is a metric connection with totally skew-symmetric torsion.

The 3-form $T = \frac{1}{6} * (d\varphi \wedge \varphi)\varphi - *d\varphi + *(\theta \wedge \varphi)$ is the torsion of ∇ and the Lee form is given by $\theta = -\frac{1}{3} * (*d\varphi \wedge \varphi)$.

Remark

In terms of the intrinsic torsion forms τ_i:

$$T = \frac{1}{6}\tau_0\varphi + *(\tau_1 \wedge \varphi) - \tau_3, \quad \theta = 4\tau_1.$$

In particular:

$T = 0 \iff \nabla^{LC}\varphi = 0 \iff d\varphi = 0$ and $d(*\varphi) = 0$.

$\theta = 0 \iff \varphi$ is coclosed.

Relation with the Hull-Strominger system

(M^7, φ) compact spin manifold, P principal G-bundle over M^7

The Hull-Strominger system is given by

$$\begin{cases} d\varphi \wedge \varphi = 0, \quad d(*\varphi) = -4df \wedge *\varphi, \\ F_A \wedge *\varphi = 0, \quad R_\nabla \wedge *\varphi = 0, \\ d * (-d\varphi - 4df \wedge \varphi) = \frac{\alpha'}{4} (\mathrm{tr}(F_A \wedge F_A) - (\mathrm{tr}(R_\nabla \wedge R_\alpha)), \end{cases}$$

where $f \in \mathcal{C}^\infty(M^7)$, ∇ and A are connections respectively on TM and P and $\alpha' > 0$ is the square of the string lenght.

Remark

- If φ is a solution, then $d(*\varphi) = \theta \wedge *\varphi$, with $\theta = -4df$ and $d\varphi \wedge \varphi = 0$, $T = -*(d\varphi + 4df \wedge \varphi)$.
- If additionally $dT = 0$, then $f = const \hookrightarrow T = 0 \hookrightarrow \varphi$ is parallel [Gauntlett, Martelli, Parkis, Waldram; Clarke, Garcia-Fernandez, Tipler].

If we have only $d\theta = 0$, this is not anymore true!

$\hookrightarrow$ New system of equations (twisted G_2-equations):

$$d\varphi \wedge \varphi = 0, \quad d(*\varphi) = \theta \wedge *\varphi, \quad d\theta = 0, \quad dT = 0.$$

Definition

A G_2 T-structure φ such $dT = 0$ will be called strong.

Remark

The twisted G_2-equations are the G_2-analogue of the twisted CY equations for $SU(n)$-structures (ω, Ψ), introduced in relation to $(0,2)$-symmetry:

$$d\Psi = \theta \wedge \Psi, \quad dd^c\omega = 0, \quad d\theta_\omega = 0,$$

where $\theta_\omega = -J\partial\omega$ is the Lee form [Garcia-Fernandez, Rubio, Shahbazi, Tipler].

Proposition (F, Merchan, Raffero, 2023)

If N^{2n}, $n = 2$ or 3, has a $SU(n)$-structure solving the twisted CY equations, then $M^7 = N^{2n} \times \mathbb{R}^{7-2n}$ has a strong G_2T-structure solving the twisted G_2-equations with $\theta = \theta_\omega$ and $T = d^c\omega$.

Using the expression of the scalar curvature in terms of τ_i

Proposition (F, Merchan, Raffero, 2023)

If φ is a strong G_2T-structure, then

$$Scal(g_\varphi) = 10\delta\tau_1 + \frac{49}{24}\tau_0^2 + 24|\tau_1|^2.$$

Some restrictions

• On a unimodular Lie group, if φ is a left-invariant strong G_2T-structure, then $Scal(g_\varphi) \geq 0$

$\hookrightarrow$ A 7-dim compact solvmanifold $\Gamma \backslash G$ cannot admit any invariant non-parallel strong G_2T-structure.

• For $n = 2, 3$ every invariant $SU(n)$-structures solving the twisted CY equations on $2n$-dim solvmanifolds must be CY.

• If (ω, Ψ) is a $SU(3)$-structure on N^6, such that

$$\varphi = \omega \wedge \eta + Re(\Psi)$$

is a strong G_2T-structure on $N^6 \times S^1$ with Lee form $\theta = \lambda\eta$, then $b_3(N^6) > 0$.

Theorem (F, Merchan, Raffero, 2023)

Let M^7 be a connected compact homogeneous space of the almost effective action of a connected Lie group.

If M^7 has an invariant strong G_2T-structure, then M^7 is (up to a covering) diffeomorphic either to $S^3 \times T^4$ or to $S^3 \times S^3 \times S^1$.

For the proof we use the classification of compact homogeneous spaces admitting an invariant G_2-structure [Lě, Munir; Reidegeld] and we apply our general results on strong G_2T-structures.

Remark

On $S^3 \times T^4 = SU(2) \times U(1)^4$, $S^3 \times S^3 \times S^1 = SU(2)^2 \times U(1)$ we also construct strong G_2T-structures solving the twisted G_2-equations (the associated G_2-connection ∇ is flat!).

Some open problems

- Do there exist 7-manifolds admitting a strong G_2T-structure inducing a non-flat G_2-connection ∇?
- If φ_0 is a strong G_2T-structure on a 7-manifold M^7, does the Laplacian coflow

$$\partial_t \psi(t) = \Delta_{\psi(t)} \psi(t), \quad \psi(0) = \psi_0 = *\varphi_0,$$

preserve the strong G_2T condition? Is there any geometric flow preserving the condition $\tau_2 = 0$?

Remark

For the two examples $S^3 \times T^4$ and $S^3 \times S^3 \times S^1$ giving solutions to the twisted G_2-equations, the solution $\psi(t)$ exists for all positive time and defines a strong G_2T-structure.

Moreover, along the flow the 7-manifolds split as a Riemannian product $Y^3 \times X^4$, with Y^3 an associative submanifold and X^4 a coassociative submanifold.

THANK YOU VERY MUCH FOR THE ATTENTION!!