Collapsed Manifolds With Local Ricci Bounded Covering Geometry

Xiaochun Rong
Abstract

Collapsed manifolds with local bounded covering geometry (i.e., sectional curvature bounded in absolute value) has been well-studied; the basic discovery by Cheeger-Fukaya-Gromov is the existence of a compatible local nilpotent symmetry structures whose orbits point to collapsed directions.

In this talk, we generalize the structural result to collapsed manifolds with local Ricci bounded covering geometry. Our construction of local nilpotent symmetry structures does not reply on the work of Cheeger-Fukaya-Gromov; which also gives an alternative approach to the structural results.
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Motivation and Questions

• A sequence Calabi-Yau or $G_2$-manifolds, $M_n \to X$, $\text{vol}(M_n) \to 0$.

• Conjecture?: $\exists$ manifold, $X_0 \subseteq X$ (of large measure), and $U_i \subseteq M_i$ s.t $\exists$ a torus/nilpotent fibration, $f_i: U_i \to X_0$.

• Questions: How to identify $X_0$? Why torus/nilpotent fibration?
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Collapsible with Bounded Sectional Curvature

- $M$ is $\epsilon$-collapsed $\iff \forall x \in M, \text{vol}(B_1(x)) < \epsilon$, a bound on 'curvature'.

- (Scaling) All compact flat manifolds can collapse to a point.

- (Berger, 80's) Let $S_1 \to S_3 \to S_2$ be the Hopf fibration. $\Rightarrow$ one parameter family of metrics, $g = \epsilon^2 ds^2 + (ds^2)_{\perp}$, satisfies $\epsilon^2 \leq \sec g \leq 4 - 3\epsilon^2$, $\text{vol}(S_3, g) \to 0$, $d_{GH}(S_3, g, S_2) \to 0$.

- (Gromov) (Inhomogeneous scaling) A nilpotent manifold, $N/\Gamma$, with $N, N_1, \cdots, N_k, e, N_i + 1 = [N, N_i]$. Construct an inhomogeneous scaling $g$, by $\epsilon, \epsilon^2, \cdots, \epsilon^k$, $|\sec g| \leq 1$, $\text{diam}(g) \to 0$. 

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Nilpotent Fibrations on Collapsed Manifolds

• (Gromov, 78), also (Ruh, 82)

\[ \exists \epsilon(n), c(n) > 0 \text{ s.t. if a compact } n\text{-manifold } M \text{ satisfies } |\text{sec } M| \leq 1, \text{ diam}(M) < \epsilon(n), \Rightarrow M \text{ is diffeo. to an infra-nilmanifold } N/\Gamma, \Gamma < N \rtimes \text{Aut}(N), [\Gamma, \Gamma] \cap N \leq c(n). \]

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Let \( M_i \) GH \( \to Y \) s.t. \( |\text{sec } M_i| \leq 1 \) and \( \text{diam}(M_i) \leq d \).

1) If \( X \) is a Riem., \( \exists \) a fibration, \( f_i: M_i \to X \), s.t. \( f_i \) is \( \epsilon_i \)-GHA, \( f_i \)-fiber is infra-nilmanifold.

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\[ \text{Be} - c(n) r(f_i(x_i)) \subseteq f_i(Br(x_i)) \subseteq \text{Be} c(n) r(f_i(x_i)). \]
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\[ f_i \text{ is } \epsilon_i\text{-GHA, } f_i\text{-fiber is infra-nilmanifold, } |\text{II}_{f_i}| \leq c(n). \]

2. If \( X \) is not Riem., \( \Rightarrow \exists \) a singular fibration, \( f_i : M_i \to X \), s.t.

\[ f_i \text{ is } \epsilon_i\text{-GHA, } f_i\text{-fiber is infra-nilmanifold,} \]
Nilpotent Fibrations on Collapsed Manifolds

- (Gromov, 78), also (Ruh, 82)

∃ ϵ(n), c(n) > 0 s.t. if a compact n-manifold M satisfies

|sec_M| ≤ 1, diam(M) < ϵ(n),

⇒ M is diffeo. to an infra-nilmanifold i.e., N/Γ,

Γ < N ⋊ Aut(N), [Γ, Γ ∩ N] ≤ c(n).

- (Fukaya, Cheeger-Fukaya-Gromov, 86, 91).

Let M_i \xrightarrow{GH} Y s.t. |sec_{M_i}| ≤ 1 and diam(M_i) ≤ d.

(1) If X is a Riem., ⇒ ∃ a fibration, f_i : M_i → X, s.t.

f_i is ϵ_i-GHA, f_i-fiber is infra-nilmanifold, |II_{f_i}| ≤ c(n).

(2) If X is not Riem., ⇒ ∃ a singular fibration, f_i : M_i → X, s.t.

f_i is ϵ_i-GHA, f_i-fiber is infra-nilmanifold,

\[ B_{e^{-c(n)}r}(f_i(x_i)) \subseteq f_i(B_r(x_i)) \subseteq B_{e^{c(n)}r}(f_i(x_i)). \]
Collapsed with Weak Curvature Conditions?

• (Yamaguchi, 92) If \( M_i \xrightarrow{GH} N \) s.t. \( \sec M_i \geq -1 \) and \( \text{diam}(M_i) \leq d \), \( \exists \) fibration maps, \( f_i : M_i \to N \), s.t. \( f_i \) is \( \epsilon_i \)-GHA, \( f_i \)-fiber is almost non-negatively curved.

• (Anderson, 90) \( (M_n, g_i) \xrightarrow{GH} T_{n-1} \) s.t. \( |\text{Ric} g_i| \leq \delta \) (no fibration!)

• (Dai-Wei-Ye (96), Petersen-Wei-Ye (99), Naber-Zhang (16), Huang-Kong-Rong-Xu (18)) A collapsed manifold with bounded Ricci curvature + 'various' conditions, \( \Rightarrow \) either \( M \) satisfies \( |\sec M| \leq 1 \), or \( \exists \) a nearby metric \( g \epsilon \) with \( |\sec g \epsilon| \leq 1 \).

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Local Rewinding Volume of Balls

Local rewinding of a $r$-ball, $B_r(x) \subseteq M$, is $\tilde{B}_r(\tilde{x})$, where $\pi: (\tilde{B}_r(x), \tilde{x}) \to (B_r(x), x)$ denotes the Riemannian universal cover.

We call $\text{vol}(B_r(\tilde{x}))$ the local rewinding volume of $B_r(x)$, denoted $\tilde{\text{vol}}(B_r(x)) = \text{vol}(B_r(x))$.

We call the $C_\alpha$-Harmonic radius of $B_r^2(\tilde{x})$ the local rewinding $C_\alpha$-Harmonic radius of $B_r(x)$, denoted by $\tilde{r}_{\alpha h}(B_r(x)) = \inf\{r_{\alpha h}(y), y \in B_r^2(\tilde{x})\}$.

$\tilde{r}_{\alpha h}(B_r(x)) \geq r_0 > 0 \Rightarrow \tilde{\text{vol}}(B_r(x)) \geq \delta(r_0) > 0$; (⇒).
Local Rewinding Volume of Balls

- (Local rewinding of $B_r(x)$)

\[ \text{Local Rewinding Volume of Balls} \]

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\[ (\Leftarrow) \]
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Local Rewinding Volume of Balls

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Duke University Workshop on The Structure of Collapsed Special Holonomy Space, April 11, 2018
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Collapsing With Local Bounded Covering Geometry

∃ \epsilon(n), i(n) > 0 s.t. if an n-manifold M satisfies |\text{sec} M| \leq 1, then for any x \in M, the local rewinding of B_{\epsilon(n)}(x) satisfies injrad(B_{\epsilon(n)}(\tilde{x})) \geq i(n).

(S^3, g_{\epsilon}) \toGH [0, \pi_2], |\text{sec} g_{\epsilon}| \leq 1.

Notice \tilde{\text{vol}}(B_{1/10}(x)) \geq v > 0, \text{rank}(\pi_1(B_{1/10}(x))) = 1, x \to 0.
Collapsing With Local Bounded Covering Geometry

- (Cheeger-Fukaya-Gromov, 91)

\[ \exists \epsilon(n), i(n) > 0 \text{ s.t.} \]
Collapsing With Local Bounded Covering Geometry

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\[ (S^3, g_\epsilon) \xrightarrow{GH} \left[ 0, \frac{\pi}{2} \right], \quad |\sec_{g_\epsilon}| \leq 1. \]
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$\exists \epsilon(n), i(n) > 0$ s.t. if an $n$-manifold $M$ satisfies $|\sec_M| \leq 1$, \Rightarrow for any $x \in M$, the local rewinding of $B_{\epsilon(n)}(x)$ satisfies

$$\text{injrad}(B_{\epsilon(n)}(\tilde{x})) \geq i(n).$$

- $(S^3, g_\epsilon) \xrightarrow{GH} [0, \frac{\pi}{2}], |\sec_{g_\epsilon}| \leq 1.$

Notice

$$\tilde{\text{vol}}(B_{\frac{1}{10}}(x)) \geq v > 0, \quad \text{rank}(\pi_1(B_{\frac{1}{10}}(x))) = 1, \quad x \to 0.$$
Ricci Local Bounded Covering Geometry

• $M$ satisfies LRBCG, if there are constants, $\rho > 0$ s.t.
  \[ \text{Ric}_M \geq - (n - 1), \]
  \[ \tilde{\text{vol}}(B_\rho(x)) \geq \delta(\rho) > 0, \quad \forall x \in M. \]

• Local bounded covering geometry $\Rightarrow$ local Ricci bounded covering geometry, \(\Leftarrow\).

Lem 1. \(M\) complete, $\text{Ric}_M \geq - (n - 1)$. If $M$ admits a Killing $N$-structure whose orbits point to all collapsed directions, $\Rightarrow$ $\exists \rho > 0$ s.t.
  \[ \tilde{\text{vol}}(B_\rho(x)) \geq \delta(\rho) > 0, \quad \forall x \in M. \]

• In this talk, "$|\text{Ric}_M| \leq n - 1". Most of work can be extended to "$\text{Ric}_M \geq - (n - 1)$".
Ricci Local Bounded Covering Geometry

• (Ricci local bounded covering geometry)

M satisfies LRBCG, if there are constants, \(\rho > 0\) s.t.

\[
\text{Ric}_M \geq -(n-1), \quad \tilde{\text{vol}}(B_{\rho}(x)) \geq \delta(\rho) > 0, \quad \forall x \in M.
\]

• Local bounded covering geometry \(\Rightarrow\) local Ricci bounded covering geometry, \((\Leftarrow)\).

Lem 1. \(M\) complete, \(\text{Ric}_M \geq -(n-1)\). If \(M\) admits a Killing N-structure whose orbits point to all collapsed directions, \(\Rightarrow\) \(\exists \rho > 0\) s.t.

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\tilde{\text{vol}}(B_{\rho}(x)) \geq \delta(\rho) > 0, \quad \forall x \in M.
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• In this talk, \(|\text{Ric}_M| \leq n-1\). Most of work can be extended to \(\text{Ric}_M \geq -(n-1)\).
Ricci Local Bounded Covering Geometry

- \( (\text{Ricci local bounded covering geometry}) \)

\( M \) satisfies LRBCG,
Ricci Local Bounded Covering Geometry

• (Ricci local bounded covering geometry)

\( M \) satisfies LRBCG, if there are constants, \( \rho > 0 \) s.t.
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$M$ satisfies LRBCG, if there are constants, $\rho > 0$ s.t.

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Ricci Local Bounded Covering Geometry

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- Local bounded covering geometry $\Rightarrow$ local Ricci bounded covering geometry, ($\Leftarrow \Rightarrow$).

**Lem 1.** $M$ complete, $\text{Ric}_M \geq -(n - 1)$. If $M$ admits a Killing $N$-structure whose orbits point to all collapsed directions,
Ricci Local Bounded Covering Geometry

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(Ricci local bounded covering geometry)

$\mathcal{M}$ satisfies LRBCG, if there are constants, $\rho > 0$ s.t.

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Ricci Local Bounded Covering Geometry

- (Ricci local bounded covering geometry) $M$ satisfies LRBCG, if there are constants, $\rho > 0$ s.t.
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Main Results

Thm A. (Almost Ricci flat with non-collapsed universal cover)

\[ \exists \epsilon(n, v), c(n) > 0 \text{s.t. if a compact } n \text{-manifold } M \text{satisfies } |\text{Ric}_M| \leq n - 1, \text{ then } \tilde{\text{vol}}(B_1(p)) \geq v > 0, \text{ then } \text{diam}(M) < \epsilon(n, v), \Rightarrow M \text{ is diffeo. to an infra-nilmanifold.} \]

\[ \text{• Thm A does not hold if removing } \tilde{\text{vol}}(B_1(p)) \geq v. \]

Cor. (Gromov, 78)

\[ \exists \epsilon(n), c(n) > 0 \text{s.t. if a compact } n \text{-manifold } M \text{satisfies } |\text{sec}_M| \leq 1, \text{ then } \text{diam}(M) < \epsilon(n), \Rightarrow M \text{ is diffeo. to an infra-nilmanifold.} \]
Main Results

**Thm A.** (Almost Ricci flat with non-collapsed universal cover)
Main Results

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$\exists \epsilon(n, v), c(n) > 0$ s.t.

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Main Results

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∃ ε(n, ν), c(n) > 0 s.t. if a compact n-manifold M satisfies
Main Results

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Main Results

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Main Results

**Thm A.** (Almost Ricci flat with non-collapsed universal cover)

∃ \( \epsilon(n, \nu), c(n) > 0 \) s.t. if a compact \( n \)-manifold \( M \) satisfies

\[
|\text{Ric}_M| \leq n - 1, \quad \tilde{\text{vol}}(B_1(p)) \geq \nu > 0, \quad \text{diam}(M) < \epsilon(n, \nu),
\]

\( \Rightarrow \) \( M \) is diffeo. to an infra-nilmanifold.

- **Thm A** does not hold if removing \( \tilde{\text{vol}}(B_1(p)) \geq \nu \).

**Cor.** (Gromov, 78)

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\]

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Main Results
Main Results

**Thm B. (Fibration)**

\[ M_i \overset{GH}{\longrightarrow} N, \]

\[ \text{diam}(f_i\text{-fiber}) \to 0. \]
Main Results

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Main Results

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\[\Rightarrow \exists \text{ a fibration, } f_i : M_i \to Y, \text{ s.t.}\]
Main Results

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- \( \text{diam}(f_i\text{-fiber}) \to 0. \)
Partially Ricci Local Bounded Covering Geometry

\[ \text{For } \mu > 0, \text{ let } V_{-\mu}(R) = \{ x \in \mathbb{R}, x \rightarrow \tilde{\text{vol}}(B_{\eta}(x)) \geq \delta(\eta) > 0, x \in M_i \}. \]

\[ \text{If } V_{-\mu} \neq \emptyset, \Rightarrow \exists \text{ open } U_i(\mu) \subset M_i \text{ s.t. } U_i(\mu) \text{ GH } \rightarrow V_{-\mu}(R). \]

\[ \text{• (Gross-Wilson) Collapsing Calabi-Yau metrics, } (K_3, g_t) \text{ GH } \rightarrow (S^2, d_\infty), \text{ and } S = S^2 - R \text{ consists of isolated cone points.} \]

\[ \text{Given small } \eta > 0, \Rightarrow V_{-\mu}(R) = S^2 - B_{\eta}(S). \]
Partially Ricci Local Bounded Covering Geometry

• (Collapsing with partially Ricci bounded covering geometry)

\[ M_i \xrightarrow{GH} X, \]
Partially Ricci Local Bounded Covering Geometry

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For \( \mu > 0 \), let

\[ V_{-\mu}(\mathcal{R}) = \{ x \in \mathcal{R}, \ x_i \to x, \ \tilde{\text{vol}}(B_\eta(x_i)) \geq \delta(\eta) > 0, \ x_i \in M_i \}. \]
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If \( V_{-\mu} \neq \emptyset \), \( \Rightarrow \) \( \exists \) open \( U_i(\mu) \subset M_i \) s.t.

\[ U_i(\mu) \xrightarrow{GH} V_{-\mu}(\mathcal{R}). \]
Partially Ricci Local Bounded Covering Geometry

• (Collapsing with partially Ricci bounded covering geometry)

\[ \mathcal{M}_i \xrightarrow{GH} X, \quad |\text{Ric}_{\mathcal{M}_i}| \leq n - 1. \]

For \( \mu > 0 \), let

\[ \mathcal{V}_{-\mu}(\mathcal{R}) = \{ x \in \mathcal{R}, \ x_i \to x, \ \tilde{\text{vol}}(B_{\eta}(x_i)) \geq \delta(\eta) > 0, \ x_i \in \mathcal{M}_i \}. \]

If \( \mathcal{V}_{-\mu} \neq \emptyset \), \( \Rightarrow \exists \) open \( U_i(\mu) \subset \mathcal{M}_i \) s.t.

\[ U_i(\mu) \xrightarrow{GH} \mathcal{V}_{-\mu}(\mathcal{R}). \]

• (Gross-Wilson) Collapsing Calabi-Yau metrics,
Partially Ricci Local Bounded Covering Geometry

- (Collapsing with partially Ricci bounded covering geometry)

\[ M_i \xrightarrow{GH} X, \quad |\text{Ric}_{M_i}| \leq n - 1. \]

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If \( V_{-\mu} \neq \emptyset \), then there exists an open \( U_i(\mu) \subset M_i \) such that

\[ U_i(\mu) \xrightarrow{GH} V_{-\mu}(\mathcal{R}). \]

- (Gross-Wilson) Collapsing Calabi-Yau metrics,

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Partially Ricci Local Bounded Covering Geometry

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If \( \mathcal{V}_{-\mu} \neq \emptyset \), \( \Rightarrow \exists \) open \( U_i(\mu) \subset M_i \) s.t.

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• (Gross-Wilson) Collapsing Calabi-Yau metrics,

\[ (K3, g_t) \xrightarrow{GH} (S^2, d_\infty), \]

and \( \mathcal{S} = S^2 - \mathcal{R} \) consists of isolated cone points. Given small \( \eta > 0 \), \( \Rightarrow \)

\[ \mathcal{V}_{-\mu}(\mathcal{R}) = S^2 - B_\eta(\mathcal{S}). \]
Main Results

Thm C.

Einstein $E_{\text{GH}} \rightarrow X$, $V - \mu(\mathbb{R}) \neq \emptyset$.

$V - \mu(\mathbb{R})$ is a Riemannian manifold, and $\exists$ a fiber bundle map, $f_i: U_i(\mu) \rightarrow V - \mu(\mathbb{R})$, $f_i$ is $\epsilon_i$-GHA, $f_i$-fiber is infra-nilmanifold, $\text{diam}(f_i$-fiber) $\rightarrow 0$. 

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Main Results

**Thm C.** (Partial fibration)

\[ \text{Einstein } M \xrightarrow{GH} X, \quad \text{and } V^- \mu(\mathbb{R}) \neq \emptyset. \]

\[ \Rightarrow V^- \mu(\mathbb{R}) \text{ is a Riemannian manifold, and } \exists \text{ a fiber bundle map, } f_i: U_i(\mu) \rightarrow V^- \mu(\mathbb{R}), \]

\[ f_i \text{ is } \epsilon_i^{-}\text{GHA}, f_i^{-}\text{fiber is infra-nilmanifold}, \quad \text{diam}(f_i^{-}\text{fiber}) \rightarrow 0. \]
Main Results

**Thm C. (Partial fibration)**

\[ \text{Einstein } M_i \xrightarrow{GH} X, \quad V_{-\mu}(\mathcal{R}) \neq \emptyset. \]
Main Results

Thm C. (Partial fibration)

\[
\text{Einstein } M_i \stackrel{GH}{\longrightarrow} X, \quad V_{-\mu}(\mathcal{R}) \neq \emptyset.
\]

\[\Rightarrow V_{-\mu}(\mathcal{R}) \text{ is a Riemannian manifold,}\]
Main Results

**Thm C. (Partial fibration)**

Einstein $M_i \overset{GH}{\to} X$, $V_{-\mu}(\mathcal{R}) \neq \emptyset$.

$\Rightarrow V_{-\mu}(\mathcal{R})$ is a Riemannian manifold, and $\exists$ a fiber bundle map, $f_i : U_i(\mu) \to V_{-\mu}(\mathcal{R})$, 
Main Results

**Thm C.** (Partial fibration)

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Main Results

**Thm C.** *(Partial fibration)*

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Main Results

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\text{Einstein } M_i \xrightarrow{GH} X, \quad V_{-\mu}(R) \neq \emptyset.
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\[f_i : U_i(\mu) \rightarrow V_{-\mu}(R),\]

\[f_i \text{ is } \epsilon_i\text{-GHA, } f_i\text{-fiber is infra-nilmanifold, } \text{diam}(f_i\text{-fiber}) \rightarrow 0.\]
A Criterion for Nilmanifolds

Lem. 1
A compact manifold \( M \) is diffeomorphic to a nilmanifold \( \iff \)

1. \( M \) admits an iterated bundles over tori, i.e.,
   \[
   M_1 \to M \to T_k^1, \\
   M_2 \to M_1 \to T_k^2, \\
   \ldots \\
   M_s \to M_{s-1} \to \ldots \to \text{pt},
   \]

2. \( \pi_1(M_i)/\pi_1(M) \) s.t. the holonomy representation via conjugation, \( \phi_i: \pi_1(M) \to \text{Aut}(\pi_1(M_{i-1})/\pi_1(M_i)) \), is trivial, \( 1 \leq i \leq s \).
A Criterion for Nilmanifolds

• (Nakayama, 14, Belegragdek) A manifold \( M \) is diffeom. to nilpotent iff \( M \) admits an iterated principle circle bundles,

\[ S^1 \to M \to M_1, \]
A Criterion for Nilmanifolds

- (Nakayama, 14, Belegragdek) A manifold $M$ is diffeom. to nilpotent iff $M$ admits an iterated principle circle bundles,

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Lem. 1

A cpct. $M$ is diffeo. to a nilmanifold $\iff$

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$$M_1 \to M \to T_k_1, \quad M_2 \to M_1 \to T_k_2,$$

2. $\pi_1(M_i)/\pi_1(M)$ s.t. the holonomy representation via conjugation,

$$\phi_i: \pi_1(M) \to \text{Aut}(\pi_1(M_{i-1})/\pi_1(M_i)),$$

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A Criterion for Nilmanifolds

● (Nakayama, 14, Belegragdek) A manifold $M$ is diffeom. to nilpotent iff $M$ admits an iterated principle circle bundles, $S^1 \to M \to M_1, \quad S^1 \to M_1 \to M_2, \quad \cdots$, 

Lem. 1 A cpct. $M$ is diffeo. to a nilmanifold $\iff$

1. $M$ admits an iterated bundles over tori i.e., $M_1 \to M \to T_{k_1}$, $M_2 \to M_1 \to T_{k_2}$, $\cdots$, $\{pt\} \to M \to T_{k_s}$. 

2. $\pi_1(M_i) \to \pi_1(M)$ s.t the holonomy representation via conjugation, $\phi_i : \pi_1(M) \to \text{Aut}(\pi_1(M_{i-1})/\pi_1(M_i))$, is trivial, $1 \leq i \leq s$. 

A Criterion for Nilmanifolds

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A Criterion for Nilmanifolds

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Lem. 1 A cpct. $M$ is diffeo. to a nilmanifold
A Criterion for Nilmanifolds

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**Lem. 1** A cpct. \( M \) is diffeo. to a nilmanifold \( \Leftrightarrow \)
A Criterion for Nilmanifolds

- (Nakayama, 14, Belegragdek) A manifold $M$ is diffeom. to nilpotent iff $M$ admits an iterated principle circle bundles,

$$S^1 \to M \to M_1, \quad S^1 \to M_1 \to M_2, \quad \cdots, \quad S^1 \to M_n \to \text{pt}.$$ 

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A Criterion for Nilmanifolds

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(2) $\pi_1(M_i) \triangleleft \pi_1(M)$ s.t the holonomy representation via conjugation, $\phi_i : \pi_1(M) \to \text{Aut}(\pi_1(M_{i-1})/\pi_1(M_i))$, is trivial, $1 \leq i \leq s$. 
Tools — The Cheeger-Colding Theory

- A theory on degeneration of Ricci limit spaces.

- (Regular point) \( x \in X \) is called regular \( \iff \) tangent cone at \( x \) is unique and isometric to \( \mathbb{R}^k \).

- (Cheeger-Colding, 96) Let \( M_i \rightharpoonup X \) s.t. \( \operatorname{Ric} M_i \geq - (n - 1) \).

  \[ \Rightarrow \]

  1. \( \mathbb{R} \) is dense in \( X \) and has a full Radon measure determined by the renormalized volume.

  2. If \( \operatorname{vol} M_i \geq v > 0 \),

     \[ \Rightarrow \]

     \( \operatorname{Haus}^n (\mathbb{R}) = \operatorname{Haus}^n (X) \).

  3. If \( \operatorname{vol} M_i \geq v > 0 \) & \( \operatorname{Ric} M_i \leq n - 1 \),

     \[ \Rightarrow \]

     \( \mathbb{R} \) is \( C^{1,\alpha} \)-manifold.

  4. (Colding-Naber, 12) \( \operatorname{Isom} (X) \) is a Lie group.
Tools — The Cheeger-Colding Theory

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Tools — The Cheeger-Colding Theory

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(\[
M_i \xrightarrow{GH} X \text{ s.t. } \text{Ric} \geq -(n-1) \implies \]
\[
\text{(1) } \mathbb{R} \text{ is dense in } X \text{ and has a full Radon measure determined by the renormalized volume.}
\]
\[
\text{(2) If } \text{vol}(M_i) \geq v > 0 \implies \text{Haus}_n(\mathbb{R}) = \text{Haus}_n(X).
\]
\[
\text{(3) If } \text{vol}(M_i) \geq v > 0 \text{ & } \text{Ric} \leq n-1 \implies \mathbb{R} \text{ is } C^{1,\alpha} \text{-manifold.}
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Tools — The Cheeger-Colding Theory

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Tools — The Cheeger-Colding Theory

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- (Cheeger-Colding, 96)
  
  Let $M_i \xrightarrow{GH} X$ s.t. $\text{Ric}_{M_i} \geq -(n - 1)$.

  \begin{enumerate}
  \item $\mathcal{R}$ is dense in $X$ and has a full Radon measure determined by the renormalized volume.
  \item $\text{Haus}^n(\mathcal{R}) = \text{Haus}^n(X)$.
  \item If $\text{vol}(M_i) \geq v > 0$, then $\text{Haus}^n(\mathcal{R}) = \text{Haus}^n(X)$.
  \item (Colding-Naber, 12) $\text{Isom}(X)$ is a Lie group.
  \end{enumerate}
Tools — The Cheeger-Colding Theory

- A theory on degeneration of Ricci limit spaces.
- (Regular point) \( x \in X \) is called regular \( \iff \) tangent cone at \( x \) is unique and isometric to \( \mathbb{R}^k \).

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Tools — The Cheeger-Colding Theory

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\[
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1) \text{\( \mathcal{R} \) is dense in \( X \) and has a full Radon measure determined by the renormalized volume.} \\
2) \text{If vol}(M_i) \geq v > 0, \Rightarrow \text{Haus}^n(\mathcal{R}) = \text{Haus}^n(X). \\
3) \text{If vol}(M_i) \geq v > 0 \& \text{Ric}_{M_i} \leq n-1,
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Tools — The Cheeger-Colding Theory

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Tools — The Cheeger-Colding Theory

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  Let $M_i \xrightarrow{GH} X$ s.t. $\text{Ric}_{M_i} \geq -(n - 1)$. $\Rightarrow$

  (1) $\mathcal{R}$ is dense in $X$ and has a full Radon measure determined by the renormalized volume.
  (2) If $\text{vol}(M_i) \geq \nu > 0$, $\Rightarrow$ $\text{Haus}^n(\mathcal{R}) = \text{Haus}^n(X)$.
  (3) If $\text{vol}(M_i) \geq \nu > 0$ & $\text{Ric}_{M_i} \leq n - 1$, $\Rightarrow$ $\mathcal{R}$ is $C^{1,\alpha}$-manifold.
  (4) (Colding-Naber, 12) $\text{Isom}(X)$ is a Lie group.
Center of mass: \( M_{\text{cpct. Riem.}} \), \( P(M) := \text{space of prob. measures on } M \).

\( \forall \mu \in P(M) \), the minimum point of the function, \( x \to \int_M \frac{1}{2} d(x, y)^2 \mu \), is called the center of mass of \( \mu \).

Let \( \rho = \text{Convrad}(M) \).

\( \Rightarrow \forall \) countable subset \( A \subset B \), \( \rho(x) \) has a unique center of mass.

Lemma (Palais, 61; Grove-Karcher, 73)

\( M_{\text{Riem.}}, G_{\text{cpct. Lie group}}, \) two \( G \)-actions on \( M \); one by isometries.

\( \Rightarrow \) two \( G \)-actions are \( C_0 \)-close relative to bounds on \( \text{sec}(M), \text{injrad}(M) \), \( \Rightarrow \) two \( G \)-actions are conjugate.
Tools — Center of Mass

• Center of mass:

\[ P(M) := \text{space of prob. measures on } M. \]

\[ \forall \mu \in P(M), \text{the minimum point of the function}, \]

\[ x \rightarrow \int_M 1 \frac{1}{2} d(\mu), \]

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\[ \text{Lem. (Palais, 61; Grove-Karcher, 73)} \]

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Center of mass: \(M\) cpct. Riem., \(P(M) := \) space of prob. measures on \(M\).
Tools — Center of Mass

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Center of mass: $\mathcal{M}$ cpct. Riem., $P(\mathcal{M}) :=$ space of prob. measures on $\mathcal{M}$. $\forall \mu \in P(\mathcal{M})$, the minimum point of the function,

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Center of mass: \( M \) cpct. Riem., \( P(M) := \) space of prob. measures on \( M \). \( \forall \mu \in P(M) \), the minimum point of the function,

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Tools — Center of Mass

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Tools — Center of Mass

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**Lem.** (Palais, 61; Grove-Karcher, 73)

$\mathcal{M}$ Riem., $\mathcal{G}$ cpct. Lie group, two $\mathcal{G}$-actions on $\mathcal{M}$; one by isometries.
Tools — Center of Mass

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**Lem.** (Palais, 61; Grove-Karcher, 73)

$M$ Riem., $G$ cpct. Lie group, two $G$-actions on $M$; one by isometries. If two $G$-action are $C^0$-close relative to bounds on $\text{sec}_M$, $\text{injrad}(M)$,
Tools — Center of Mass

• Center of mass: \( M \) cpt. Riem., \( P(M) := \) space of prob. measures on \( M \). \( \forall \mu \in P(M) \), the minimum point of the function,

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• Let \( \rho = \text{Convrad}(M) \). \( \Rightarrow \forall \) countable subset \( A \subset B_\rho(x) \) has a unique center of mass.

Lem. (Palais, 61; Grove-Karcher,73)

\( M \) Riem., \( G \)cpt. Lie group, two \( G \)-actions on \( M \); one by isometries. If two \( G \)-action are \( C^0 \)-close relative to bounds on \( \text{sec}_M \), \( \text{injrad}(M) \), \( \Rightarrow \) two \( G \)-actions are conjugate.
Tools — Equivariant GH-Convergence

Lem. Given \((X_i, p_i)\) GH-\(\rightarrow\) \((X, p)\), \(G_i\) closed subgp. of Isom \((X_i)\), \(\Rightarrow\) \((X_i, p_i, \Gamma_i)\) GH-\(\rightarrow\) \((X, p, G)\), closed subgp. \(G \subset\) Isom \((X)\).

Lem. If \((X_i, p_i, \Gamma_i)\) GH-\(\rightarrow\) \((X, p, G)\), \(\Rightarrow\) \((X_i/\Gamma_i, \bar{p}_i)\) GH-\(\rightarrow\) \((X/G, \bar{p})\).

Lem. Given \((\tilde{X}_i, \tilde{p}_i, \Gamma_i)\) GH-\(\rightarrow\) \((\tilde{X}, \tilde{p}, G)\) \(\mid \downarrow \pi_i \downarrow \mid \downarrow \pi\) \((X_i, p_i)\) GH-\(\rightarrow\) \(X = (X, p)\), \((X_i\text{ is cpct.})\) \(\Rightarrow\exists \epsilon > 0\) s.t. \(\Gamma_i(\epsilon)/\Gamma_i \sim = G/G_0\).
Tools — Equivariant GH-Convergence

- Equivariant GH convergence (Fukaya-Yamaguchi, 91).

Lem.
Given \((X_i, p_i)\) GH \(\to\) \((X, p)\), closed subgp. of Isom \((X_i)\), \(\implies\) \((X_i, p_i, \Gamma_i)\) GH \(\to\) \((X, p, G)\), closed subgp. \(G \subset\) Isom \((X)\).

Lem.
If \((X_i, p_i, \Gamma_i)\) GH \(\to\) \((X, p, G)\), \(\implies\) \((X_i/\Gamma_i, \bar{p}_i)\) GH \(\to\) \((X/G, \bar{p})\).

Lem.
Given \((\tilde{X}_i, \tilde{p}_i, \Gamma_i)\) GH \(\to\) \((\tilde{X}, \tilde{p}, G)\) \(\downarrow\) \((X_i, p_i)\) GH \(\to\) \((X, p)\) \(=\) \((\tilde{X}, \tilde{p})\), \((X_i\text{ is cpc})\), \(\exists\) \(\epsilon > 0\) s.t. \(\Gamma_i(\epsilon)/\Gamma_i = G/G_0\).
Tools — Equivariant GH-Convergence

• Equivariant GH convergence (Fukaya-Yamaguchi, 91).

Lem. Given $(X_i, p_i) \xrightarrow{GH} (X, p)$, $G_i$ closed subgp. of Isom$(X_i)$,
Tools — Equivariant GH-Convergence

- Equivariant GH convergence (Fukaya-Yamaguchi, 91).

Lem. Given \((X_i, p_i) \overset{GH}{\longrightarrow} (X, p), G_i \text{ closed subgp. of } \text{Isom}(X_i),\)
\Rightarrow (X_i, p_i, \Gamma_i) \overset{GH}{\longrightarrow} (X, p, G), \text{ closed subgp. } G \subset \text{Isom}(X).
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 \(\Rightarrow (X_i, p_i, \Gamma_i) \overset{GH}{\rightarrow} (X, p, G), \text{ closed subgp. } G \subset \text{Isom}(X).\)

Lem.
If \((X_i, p_i, \Gamma_i) \overset{GH}{\rightarrow} (X, p, G),\)
Tools — Equivariant GH-Convergence

- Equivariant GH convergence (Fukaya-Yamaguchi, 91).

**Lem.** Given \((X_i, p_i) \xrightarrow{GH} (X, p)\), \(G_i\) closed subgp. of \(\text{Isom}(X_i)\),
\[ \Rightarrow (X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G)\), closed subgp. \(G \subset \text{Isom}(X)\).

**Lem.**
If \((X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G)\),
\[ \Rightarrow (X_i/\Gamma_i, \bar{p}_i) \xrightarrow{GH} (X/G, \bar{p})\].

Tools — Equivariant GH-Convergence

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**Lem.** Given \((X_i, p_i) \overset{GH}{\rightarrow} (X, p)\), \(G_i\) closed subgp. of \(\text{Isom}(X_i)\), \(\Rightarrow (X_i, p_i, \Gamma_i) \overset{GH}{\rightarrow} (X, p, G)\), closed subgp. \(G \subset \text{Isom}(X)\).

**Lem.**
If \((X_i, p_i, \Gamma_i) \overset{GH}{\rightarrow} (X, p, G)\), \(\Rightarrow (X_i/\Gamma_i, \bar{p}_i) \overset{GH}{\rightarrow} (X/G, \bar{p})\).

**Lem.** Given

\[
\begin{align*}
(\tilde{X}_i, \tilde{p}_i, \Gamma_i) & \overset{GH}{\rightarrow} (\tilde{X}, \tilde{p}, G) \\
\downarrow \pi_i & \quad \quad \downarrow \pi \\
(X_i, p_i) & \overset{GH}{\rightarrow} X = (X, p),
\end{align*}
\]
- Equivariant GH convergence (Fukaya-Yamaguchi, 91).

**Lem.** Given \((X_i, p_i) \xrightarrow{GH} (X, p)\), \(G_i\) closed subgp. of \(\text{Isom}(X_i)\),
\[\Rightarrow (X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G),\] closed subgp. \(G \subset \text{Isom}(X)\).

**Lem.**
If \((X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G)\),
\[\Rightarrow (X_i/\Gamma_i, \bar{p}_i) \xrightarrow{GH} (X/G, \bar{p}).\]

**Lem.**
Given
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(\tilde{X}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} (\tilde{X}, \tilde{p}, G) \\
\downarrow \pi_i & \quad \downarrow \pi \\
(X_i, p_i) & \xrightarrow{GH} X = (X, p),
\end{align*}
\]
(X is cpct.)
Tools — Equivariant GH-Convergence

- Equivariant GH convergence (Fukaya-Yamaguchi, 91).

**Lem.** Given $(X_i, p_i) \xrightarrow{GH} (X, p)$, $G_i$ closed subgp. of $\text{Isom}(X_i)$, $\Rightarrow (X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G)$, closed subgp. $G \subset \text{Isom}(X)$.

**Lem.**

If $(X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G)$, $\Rightarrow (X_i/\Gamma_i, \bar{p}_i) \xrightarrow{GH} (X/G, \bar{p})$.

**Lem.** Given

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(\tilde{X}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G) \\
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(X_i, p_i) \xrightarrow{GH} X = (X, p),
$$

$\Rightarrow \exists \epsilon > 0$ s.t. $\Gamma_i(\epsilon) \triangleleft \Gamma_i$.  

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Tools — Equivariant GH-Convergence

- Equivariant GH convergence (Fukaya-Yamaguchi, 91).

**Lem.** Given \((X_i, p_i) \xrightarrow{GH} (X, p), G_i\) closed subgp. of \(\text{Isom}(X_i)\),

\[ (X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G), \text{closed subgp. } G \subset \text{Isom}(X). \]

**Lem.** If \((X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G), \Rightarrow (X_i/\Gamma_i, \bar{p}_i) \xrightarrow{GH} (X/G, \bar{p}).\)

**Lem.** Given

\[
\begin{align*}
(\tilde{X}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} (\tilde{X}, \tilde{p}, G) \\
\downarrow \pi_i & \quad \downarrow \pi \\
(X_i, p_i) & \xrightarrow{GH} X = (X, p),
\end{align*}
\]

\[ \Rightarrow \exists \epsilon > 0 \text{ s.t. } \Gamma_i(\epsilon) \triangleleft \Gamma_i, \Gamma_i/\Gamma_i(\epsilon) \cong G/G_0. \]
Construction of N-Structures via $\tilde{r}_h^\alpha(B_\rho(x)) \geq \delta(\rho)$
Construction of N-Structures via $\tilde{r}_h^\alpha(B_\rho(x)) \geq \delta(\rho)$

**Thm.** (Local rewinding Harmonic radius bounded below)

$\exists \epsilon(n, \rho) > 0$ s.t. if a compact $n$-manifold $M$ satisfies

$\tilde{r}_h^\alpha(B_\rho(x)) \geq \delta(\rho) > 0, \quad \text{diam}(M) < \epsilon(n, \nu),$

$\Rightarrow M$ is diffeo. to an infra-nilmanifold.
Sketch of Proof of Thm
Sketch of Proof of Thm

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G)\]

\[\downarrow \pi_i \quad \downarrow \pi\]

\[M_i \xrightarrow{GH} \text{pt.}\]

Because \(\text{vol}(B_1(\tilde{p}_i)) \geq v\), \(\Rightarrow \dim H(\tilde{X}) = n\), and \(G\) acts transitively on \(\tilde{X}\). \(\Rightarrow \) all points in \(\tilde{X}\) are regular.

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C_\alpha} (\tilde{X}, \tilde{p}, G)\]

\[\downarrow \pi_i \quad \downarrow \pi\]

\[r_i - 1 \xrightarrow{C_\alpha} (R^n, \tilde{p}', G')\]

\[M_i \xrightarrow{GH} \text{pt.}\]
Sketch of Proof of Thm

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G)\]

\[\downarrow \pi_i \quad \downarrow \pi\]

\[M_i \xrightarrow{GH} \text{pt.}\]

Because \(\text{vol}(B_1(\tilde{p}_i)) \geq v\),
Sketch of Proof of Thm

$$(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G)$$

$$M_i \xrightarrow{GH} \text{ pt.}$$

Because $\text{vol}(B_1(\tilde{p}_i)) \geq v$, $\Rightarrow \dim_H(\tilde{X}) = n$, and $G$ acts transitively on $\tilde{X}$. 
Sketch of Proof of Thm

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G)\]

\[
\begin{array}{c}
M_i \xrightarrow{GH} \text{pt.} \\
\pi_i \downarrow \quad \downarrow \pi
\end{array}
\]

Because \(\text{vol}(B_1(\tilde{p}_i)) \geq v\), \(\Rightarrow \) \(\dim_H(\tilde{X}) = n\), and \(G\) acts transitively on \(\tilde{X}\). \(\Rightarrow \) all points in \(\tilde{X}\) are regular.
Sketch of Proof of Thm

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G)\]

\[\downarrow \pi_i \quad \quad \quad \downarrow \pi\]

\[M_i \xrightarrow{GH} \text{pt.}\]

Because \(\text{vol}(B_1(\tilde{p}_i)) \geq v\), \(\Rightarrow \dim_H(\tilde{X}) = n\), and \(G\) acts transitively on \(\tilde{X}\). \(\Rightarrow\) all points in \(\tilde{X}\) are regular. \(\Rightarrow\)

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\tilde{X}, \tilde{p}, G)\]

\[\downarrow \pi_i \quad \quad \quad \downarrow \pi\]

\[M_i \xrightarrow{GH} \text{pt.}\]
Sketch of Proof of Thm

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G) \]

\[
\begin{array}{c}
\downarrow \pi_i \\
M_i
\end{array}
\begin{array}{c}
\downarrow \pi \\
\text{pt.}
\end{array}
\]

Because \(\text{vol}(B_1(\tilde{p}_i)) \geq v\), \(\Rightarrow\) \(\dim_H(\tilde{X}) = n\), and \(G\) acts transitively on \(\tilde{X}\). \(\Rightarrow\) all points in \(\tilde{X}\) are regular. \(\Rightarrow\)

\[(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\tilde{X}, \tilde{p}, G) \quad (r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G') \]

\[
\begin{array}{c}
\downarrow \pi_i \\
M_i
\end{array}
\begin{array}{c}
\downarrow \pi \\
\text{pt.}
\end{array}
\begin{array}{c}
\downarrow \pi_i \\
r_i^{-1} M_i
\end{array}
\begin{array}{c}
\downarrow \pi \\
X
\end{array}
\]

\(r_i \to 0\) as \(i \to \infty\).
Sketch of Proof of Thm

$$(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{GH} (\tilde{X}, \tilde{p}, G)$$

$$\downarrow \pi_i \quad \downarrow \pi$$

Because $\text{vol}(B_1(\tilde{p}_i)) \geq v$, $\Rightarrow \dim_H(\tilde{X}) = n$, and $G$ acts transitively on $\tilde{X}$. $\Rightarrow$ all points in $\tilde{X}$ are regular. $\Rightarrow$

$$(\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\tilde{X}, \tilde{p}, G) \quad (r_i^{-1}\tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G')$$

$$\downarrow \pi_i \quad \downarrow \pi \quad \Rightarrow \quad \downarrow \pi_i \quad \downarrow \pi$$

$M_i \xrightarrow{GH} \text{pt.} \quad r_i^{-1}M_i \xrightarrow{GH} X,$

where $r_i = \text{diam}(M_i) \to 0$ as $i \to \infty$. 
Sketch of Proof of Thm

\[
(r_i^{-1} \hat{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)
\]

\[
r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k.
\]
Sketch of Proof of Thm

\[(r_i^{-1} \hat{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)\]

\[\Rightarrow \quad \downarrow \pi_i \quad \quad \downarrow \pi \]

\[r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k.\]

**Lem. 2** Let \(M_i \xrightarrow{GH} Y\) (manifold) s.t.
Sketch of Proof of Thm

\[ (r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G) \]

\[ \Rightarrow \quad \downarrow \pi_i \quad \downarrow \pi \]

\[ r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k. \]

**Lem. 2** Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[ \tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i. \]
Sketch of Proof of Thm

\[(r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)\]

\[\Rightarrow \quad \downarrow \pi_i \quad \downarrow \pi \quad \rightarrow \quad \tilde{r}_i^{-1} \hat{M}_i \xrightarrow{GH} T^k.\]

**Lem. 2** Let \(M_i \xrightarrow{GH} Y\) (manifold) s.t.

\[\tilde{r}_h^\alpha (B_\rho (x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.\]

\[\Rightarrow \quad \exists \text{ a fiber bundle, } M_i, 1 \rightarrow M_i \xrightarrow{f_i} Y, \text{ s.t.}\]
Sketch of Proof of Thm

\[(r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)\]

\[\Rightarrow \quad \tilde{r}^\alpha_{\gamma} \left( B_{\rho}(x_i) \right) \geq \delta > 0, \quad \forall x_i \in M_i.\]

Lem. 2 Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[\Rightarrow \exists \text{ a fiber bundle, } M_{i,1} \to M_i \xrightarrow{f_i} Y, \text{ s.t. }\]

\[\tilde{r}^\alpha_{\gamma}(M_{i,1}) \geq \delta'(\rho, \delta),\]
Sketch of Proof of Thm

\[
(r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)
\]

\[
\Rightarrow \quad \downarrow \pi_i \quad \downarrow \pi
\]

\[
r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k.
\]

Lem. 2 Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[
\tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.
\]

\[
\Rightarrow \exists \text{ a fiber bundle, } M_{i,1} \rightarrow M_i \xrightarrow{f_i} Y, \text{ s.t.}
\]

\[
\tilde{r}_h^\alpha(M_{i,1}) \geq \delta'(\rho, \delta), \quad \text{diam}(M_{i,1}) \rightarrow 0.
\]
Sketch of Proof of Thm

\[ (r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G) \]

\[ \Rightarrow \quad \downarrow \pi_i \downarrow \pi \]

\[ r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k. \]

**Lem. 2** Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[ \tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i. \]

\[ \Rightarrow \exists \text{ a fiber bundle, } M_{i,1} \to M_i \xrightarrow{f_i} Y, \text{ s.t.} \]

\[ \tilde{r}_h^\alpha(M_{i,1}) \geq \delta'(\rho, \delta), \quad \text{diam}(M_{i,1}) \to 0. \]

**Proof.** (i) Assume \( \delta = \text{convrad}(T^k) < \rho. \)
Sketch of Proof of Thm

\[(r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)\]

\[\Rightarrow \quad \downarrow \pi_i \quad \quad \downarrow \pi\]

\[r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k.\]

**Lem. 2** Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[\tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.\]

\[\Rightarrow \exists \text{ a fiber bundle}, \ M_{i,1} \rightarrow M_i \xrightarrow{f_i} Y, \ s.t.\]

\[\tilde{r}_h^\alpha(M_{i,1}) \geq \delta'(\rho, \delta), \quad \text{diam}(M_{i,1}) \rightarrow 0.\]

**Proof.** (i) Assume \( \delta = \text{con} \\text{vrad}(T^k) < \rho \). Averaging diffeo.

\[\tilde{f}_i : B_1(\tilde{p}_i) \rightarrow B_1(\tilde{p}) \ \text{over} \ \Gamma_i(x) \cap B_1(\tilde{p}_i),\]
Sketch of Proof of Thm

\[(r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)\]

\[\Rightarrow \quad \downarrow \pi_i \quad \downarrow \pi \quad r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k.\]

**Lem. 2** Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[\tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.\]

\[\Rightarrow \exists \text{ a fiber bundle, } M_{i,1} \rightarrow M_i \xrightarrow{f_i} Y, \text{ s.t.} \]

\[\tilde{r}_h^\alpha(M_{i,1}) \geq \delta'(\rho, \delta), \quad \text{diam}(M_{i,1}) \rightarrow 0.\]

**Proof.** (i) Assume \( \delta = \text{convrad}(T^k) < \rho. \) Averaging diffeo.

\(\tilde{f}_i : B_1(\tilde{p}_i) \rightarrow B_1(\tilde{p})\) over \( \Gamma_i(x) \cap B_1(\tilde{p}_i), \Rightarrow f_i : B_{\frac{\rho}{2}}(x_i) \rightarrow T^k.\)
Sketch of Proof of Thm

\[
(r_i^{-1}\hat{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)
\]

⇒ \[
\begin{array}{c}
\downarrow \pi_i \\
\end{array}
\]

\[
r_i^{-1}\hat{M}_i \xrightarrow{GH} T^k.
\]

**Lem. 2** Let \(M_i \xrightarrow{GH} Y\) (manifold) s.t.

\[\tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.\]

⇒ \exists a fiber bundle, \(M_i,1 \to M_i \xrightarrow{f_i} Y\), s.t.

\[\tilde{r}_h^\alpha(M_i,1) \geq \delta'(\rho, \delta), \quad \text{diam}(M_i,1) \to 0.\]

**Proof.** (i) Assume \(\delta = \text{convrad}(T^k) < \rho\). Averaging diffeo.

\(\tilde{f}_i : B_1(\tilde{p}_i) \to B_1(\tilde{p})\) over \(\Gamma_i(x) \cap B_1(\tilde{p}_i), \Rightarrow f_i : B_{\rho/2}(x_i) \to T^k.\)

(ii) Cover \(T^k\) with finite \(B_{\rho/2}(z_i),\)
Sketch of Proof of Thm

\[(r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G)\]

\[
\Rightarrow \\
\downarrow \pi_i \\
\tilde{r}_i^{-1} \hat{M}_i \xrightarrow{GH} T^k.
\]

**Lem. 2** Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[\tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.\]

\[
\Rightarrow \exists \text{ a fiber bundle, } M_{i,1} \rightarrow M_i \xrightarrow{f_i} Y, \text{ s.t.} \\
\tilde{r}_h^\alpha(M_{i,1}) \geq \delta'(\rho, \delta), \quad \text{diam}(M_{i,1}) \rightarrow 0.
\]

**Proof.** (i) Assume \( \delta = \text{convrad}(T^k) < \rho \). Averaging diffeo.

\[\tilde{f}_i : B_1(\tilde{p}_i) \rightarrow B_1(\tilde{p}) \text{ over } \Gamma_i(x) \cap B_1(\tilde{p}_i), \Rightarrow f_i : B_{\frac{\rho}{2}}(x_i) \rightarrow T^k.\]

(ii) Cover \( T^k \) with finite \( B_{\frac{\rho}{2}}(z_i) \), by (i) \( \Rightarrow B_{\frac{\rho}{2}}(x_{ij}) \xrightarrow{f_{ij}} B_{\frac{\rho}{2}}(z_i)\).
Sketch of Proof of Thm

\[
(r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G) \\
\Rightarrow \quad \downarrow \pi_i \quad \downarrow \pi \\
(r_i^{-1} \hat{M}_i) \xrightarrow{GH} T^k.
\]

**Lem. 2** Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[
\tilde{r}_h^\alpha (B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.
\]

\( \Rightarrow \) \( \exists \) a fiber bundle, \( M_{i,1} \rightarrow M_i \xrightarrow{f_i} Y \), s.t.

\[
\tilde{r}_h^\alpha (M_{i,1}) \geq \delta'(\rho, \delta), \quad \text{diam}(M_{i,1}) \rightarrow 0.
\]

**Proof.** (i) Assume \( \delta = \text{conrad}(T^k) < \rho \). Averaging diffeo.

\( \tilde{f}_i : B_1(\tilde{p}_i) \rightarrow B_1(\tilde{p}) \) over \( \Gamma_i(x) \cap B_1(\tilde{p}_i) \), \( \Rightarrow \) \( f_i : B_{\rho/2}(x_i) \rightarrow T^k \).

(ii) Cover \( T^k \) with finite \( B_{\rho/2}(z_i) \), by (i) \( \Rightarrow \) \( B_{\rho/2}(x_{ij}) \xrightarrow{f_{ij}} B_{\rho/2}(z_i) \).

Gluing together \( \{f_{ij}\} \), via center of mass,
Sketch of Proof of Thm

\[ (r_i^{-1} \tilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{C^\alpha} (\mathbb{R}^n, \tilde{p}', G) \]

\[ \Rightarrow \quad \xrightarrow{\pi_i} \quad \xrightarrow{\pi} \quad r_i^{-1} \hat{M}_i \xrightarrow{GH} T^k. \]

**Lem. 2** Let \( M_i \xrightarrow{GH} Y \) (manifold) s.t.

\[ \tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i. \]

\[ \Rightarrow \exists \text{ a fiber bundle, } M_{i,1} \rightarrow M_i \xrightarrow{f_i} Y, \text{ s.t.} \]

\[ \tilde{r}_h^\alpha(M_{i,1}) \geq \delta'(\rho, \delta), \quad \text{diam}(M_{i,1}) \rightarrow 0. \]

**Proof.** (i) Assume \( \delta = \text{con} \text{rad}(T^k) < \rho \). Averaging diffeo.

\[ \tilde{f}_i : B_1(\tilde{p}_i) \rightarrow B_1(\tilde{p}) \text{ over } \Gamma_i(x) \cap B_1(\tilde{p}_i), \Rightarrow f_i : B_{\frac{\rho}{2}}(x_i) \rightarrow T^k. \]

(ii) Cover \( T^k \) with finite \( B_{\frac{\rho}{2}}(z_i) \), by (i) \( \Rightarrow B_{\frac{\rho}{2}}(x_{ij}) \xrightarrow{f_{ij}} B_{\frac{\rho}{2}}(z_i). \)

Gluing together \{f_{ij}\}, via center of mass, \( \Rightarrow f_i : \hat{M}_i \rightarrow T^k. \) \( \square \)
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).
Sketch of Proof of Thm

● Almost a tower of fibrations over tori (by Lem. 2).

\[ \mathcal{M}_1 \to \hat{\mathcal{M}} \to T^{k_1}, \]
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \to \hat{M} \to T^{k_1}, \quad M_2 \to \hat{M}_1 \to T^{k_2}, \]
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \to \hat{M} \to T^{k_1}, \ M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \]
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \to \hat{M} \to T^{k_1}, \quad M_2 \to \hat{M}_1 \to T^{k_2}, \quad \cdots, \quad \{pt\} \to \hat{M}_s \to T^{k_s}. \]
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \to \hat{M} \to T^{k_1}, \; M_2 \to \hat{M}_1 \to T^{k_2}, \; \cdots, \; \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

Lem. 3 \( M \) admits the bundles over tori,
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \rightarrow \hat{M} \rightarrow T^{k_1}, \ M_2 \rightarrow \hat{M}_1 \rightarrow T^{k_2}, \ \cdots, \ \{\text{pt}\} \rightarrow \hat{M}_s \rightarrow T^{k_s}. \]

**Lem. 3** \( M \) admits the bundles over tori, \( \Rightarrow \hat{M} \equiv \mathbb{R}^n. \)
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \to \hat{M} \to T^{k_1}, \ M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \ \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

**Lem. 3** \( M \) admits the bundles over tori, \( \Rightarrow \hat{M} \ \text{diffeo} \sim \mathbb{R}^n. \)

**Proof.** Induction on \( s \), starting \( s = 2: \ T^{k_1} \to M \to T^{k_2}. \)

\[
\begin{array}{ccc}
T^{k_1} & \longrightarrow & \pi^* \hat{M} \longrightarrow \mathbb{R}^k \\
\downarrow \text{id} & \downarrow \pi_i & \downarrow \pi \\
T^{k_1} & \longrightarrow & \hat{M} \longrightarrow \text{GH} \longrightarrow T^k.
\end{array}
\]
Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \to \hat{M} \to T^{k_1}, \ M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

**Lem. 3** \( M \) admits the bundles over tori, \( \Rightarrow \hat{M} \cong \mathbb{R}^n. \)

**Proof.** Induction on \( s \), starting \( s = 2: T^{k_1} \to M \to T^{k_2}. \)

\[
\begin{array}{ccc}
T^{k_1} & \longrightarrow & \pi^* \hat{M} \\
\downarrow \text{id} & & \downarrow \pi_i \\
T^{k_1} & \longrightarrow & \hat{M}
\end{array}
\]

\[ \Rightarrow \ 
\begin{array}{ccc}
\pi^* \hat{M} & \cong & \mathbb{R}^k \times T^{k_1}, \\
\end{array}
\]

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- Almost a tower of fibrations over tori (by Lem. 2).

\[ \mathcal{M}_1 \to \hat{\mathcal{M}} \to T^{k_1}, \mathcal{M}_2 \to \hat{\mathcal{M}}_1 \to T^{k_2}, \ldots, \{\text{pt}\} \to \hat{\mathcal{M}}_s \to T^{k_s}. \]

**Lem. 3** \( \mathcal{M} \) admits the bundles over tori, \( \Rightarrow \hat{\mathcal{M}} \diffeo \simeq \mathbb{R}^n. \)

**Proof.** Induction on \( s \), starting \( s = 2: T^{k_1} \to \mathcal{M} \to T^{k_2}. \)

\[
\begin{array}{ccc}
T^{k_1} & \longrightarrow & \pi^* \hat{\mathcal{M}} \longrightarrow \mathbb{R}^k \\
\downarrow \text{id} & & \downarrow \pi_i & & \downarrow \pi \\
T^{k_1} & \longrightarrow & \hat{\mathcal{M}} & \overset{\text{GH}}{\longrightarrow} & T^k.
\end{array}
\]

\[
\Rightarrow \pi^* \hat{\mathcal{M}} \simeq \mathbb{R}^k \times T^{k_1},
\]

By induction, \( \Rightarrow \hat{\mathcal{M}}_1 \diffeo \simeq \mathbb{R}^m. \)
Sketch of Proof of Thm

Almost a tower of fibrations over tori (by Lem. 2).

\[ M_1 \to \hat{M} \to T^{k_1}, \; M_2 \to \hat{M}_1 \to T^{k_2}, \; \cdots, \; \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

Lem. 3 \( M \) admits the bundles over tori, \( \Rightarrow \hat{M} \; \text{diffeo} \sim \; \mathbb{R}^n. \)

Proof. Induction on \( s \), starting \( s = 2: \; T^{k_1} \to M \to T^{k_2}. \)

\[
\begin{array}{ccc}
T^{k_1} & \longrightarrow & \pi^* \hat{M} \\
\downarrow \text{id} & & \downarrow \pi_i \\
T^{k_1} & \longrightarrow & \hat{M} \\
 & & \xrightarrow{GH} \\
 & & T^k.
\end{array}
\]

\[ \Rightarrow \pi^* \hat{M} \simeq \mathbb{R}^k \times T^{k_1}, \]

By induction, \( \Rightarrow \hat{M}_1 \; \text{diffeo} \sim \mathbb{R}^m. \) Similar to the above, \( \Rightarrow \hat{M} \; \text{diffeo} \sim \mathbb{R}^n. \)
Sketch of Proof of Thm

• Almost a tower of fibrations over tori:

\[ M_1 \to \hat{M} \to T^{k_1}, M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]
Sketch of Proof of Thm

- Almost a tower of fibrations over tori:
  \[ M_1 \to \hat{M} \to T^{k_1}, \ M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

**Lem. 4** \( \hat{M}_i \) can be chosen s.t. \( \pi_1(\hat{M}_i) \triangleleft \pi_1(\hat{M}), \ 1 \leq i \leq s. \)
Sketch of Proof of Thm

- Almost a tower of fibrations over tori:
  \[ M_1 \to \hat{M} \to T^{k_1}, \ M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \ \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

**Lem. 4** \( \hat{M}_i \) can be chosen s.t. \( \pi_1(\hat{M}_i) \triangleleft \pi_1(\hat{M}), \ 1 \leq i \leq s. \)

**Proof.** (i) \( \pi_1(M) \) has a set of generators of length \( \leq 3 \text{diam}(M), \)
Sketch of Proof of Thm

- Almost a tower of fibrations over tori:

\[ \mathcal{M}_1 \rightarrow \hat{\mathcal{M}} \rightarrow T^{k_1}, \mathcal{M}_2 \rightarrow \hat{\mathcal{M}}_1 \rightarrow T^{k_2}, \ldots, \{\text{pt}\} \rightarrow \hat{\mathcal{M}}_s \rightarrow T^{k_s}. \]

Lemma 4 \( \hat{\mathcal{M}}_i \) can be chosen s.t. \( \pi_1(\hat{\mathcal{M}}_i) \triangleleft \pi_1(\hat{\mathcal{M}}), 1 \leq i \leq s. \)

Proof. (i) \( \pi_1(\mathcal{M}) \) has a set of generators of length \( \leq 3\text{diam}(\mathcal{M}) \),
(ii) \( \pi_1(\mathcal{M}) \) can be generated by short generators of \( \# \leq l(n) \).
Sketch of Proof of Thm

- Almost a tower of fibrations over tori:
  \[ M_1 \to \hat{M} \to T^{k_1}, \ M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \ \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

**Lem. 4** \( \hat{M}_i \) can be chosen s.t. \( \pi_1(\hat{M}_i) \triangleleft \pi_1(\hat{M}), \ 1 \leq i \leq s. \)

**Proof.** (i) \( \pi_1(M) \) has a set of generators of length \( \leq 3\text{diam}(M), \)
(ii) \( \pi_1(M) \) can be generated by short generators of \( \# \leq l(n). \)
(i) and (ii) \( \Rightarrow \exists \Lambda_1 \triangleleft \pi_1(\hat{M}_1), \)
Sketch of Proof of Thm

• Almost a tower of fibrations over tori:
  \( \mathcal{M}_1 \to \hat{\mathcal{M}} \to \mathcal{T}^{k_1}, \mathcal{M}_2 \to \hat{\mathcal{M}}_1 \to \mathcal{T}^{k_2}, \ldots, \{\text{pt}\} \to \hat{\mathcal{M}}_s \to \mathcal{T}^{k_s} \).

Lem. 4 \( \hat{\mathcal{M}}_i \) can be chosen s.t. \( \pi_1(\hat{\mathcal{M}}_i) \triangleleft \pi_1(\hat{\mathcal{M}}), 1 \leq i \leq s \).

Proof. (i) \( \pi_1(\mathcal{M}) \) has a set of generators of length \( \leq 3\text{diam}(\mathcal{M}) \),
(ii) \( \pi_1(\mathcal{M}) \) can be generated by short generators of \( \# \leq l(n) \).
(i) and (ii) \( \Rightarrow \exists \Lambda_1 \triangleleft \pi_1(\hat{\mathcal{M}}_1), [\pi_1(\hat{\mathcal{M}}_1) : \Lambda_1] \leq C(n, d) \),
Sketch of Proof of Thm

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(i) and (ii) \( \Rightarrow \) \( \exists \Lambda_1 \triangleleft \pi_1(\hat{M}_1), \ [\pi_1(\hat{M}_1) : \Lambda_1] \leq C(n, d), \)

\[
\begin{array}{ccc}
M'_2 & \longrightarrow & \tilde{M}_1/\Lambda_1 \\
\downarrow & & \downarrow \pi \\
M_2 & \longrightarrow & \hat{M}_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
M'_2 & \longrightarrow & \tilde{M}_1/\Lambda_1 \\
\downarrow & & \downarrow \pi \\
M_2 & \longrightarrow & \hat{M}_1 \\
\end{array} \longrightarrow \quad f \quad \longrightarrow \quad T^{k_2}
\]

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Sketch of Proof of Thm

- Almost a tower of fibrations over tori:
  \[ M_1 \to \hat{M} \to T^{k_1}, M_2 \to \hat{M}_1 \to T^{k_2}, \ldots, \{\text{pt}\} \to \hat{M}_s \to T^{k_s}. \]

**Lem. 4** \( \hat{M}_i \) can be chosen s.t. \( \pi_1(\hat{M}_i) \triangleleft \pi_1(\hat{M}), 1 \leq i \leq s \).

**Proof.** (i) \( \pi_1(M) \) has a set of generators of length \( \leq 3\text{diam}(M) \),
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(i) and (ii) \( \Rightarrow \exists \Lambda_1 \triangleleft \pi_1(\hat{M}_1), [\pi_1(\hat{M}_1) : \Lambda_1] \leq C(n, d), \)

\[
\begin{align*}
  M_2' & \longrightarrow \tilde{M}_1/\Lambda_1 \quad \xrightarrow{f} \quad T^{k_2} \\
  \downarrow & \quad \downarrow \pi \quad \downarrow \\
  M_2 & \longrightarrow \hat{M}_1 \quad \xrightarrow{f} \quad T^{k_2}. \\
\end{align*}
\]

\( \Rightarrow M_1 \to \hat{M} \to T^{k_1} \) and \( M_2' \to \tilde{M}_1/\Lambda_1 \to T^{k_2} \) satisfies
\[
\pi_1(M_1) \triangleleft \pi_1(\hat{M}),
\]
Sketch of Proof of Thm

- Almost a tower of fibrations over tori:
  \[ M_1 \rightarrow \hat{M} \rightarrow T^{k_1}, \quad M_2 \rightarrow \hat{M}_1 \rightarrow T^{k_2}, \ldots, \{\text{pt}\} \rightarrow \hat{M}_s \rightarrow T^{k_s}. \]

**Lem. 4** \( \hat{M}_i \) can be chosen s.t. \( \pi_1(\hat{M}_i) \triangleleft \pi_1(\hat{M}), \ 1 \leq i \leq s. \)

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(ii) \( \pi_1(M) \) can be generated by short generators of \( \# \leq l(n) \).
(i) and (ii) \( \Rightarrow \exists \Lambda_1 \triangleleft \pi_1(\hat{M}_1), [\pi_1(\hat{M}_1) : \Lambda_1] \leq C(n, d), \)
\[
\begin{array}{ccc}
M'_2 & \longrightarrow & \tilde{M}_1/\Lambda_1 \\
\downarrow & & \downarrow \pi \\
M_2 & \longrightarrow & \hat{M}_1
\end{array}
\xrightarrow{f} \quad T^{k_2}
\]
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M'_2 & \longrightarrow & \tilde{M}_1/\Lambda_1 \\
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\end{array}
\xrightarrow{f} \quad T^{k_2}.
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\( \Rightarrow M_1 \rightarrow \hat{M} \rightarrow T^{k_1} \) and \( M'_2 \rightarrow \tilde{M}_1/\Lambda_1 \rightarrow T^{k_2} \) satisfies
\[
\pi_1(M_1) \triangleleft \pi_1(\hat{M}), \quad \pi_1(\tilde{M}/\Lambda_1) \triangleleft \pi_1(\hat{M}).
\]
Sketch of Proof of Thm

Let $K_i = \ker(\phi_i)/\pi_1(M)$. By Lem. 4, $|\pi_1(\hat{M})/K_i| = a_i$. Let $K = \bigcap_{i=1}^{s} K_i/\pi_1(M)$. Then $[\pi_1(\hat{M}) : K] \leq a_1 \cdots a_s < \infty$.

Let $\hat{M} = \tilde{M}/K$. This gives a tower of bundles over tori: $M_1 \to \hat{M} \to T_{k_1}$, $M_2 \to M_1 \to T_{k_2}$, $\cdots$, $M_s \to M_{s-1} \to T_{k_s}$, s.t. $\phi_i : \pi_1(M) \to \Aut(\pi_1(M_i)/\pi_1(M_i+1))$ is trivial.
Sketch of Proof of Thm

\[ \phi_i : \pi_1(\hat{M}) \to \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})), \]
Sketch of Proof of Thm

\[ \phi_i : \pi_1(\hat{M}) \to \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})) \], by conjugation.
Sketch of Proof of Thm

$\Rightarrow \phi_i : \pi_1(\hat{M}) \rightarrow \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1}))$, by conjugation.

**Lem. 5** $\text{Im}(\phi_i)$ is finite.
Sketch of Proof of Thm

\[ \phi_i : \pi_1(\hat{M}) \rightarrow \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})), \] by conjugation.

**Lem. 5** \( \text{Im}(\phi_i) \) is finite.

**Completion of proof of Thm:**

Let \( K_i = \ker(\phi_i) \lhd \pi_1(M) \).
Sketch of Proof of Thm

\[ \Rightarrow \phi_i : \pi_1(\hat{M}) \to \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})) \], by conjugation.

**Lem. 5** $\text{Im}(\phi_i)$ is finite.

**Completion of proof of Thm:**

Let $K_i = \ker(\phi_i) \triangleleft \pi_1(M)$. By **Lem. 4** \[ |\pi_1(\hat{M})/K_i| = a_i. \]
Sketch of Proof of Thm

⇒ \( \phi_i : \pi_1(\hat{M}) \to \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})) \), by conjugation.

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Sketch of Proof of Thm

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Sketch of Proof of Thm

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Let \( \hat{M} = \tilde{M}/K \).
Sketch of Proof of Thm

\[ \phi_i : \pi_1(\hat{M}) \to \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})) \], by conjugation.

**Lem. 5** \( \text{Im}(\phi_i) \) is finite.

Completion of proof of Thm:

Let \( K_i = \ker(\phi_i) \triangleleft \pi_1(M) \). By **Lem. 4** \( \Rightarrow |\pi_1(\hat{M})/K_i| = a_i \).

Let \( K = \bigcap_{i=1}^{s} K_i \triangleleft \pi_1(M) \). \( \Rightarrow [\pi_1(\hat{M}) : K] \leq a_1 \cdots a_s < \infty \).

Let \( \hat{M} = \tilde{M}/K \). \( \Rightarrow \) a tower of bundles over tori:
Sketch of Proof of Thm

\[ \Rightarrow \phi_i : \pi_1(\hat{M}) \to \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})) , \text{ by conjugation.} \]

**Lem. 5** \( \text{Im}(\phi_i) \) is finite.

**Completion of proof of Thm:**

Let \( K_i = \ker(\phi_i) \triangleleft \pi_1(M) \). By **Lem. 4** \[ \Rightarrow |\pi_1(\hat{M})/K_i| = a_i. \]

Let \( K = \bigcap_{i=1}^s K_i \triangleleft \pi_1(M) \). \[ \Rightarrow [\pi_1(\hat{M}) : K] \leq a_1 \cdots a_s < \infty. \]

Let \( \hat{M} = \tilde{M}/K \). \[ \Rightarrow \text{a tower of bundles over tori:} \]

\[ M_1 \to \hat{M} \to T^{k_1}, M_2 \to M_1 \to T^{k_2}, \ldots, M_s \to M_{s-1} \to T^{k_s}, \]
Sketch of Proof of Thm

\[ \phi_i : \pi_1(\hat{M}) \rightarrow \text{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1})) \text{, by conjugation.} \]

**Lem. 5** \( \text{Im}(\phi_i) \) is finite.

**Completion of proof of Thm:**

Let \( K_i = \ker(\phi_i) \triangleleft \pi_1(M) \). By **Lem. 4** \( |\pi_1(\hat{M})/K_i| = a_i \).

Let \( K = \bigcap_{i=1}^{s} K_i \triangleleft \pi_1(M) \). \[ \Rightarrow [\pi_1(\hat{M}) : K] \leq a_1 \cdots a_s < \infty. \]

Let \( \hat{M} = \tilde{M}/K \). \[ \Rightarrow \text{a tower of bundles over tori:} \]

\[ M_1 \rightarrow \hat{M} \rightarrow T^{k_1}, M_2 \rightarrow M_1 \rightarrow T^{k_2}, \cdots, M_s \rightarrow M_{s-1} \rightarrow T^{k_s}, \]

s.t.

\[ \phi_i : \pi_1(M) \rightarrow \text{Aut}(\pi_1(M_i)/\pi_1(M_{i+1})) \text{ is trivial.} \]
Sketch of Proof of Thm A

Proof of Lem. 5.
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling;
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling; \( \implies \) a Gromov’s short basis of \( \pi_1(M, p) \) can be chosen:

\[ \gamma_{1,1}, \cdots, \gamma_{1,k_1}, \]
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling; \(\Rightarrow\) a Gromov’s short basis of \(\pi_1(M, p)\) can be chosen:

\[\gamma_{1,1}, \ldots, \gamma_{1,k_1}, \gamma_{2,1}, \ldots, \gamma_{2,k_2},\]
Sketch of Proof of Thm A

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\[ \gamma_{1,1}, \cdots, \gamma_{1,k_1}, \gamma_{2,1}, \cdots, \gamma_{2,k_2}, \cdots, \]
Sketch of Proof of Thm A

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\[
\gamma_{1,1}, \ldots, \gamma_{1,k_1}, \gamma_{2,1}, \ldots, \gamma_{2,k_2}, \ldots, \gamma_{s,1}, \ldots, \gamma_{s,k_s},
\]
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling; \( \Rightarrow \) a Gromov’s short basis of \( \pi_1(M, p) \) can be chosen:

\[
\gamma_{1,1}, \ldots, \gamma_{1,k_1}, \gamma_{2,1}, \ldots, \gamma_{2,k_2}, \ldots, \gamma_{s,1}, \ldots, \gamma_{s,k_s},
\]

\[
|\gamma_{i,t}|/|\gamma_{j,s}| \gg 1, \quad i < j.
\]
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling; ⇒ a Gromov’s short basis of $\pi_1(M, p)$ can be chosen:

1. $\gamma_{1,1}, \ldots, \gamma_{1,k_1}, \gamma_{2,1}, \ldots, \gamma_{2,k_2}, \ldots, \gamma_{s,1}, \ldots, \gamma_{s,k_s}$,
2. $|\gamma_{i,t}| / |\gamma_{j,s}| \gg 1, \quad i < j$.
3. $\phi_i : \pi_1(M, p) \to \text{Aut}[\pi_1(M_i)/\pi_1(M_{i+1})]$. 

|π_1(M_j, p) ∩ B_ε^2(n)(p)| ≤ vol(B_−1(ε^2(n))) vol(B_−1(δ_j)),
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling; \( \Rightarrow \) a Gromov’s short basis of \( \pi_1(M, p) \) can be chosen:

1. \( \gamma_{1,1}, \ldots, \gamma_{1,k_1}, \gamma_{2,1}, \ldots, \gamma_{2,k_2}, \ldots, \gamma_{s,1}, \ldots, \gamma_{s,k_s} \),
2. \( |\gamma_{i,t}|/|\gamma_{j,s}| \gg 1, \quad i < j \).
3. \( \phi_i : \pi_1(M, p) \to \text{Aut}[\pi_1(M_i)/\pi_1(M_{i+1})] \),
4. \( \phi_i(h)(\alpha \cdot \pi_1(M_{i+1})) = (h \cdot \alpha \cdot h^{-1}) \cdot \pi_1(M_{i+1}) \).
Sketch of Proof of Thm A

**Proof of Lem. 5.** The iterated bundles over tori are obtained via successive rescaling; \( \Rightarrow \) a Gromov’s short basis of \( \pi_1(M, p) \) can be chosen:

- \( \gamma_{1,1}, \cdots, \gamma_{1,k_1}, \gamma_{2,1}, \cdots, \gamma_{2,k_2}, \cdots, \gamma_{s,1}, \cdots, \gamma_{s,k_s}, \)
- \( |\gamma_{i,t}|/|\gamma_{j,s}| >> 1, \quad i < j. \)
- \( \phi_i : \pi_1(M, p) \to \text{Aut}[\pi_1(M_i)/\pi_1(M_{i+1})], \)
- \( \phi_i(h)(\alpha \cdot \pi_1(M_{i+1})) = (h \cdot \alpha \cdot h^{-1}) \cdot \pi_1(M_{i+1}). \)
- \( e^{-c(n)} \leq \frac{d(\bar{p}, h \cdot \alpha_i \cdot h^{-1}(\bar{p}))}{d(\bar{p}, \alpha_i(\bar{p}))} \leq e^{c(n)}. \)
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling; ⇒ a Gromov’s short basis of $\pi_1(M, p)$ can be chosen:

- $\gamma_{1,1}, \ldots, \gamma_{1,k_1}, \gamma_{2,1}, \ldots, \gamma_{2,k_2}, \ldots, \gamma_{s,1}, \ldots, \gamma_{s,k_s}$,
- $|\gamma_{i,t}| / |\gamma_{j,s}| \gg 1$, $i < j$.
- $\phi_i : \pi_1(M, p) \to \text{Aut}[\pi_1(M_i)/\pi_1(M_{i+1})]$, $\phi_i(h)(\alpha \cdot \pi_1(M_{i+1})) = (h \cdot \alpha \cdot h^{-1}) \cdot \pi_1(M_{i+1})$.
- $e^{-c(n)} \leq \frac{d(\tilde{p}, h \cdot \alpha_i \cdot h^{-1}(\tilde{p}))}{d(\tilde{p}, \alpha_i(\tilde{p}))} \leq e^{c(n)}$.
- $|\pi_1(M_j, p) \cap B_{e^{2c(n)}}(\tilde{p})| \leq \frac{\text{vol}(B_{-1}(e^{2c(n)})}{\text{vol}(B_{-1}(\delta_j))}$,
- $\rho(h) : (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{k_j}) \to (\rho(h)(\tilde{\gamma}_1), \ldots, \rho(h)(\tilde{\gamma}_{k_j}))$

has at most the following number of possibilities:
Sketch of Proof of Thm A

Proof of Lem. 5. The iterated bundles over tori are obtained via successive rescaling; \( \Rightarrow \) a Gromov’s short basis of \( \pi_1(M, p) \) can be chosen:

\[
\begin{align*}
&\gamma_{1,1}, \cdots, \gamma_{1,k_1}, \gamma_{2,1}, \cdots, \gamma_{2,k_2}, \cdots, \gamma_{s,1}, \cdots, \gamma_{s,k_s}, \\
&|\gamma_{i,t}| / |\gamma_{j,s}| > > 1, \quad i < j. \\
&\phi_i : \pi_1(M, p) \to \text{Aut}[\pi_1(M_i)/\pi_1(M_{i+1})], \\
&\phi_i(h)(\alpha \cdot \pi_1(M_{i+1})) = (h \cdot \alpha \cdot h^{-1}) \cdot \pi_1(M_{i+1}), \\
&e^{-c(n)} \leq \frac{d(\bar{p}, h \cdot \alpha_i \cdot h^{-1}(\bar{p}))}{d(\bar{p}, \alpha_i(\bar{p}))} \leq e^{c(n)}. \\
\end{align*}
\]

\[|\pi_1(M_j, p) \cap B_{e^{2c(n)}}(\bar{p})| \leq \frac{\text{vol}(B_{-1}(e^{2c(n)}))}{\text{vol}(B_{-1}(\delta_j))},\]

\[\rho(h) : (\bar{\gamma}_{1}, \cdots, \bar{\gamma}_{k_j}) \to (\rho(h)(\bar{\gamma}_{1}), \cdots, \rho(h)(\bar{\gamma}_{k_j}))\]

has at most the following number of possibilities:

\[c_j = \left(\frac{\text{vol}(B_{-1}(e^{2c(n)}))}{\text{vol}(B_{-1}(\delta_j))}\right)^{k_j}.\]
Thanks For Attention!