Collapsed Manifolds With Local Ricci Bounded Covering Geometry

Xiaochun Rong

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In this talk, we generalize the structural result to collapsed manifolds with local Ricci bounded covering geometry. Our construction of local nilpotent symmetry structures does not reply on the work of Cheeger-Fukaya-Gromov; which also gives an alternative approach to the structural results.

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Construct an inhomogeneous scaling g_ϵ, by $\epsilon, \epsilon^2, ..., \epsilon^k$,

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A collapsed manfold M with bounded Ricci Curvature curvature + 'various' conditions,

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$\Rightarrow$ $V_{-\mu}(\mathcal{R})$ is a Riemannian manifold,

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Thm. (Local rewinding Harmonic radius bounded below)

$\exists\, \epsilon(n, \rho) > 0$ s.t. if a compact n-manifold M satisfies

$$\tilde{r}_h^\alpha(B_\rho(x)) \geq \delta(\rho) > 0, \quad \text{diam}(M) < \epsilon(n, v),$$

$\Rightarrow$ M is diffeo. to an infra-nilmanifold.

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where $r_i = \mathrm{diam}(M_i) \to 0$ as $i \to \infty$.

Sketch of Proof of Thm

$$\Rightarrow \qquad \begin{array}{ccc} (r_i^{-1}\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{C^\alpha} & (\mathbb{R}^n, \tilde{p}', G) \\ \downarrow \pi_i & & \downarrow \pi \\ r_i^{-1}\hat{M}_i & \xrightarrow{GH} & T^k. \end{array}$$

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Lem. 2 Let $M_i \xrightarrow{GH} Y$ (manifold) s.t.

Sketch of Proof of Thm

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Lem. 2 Let $M_i \xrightarrow{GH} Y$ (manifold) s.t.

$$\tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall x_i \in M_i.$$

Sketch of Proof of Thm

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$\Rightarrow$ $\exists$ a fiber bundle, $M_{i,1} \to M_i \xrightarrow{f_i} Y$, s.t.

Sketch of Proof of Thm

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Sketch of Proof of Thm

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Proof. (i) Assume $\delta = \mathrm{convrad}(T^k) < \rho$.

Sketch of Proof of Thm

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Proof. (i) Assume $\delta = \mathrm{convrad}(T^k) < \rho$. Averaging diffeo. $\tilde f_i : B_1(\tilde p_i) \to B_1(\tilde p)$ over $\Gamma_i(x) \cap B_1(\tilde p_i)$,

Sketch of Proof of Thm

$$\Rightarrow \quad \begin{array}{ccc} (r_i^{-1}\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{C^\alpha} & (\mathbb{R}^n, \tilde{p}', G) \\ \downarrow \pi_i & & \downarrow \pi \\ r_i^{-1}\hat{M}_i & \xrightarrow{GH} & T^k. \end{array}$$

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Proof. (i) Assume $\delta = \mathrm{convrad}(T^k) < \rho$. Averaging diffeo. $\tilde{f}_i : B_1(\tilde{p}_i) \to B_1(\tilde{p})$ over $\Gamma_i(x) \cap B_1(\tilde{p}_i)$, $\Rightarrow$ $f_i : B_{\frac{\rho}{2}}(x_i) \to T^k$.

Sketch of Proof of Thm

$$\Rightarrow \quad \begin{array}{ccc} (r_i^{-1}\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{C^\alpha} & (\mathbb{R}^n, \tilde{p}', G) \\ \downarrow \pi_i & & \downarrow \pi \\ r_i^{-1}\hat{M}_i & \xrightarrow{GH} & T^k. \end{array}$$

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(ii) Cover T^k with finite $B_{\frac{\rho}{2}}(z_i)$,

Sketch of Proof of Thm

$$\Rightarrow \qquad \begin{array}{ccc} (r_i^{-1}\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{C^\alpha} & (\mathbb{R}^n, \tilde{p}', G) \\ \Big\downarrow \pi_i & & \Big\downarrow \pi \\ r_i^{-1}\hat{M}_i & \xrightarrow{GH} & T^k. \end{array}$$

Lem. 2 Let $M_i \xrightarrow{GH} Y$ (manifold) s.t.

$$\tilde{r}_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall\, x_i \in M_i.$$

$\Rightarrow$ $\exists$ a fiber bundle, $M_{i,1} \to M_i \xrightarrow{f_i} Y$, s.t.

$$\tilde{r}_h^\alpha(M_{i,1}) \geq \delta'(\rho, \delta), \quad \operatorname{diam}(M_{i,1}) \to 0.$$

Proof. (i) Assume $\delta = \operatorname{convrad}(T^k) < \rho$. Averaging diffeo. $\tilde{f}_i : B_1(\tilde{p}_i) \to B_1(\tilde{p})$ over $\Gamma_i(x) \cap B_1(\tilde{p}_i)$, $\Rightarrow$ $f_i : B_{\frac{\rho}{2}}(x_i) \to T^k$.

(ii) Cover T^k with finite $B_{\frac{\rho}{2}}(z_i)$, by (i) $\Rightarrow$ $B_{\frac{\rho}{2}}(x_{ij}) \xrightarrow{f_{ij}} B_{\frac{\rho}{2}}(z_i)$.

Sketch of Proof of Thm

$$\Rightarrow \quad \begin{array}{ccc} (r_i^{-1}\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{C^\alpha} & (\mathbb{R}^n, \tilde{p}', G) \\ \downarrow \pi_i & & \downarrow \pi \\ r_i^{-1}\hat{M}_i & \xrightarrow{GH} & T^k. \end{array}$$

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(ii) Cover T^k with finite $B_{\frac{\rho}{2}}(z_i)$, by (i) $\Rightarrow$ $B_{\frac{\rho}{2}}(x_{ij}) \xrightarrow{f_{ij}} B_{\frac{\rho}{2}}(z_i)$. Gluing together $\{f_{ij}\}$, via center of mass,

Sketch of Proof of Thm

$$\Rightarrow \quad \begin{array}{ccc} (r_i^{-1}\tilde M_i, \tilde p_i, \Gamma_i) & \xrightarrow{C^\alpha} & (\mathbb{R}^n, \tilde p', G) \\ \downarrow \pi_i & & \downarrow \pi \\ r_i^{-1}\hat M_i & \xrightarrow{GH} & T^k. \end{array}$$

Lem. 2 Let $M_i \xrightarrow{GH} Y$ (manifold) s.t.

$$\tilde r_h^\alpha(B_\rho(x_i)) \geq \delta > 0, \quad \forall\, x_i \in M_i.$$

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(ii) Cover T^k with finite $B_{\frac{\rho}{2}}(z_i)$, by (i) $\Rightarrow$ $B_{\frac{\rho}{2}}(x_{ij}) \xrightarrow{f_{ij}} B_{\frac{\rho}{2}}(z_i)$. Gluing together $\{f_{ij}\}$, via center of mass, $\Rightarrow$ $f_i : \hat M_i \to T^k$. □

Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

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Sketch of Proof of Thm

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$$M_1 \to \hat{M} \to T^{k_1},\ M_2 \to \hat{M}_1 \to T^{k_2},\ \cdots,\ \{\mathrm{pt}\} \to \hat{M}_s \to T^{k_s}.$$

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Lem. 3 M admits the bundles over tori,

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Lem. 3 M admits the bundles over tori, $\Rightarrow \tilde{M} \overset{\text{diffeo}}{\cong} \mathbb{R}^n$.

Sketch of Proof of Thm

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Lem. 3 M admits the bundles over tori, $\Rightarrow \tilde{M} \overset{\text{diffeo}}{\cong} \mathbb{R}^n$.

Proof. Induction on s, starting $s = 2$: $T^{k_1} \to M \to T^{k_2}$.

$$\begin{array}{ccccc} T^{k_1} & \longrightarrow & \pi^*\hat{M} & \longrightarrow & \mathbb{R}^k \\ \downarrow \mathrm{id} & & \downarrow \pi_i & & \downarrow \pi \\ T^{k_1} & \longrightarrow & \hat{M} & \xrightarrow{GH} & T^k. \end{array}$$

Sketch of Proof of Thm

- Almost a tower of fibrations over tori (by Lem. 2).

$M_1 \to \hat{M} \to T^{k_1}$, $M_2 \to \hat{M}_1 \to T^{k_2}$, $\cdots$, $\{\mathrm{pt}\} \to \hat{M}_s \to T^{k_s}$.

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By induction, $\Rightarrow$ $\tilde{M}_1 \overset{\text{diffeo}}{\cong} \mathbb{R}^m$.

Sketch of Proof of Thm

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$$M_1 \to \hat{M} \to T^{k_1},\ M_2 \to \hat{M}_1 \to T^{k_2},\ \cdots,\ \{\mathrm{pt}\} \to \hat{M}_s \to T^{k_s}.$$

Lem. 3 M admits the bundles over tori, $\Rightarrow \tilde{M} \overset{\text{diffeo}}{\cong} \mathbb{R}^n$.

Proof. Induction on s, starting $s = 2$: $T^{k_1} \to M \to T^{k_2}$.

$$\begin{array}{ccccc} T^{k_1} & \longrightarrow & \pi^*\hat{M} & \longrightarrow & \mathbb{R}^k \\ \downarrow \mathrm{id} & & \downarrow \pi_i & & \downarrow \pi \\ T^{k_1} & \longrightarrow & \hat{M} & \xrightarrow{GH} & T^k. \end{array} \quad \Rightarrow \quad \pi^*\hat{M} \simeq \mathbb{R}^k \times T^{k_1},$$

By induction, $\Rightarrow \tilde{M}_1 \overset{\text{diffeo}}{\cong} \mathbb{R}^m$. Similar to the above, $\Rightarrow$ $\tilde{M} \overset{\text{diffeo}}{\cong} \mathbb{R}^n$. □

Sketch of Proof of Thm

- Almost a tower of fibrations over tori:

$$M_1 \to \hat{M} \to T^{k_1},\ M_2 \to \hat{M}_1 \to T^{k_2},\ \cdots,\ \{\text{pt}\} \to \hat{M}_s \to T^{k_s}.$$

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Lem. 4 $\hat{M}_i$ can be chosen s.t. $\pi_1(\hat{M}_i) \triangleleft \pi_1(\hat{M})$, $1 \le i \le s$.

Sketch of Proof of Thm

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Proof. (i) $\pi_1(M)$ has a set of generators of length $\le 3\mathrm{diam}(M)$,

Sketch of Proof of Thm

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Proof. (i) $\pi_1(M)$ has a set of generators of length $\le 3\text{diam}(M)$,

(ii) $\pi_1(M)$ can be generated by short generators of $\# \le I(n)$.

Sketch of Proof of Thm

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$$M_1 \to \hat{M} \to T^{k_1},\ M_2 \to \hat{M}_1 \to T^{k_2},\ \cdots,\ \{\text{pt}\} \to \hat{M}_s \to T^{k_s}.$$

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(i) and (ii) $\Rightarrow \exists\, \Lambda_1 \triangleleft \pi_1(\hat{M}_1)$,

Sketch of Proof of Thm

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(i) and (ii) $\Rightarrow \exists\, \Lambda_1 \triangleleft \pi_1(\hat{M}_1)$, $[\pi_1(\hat{M}_1) : \Lambda_1] \le C(n,d)$,

Sketch of Proof of Thm

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(ii) $\pi_1(M)$ can be generated by short generators of $\# \le l(n)$.

(i) and (ii) $\Rightarrow \exists\, \Lambda_1 \lhd \pi_1(\hat{M}_1)$, $[\pi_1(\hat{M}_1) : \Lambda_1] \le C(n,d)$,

$$\begin{array}{ccccc} M_2' & \longrightarrow & \tilde{M}_1/\Lambda_1 & \xrightarrow{f} & T^{k_2} \\ \downarrow & & \downarrow \pi & & \downarrow \\ M_2 & \longrightarrow & \hat{M}_1 & \xrightarrow{f} & T^{k_2}. \end{array}$$

Sketch of Proof of Thm

- Almost a tower of fibrations over tori:

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$\Rightarrow M_1 \to \hat{M} \to T^{k_1}$ and $M_2' \to \tilde{M}_1/\Lambda_1 \to T^{k_2}$ satisfies

$$\pi_1(M_1) \lhd \pi_1(\hat{M}),$$

Sketch of Proof of Thm

- Almost a tower of fibrations over tori:

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Proof. (i) $\pi_1(M)$ has a set of generators of length $\le 3\text{diam}(M)$,

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(i) and (ii) $\Rightarrow \exists\, \Lambda_1 \triangleleft \pi_1(\hat{M}_1)$, $[\pi_1(\hat{M}_1) : \Lambda_1] \le C(n,d)$,

$$\begin{array}{ccccc} M_2' & \longrightarrow & \tilde{M}_1/\Lambda_1 & \xrightarrow{f} & T^{k_2} \\ \downarrow & & \downarrow \pi & & \downarrow \\ M_2 & \longrightarrow & \hat{M}_1 & \xrightarrow{f} & T^{k_2}. \end{array}$$

$\Rightarrow M_1 \to \hat{M} \to T^{k_1}$ and $M_2' \to \tilde{M}_1/\Lambda_1 \to T^{k_2}$ satisfies

$$\pi_1(M_1) \triangleleft \pi_1(\hat{M}), \quad \pi_1(\tilde{M}/\Lambda_1) \triangleleft \pi_1(\hat{M}).$$

Sketch of Proof of Thm

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$\Rightarrow \phi_i : \pi_1(\hat{M}) \to \mathrm{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1}))$,

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$\Rightarrow \phi_i : \pi_1(\hat{M}) \to \mathrm{Aut}(\pi_1(\hat{M}_i)/\pi_1(M_{i+1}))$, by conjugation.

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Lem. 5 $\mathrm{Im}(\phi_i)$ is finite.

Sketch of Proof of Thm

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Completion of proof of Thm:

Let $K_i = \ker(\phi_i) \lhd \pi_1(M)$.

Sketch of Proof of Thm

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s.t.

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Thanks For Attention!