Introduction to Ricci Curvature and the Convergence Theory

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Structure of Collapsed Special Holonomy Spaces
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Elliptic Theory

- (Elliptic Inequality)

$$\begin{cases} \mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm} + \nabla^2 \mathrm{Ric} \\ \mathrm{Ric}_g \equiv \lambda \cdot g, \ \lambda \in \mathbb{R} \end{cases} \tag{1}$$

$$\Longrightarrow \ \Delta |\mathrm{Rm}| \geq -C(n) \cdot |\mathrm{Rm}|^2. \tag{2}$$

Elliptic Theory

- (Elliptic Inequality)

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$$\Longrightarrow \ \Delta |\mathrm{Rm}| \geq -C(n) \cdot |\mathrm{Rm}|^2. \tag{2}$$

- (Sobolev) Let $u \in C_0^\infty(\Omega)$,

$$\Big(\int_\Omega u^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{2n}} \leq C_S \Big(\int_\Omega |\nabla u|^2\Big)^{\frac{1}{2}}. \tag{3}$$

Elliptic Theory

- (Moser Iteration) In dimension 4, there are $\delta(C_S) > 0$ and $Q(C_S) > 0$ let u satisfy

$$\Delta u \geq -u^2, \tag{4}$$

and

$$\int_{B_2(x)} |u|^2 \leq \delta, \tag{5}$$

then

$$\sup_{B_1(x)} |u| \leq Q \cdot \left(\int_{B_2(x)} |u|^2 \right)^{\frac{1}{2}}. \tag{6}$$

Elliptic Theory

- (Croke) Let (M^n, g) satisfy $\mathrm{Ric}_g \geq -(n-1)$ and $\mathrm{Vol}(B_1(p)) \geq v > 0$, then in $B_2(p)$ we have

$$C_S \leq C_0(n, v) < \infty. \tag{7}$$

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$$C_S \leq C_0(n, v) < \infty. \tag{7}$$

- (Classical ϵ-Regularity) Let (M^4, g, p) be an Einstein manifold with $|\mathrm{Ric}_g| \leq 3$. Assume $\mathrm{Vol}(B_1(p)) \geq v > 0$, then there are constants $\epsilon(v) > 0$ and $C(v) < \infty$ such that

$$\int_{B_2(p)} |\mathrm{Rm}|^2 < \epsilon \;\Rightarrow\; \sup_{B_1(p)} |\mathrm{Rm}| \leq C(v). \tag{8}$$

Chern-Gauss-Bonnet and Integral Curvature Bounds

- Let (M^4, g) be a closed 4-manifold, then Chern-Gauss-Bonnet theorem states that

$$\chi(M^4) = \int_{M^4} P_\chi, \tag{9}$$

where

$$P_\chi \equiv \frac{1}{8\pi^2}(|\operatorname{Rm}|^2 - 4|\operatorname{Ric}|^2 + R^2). \tag{10}$$

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- If (M^4, g) is Einstein, then

$$P_\chi = \frac{1}{8\pi^2}|\operatorname{Rm}|^2. \tag{11}$$

The ϵ-Regularity Theorems and Integral Curvature Bounds

Theorem (M. Anderson)

Given $n \geq 2$, there are dimensional constants $\epsilon(n) > 0$ and $C(n) > 0$ such that the following holds. Let (M^n, g, p) be an Einstein manifold with $|\operatorname{Ric}_g| \leq n-1$, then

$$\fint_{B_2(p)} |\operatorname{Rm}|^{\frac{n}{2}} < \epsilon \Longrightarrow \sup_{B_1(p)} |\operatorname{Rm}| \leq 1. \tag{12}$$

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- This type of ϵ-regularity mostly applies in the non-collapsing case.
- There is a much stronger ϵ-regularity when $n = 4$ due to Cheeger-Tian.

The ϵ-Regularity Theorems and Integral Curvature Bounds

Theorem (Cheeger-Tian, 2005)

There exist absolute constants $\epsilon > 0$, $C < \infty$ such that the following holds. Let (M^4, g) be an Einstein 4-manifold with $\mathrm{Ric}_g \equiv \lambda \cdot g$ and $|\lambda| \leq 3$. Then

$$\int_{B_2(p)} |\mathrm{Rm}|_g^2 \, \mathrm{dvol}_g < \epsilon \Longrightarrow \sup_{B_1(p)} |\mathrm{Rm}|_g \leq C. \tag{13}$$

C^1-Harmonic Radius

Definition (C^1-harmonic coordinates)

Let $u = (u_1, \ldots, u_n) : B_r(p) \to \mathbb{R}^n$ with $u(p) = 0$ and u a diffeomorphism onto its image. We call u a C^1-harmonic coordinates system with $\|u\|_r \leq 1$ if the following properties hold:

- For each $1 \leq k \leq n$, u_k is harmonic.
- If $g_{ij} = g(\nabla u_i, \nabla u_j)$ is the metric in coordinates, then

$$|g_{ij} - \delta_{ij}|_{C^0(B_r(p))} + r|\partial g_{ij}|_{C^0(B_r(p))} < 10^{-6}, \tag{14}$$

  where the scale-invariant norms are taken in the euclidean coordinates.

C^1-Harmonic Radius

Definition (C^1-Harmonic Radius)

For $x \in M^n$ we define the harmonic radius $r_h(x)$ by

$$r_h(x) \equiv \sup\{r > 0| \quad \exists\, C^1 - \text{ harmonic coordinates } u : B_r(x) \to \mathbb{R}^n \text{ with} \|u\|_r \leq 1\}. \tag{15}$$

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Definition (Curvature Radius)

For $x \in M^n$ we define the harmonic radius $r_{|\operatorname{Rm}|}(x)$ by

$$r_h(x) \equiv \sup\Big\{r > 0 \Big| \sup_{B_r(x)} r^2 |\operatorname{Rm}| \leq 1\Big\}. \tag{16}$$

C^1-Harmonic Radius

- In harmonic coordinates, we have the following expression of Ricci tensor,

$$\mathrm{Ric}_{ij} = \frac{1}{2} g^{kl} \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} + Q\Big(\frac{\partial g_{rs}}{\partial x_m}\Big). \tag{17}$$

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- With $|\,\mathrm{Ric}_g\,| \leq n-1$, then by the standard elliptic regularity, within the C^1-harmonic radius, the metric has uniformly bounded $W^{2,p}$-norm for any $p > 1$.
- If (M^n, g) is Einstein, then $r_h(x) \geq r_0$ implies that $|\,\mathrm{Rm}\,|$ is uniformly bounded around x.

C^1-Harmonic Radius

Theorem (M. Anderson, 1990)

Let (M^n, g, p) *be a Riemannian manifold with* $|\,\mathrm{Ric}_g\,| \le n-1$. *Assume*

$$\mathrm{InjRad}(x) \ge i_0 > 0 \tag{18}$$

for every $x \in B_2(p)$, *then there exists* $r_0(n, i_0) > 0$ *such that for all* $x \in B_1(p)$,

$$r_h(x) \ge r_0 > 0. \tag{19}$$

C^1-Harmonic Radius

Theorem (M. Anderson, 1990)

Let (M^n, g, p) be a Riemannian manifold with $|\operatorname{Ric}_g| \leq n-1$ and $\operatorname{Vol}_g(B_1(p)) \geq v > 0$, then there are uniform constants $\epsilon_0(n,v) > 0$ and $r_0(n,v) > 0$ such that if

$$\int_{B_2(p)} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0, \tag{20}$$

then for all $x \in B_1(p)$,

$$r_h(x) \geq r_0 > 0. \tag{21}$$

C^1-Harmonic Radius

Theorem (Cheeger-Tian, 2005)

There exist absolute constants $\epsilon > 0$ and $r_0 > 0$ such that the following holds. Let (M^4, g) be a 4-manifold with $|\operatorname{Ric}_g| \leq 3$ and

$$\int_{B_2(p)} |\operatorname{Rm}|^2 < \epsilon, \tag{22}$$

then for every $x \in B_1(p)$,

$$r_h(\tilde{x}) \geq r_0 > 0, \tag{23}$$

where $\tilde{x} \in \widetilde{B_{10r_0}(x)}$.

ϵ-Regularity Theorems absent a Priori Integral Bounds

Theorem (M. Anderson, 1990)

There exists $\epsilon(n) > 0$ *such that if a Riemannian manifold* (M^n, g, p) *satisfies* $|\operatorname{Ric}_g| \leq (n-1)\epsilon^2$ *and*

$$\frac{\operatorname{Vol}(B_{3/2}(p))}{\operatorname{Vol}(B_{3/2}(0^n))} > 1 - \epsilon, \tag{24}$$

then

$$r_h(p) \geq 1. \tag{25}$$

ϵ-Regularity Theorems absent a Priori Integral Bounds

Theorem (Cheeger-Colding, 1997)

Given $n \geq 2$, there exists $\epsilon(n) > 0$ such that the following holds. Let (M^n, g, p) be a Riemannian manifold with $|\operatorname{Ric}_g| \leq (n-1)\epsilon^2$ and

$$d_{GH}(B_2(p), B_2(0^n)) < \epsilon, \ 0^n \in \mathbb{R}^n, \tag{26}$$

then $r_h(p) \geq 1$. In particular, if (M^n, g) is Einstein, then $\| \operatorname{Rm}_g \| \leq 1$.

Quantitative Symmetry and ϵ-Regularity

Definition

Let (X, d, p) be a metric space,

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Quantitative Symmetry and ϵ-Regularity

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- we say X is (k, ϵ, r)-symmetric at p if there exists a compact metric space Y such that

$$d_{GH}(B_{r\epsilon^{-1}}(p), B_{r\epsilon^{-1}}(0^k, y^*)) < r\epsilon, \ (0^k, y) \in \mathbb{R}^k \times C(Y), \tag{27}$$

where $C(Y)$ is a metric cone with a cone tip y^*.

Quantitative Symmetry and ϵ-Regularity

Theorem (Cheeger-Colding's Metric Cone Theorem, 1996)

Let (M_j^n, g_j, p_j) *be a sequence of non-collapsing manifolds with* $\mathrm{Ric}_{g_j} \geq -(n-1)$ *such that*

$$(M_j^n, g_j, p_j) \xrightarrow{GH} (X, d, p), \tag{28}$$

then for every $x \in X$*, each tangent cone at* x *is a metric cone.*

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then for every $x \in X$, each tangent cone at x is a metric cone.

Theorem (Cheeger-Colding's Metric Cone Theorem, 1996)

Let (X, d, p) be a non-collapsed limit space under lower Ricci, then every tangent cone over p is k-symmetric for some $k \geq 0$.

Quantitative Symmetry and ϵ-Regularity

Let (X^n, d) be a Ricci limit space.

- Let $1 \leq k \leq n$, we define

$$\mathcal{S}^k(X) \equiv \Big\{ x \in X \Big| \text{no tangent cone at } x \text{ is } (k{+}1){-}\text{symmetric} \Big\} \tag{29}$$

and

$$\mathcal{S}(X) \equiv \mathcal{S}^{n-1}(X), \; \mathcal{R}(X) \equiv X \setminus \mathcal{S}(X). \tag{30}$$

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- By definition,

$$\mathcal{S}^0(X) \subset \mathcal{S}^1(X) \subset \ldots \subset \mathcal{S}^{n-1}(X) = \mathcal{S}(X). \tag{31}$$

Quantitative Symmetry and ϵ-Regularity

Theorem (Cheeger-Colding's Stratification Theorem, 1997)

Let (X^n, d, p) be a non-collapsing Ricci-limit space, then

$$\dim_{\mathcal{H}}(\mathcal{S}^k) \leq k \tag{32}$$

and

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \ldots \subset \mathcal{S}^{n-2} = \mathcal{S}. \tag{33}$$

In particular,

$$\dim_{\mathcal{H}}(\mathcal{S}) \leq n - 2. \tag{34}$$

- The half Euclidean space $\mathbb{R}^n_+$ cannot be a non-collapsing Ricci limit space or a tangent cone in X^n.

Quantitative Symmetry and ϵ-Regularity

Theorem (Cheeger-Naber, 2014)

Let $(M^n_j, g_j, p_j) \xrightarrow{GH} (X^n_\infty, d_\infty, p_\infty)$ *satisfy* $|\operatorname{Ric}_{g_j}| \le n-1$*, then the singular set satisfies*

$$\mathcal{S}(X^n_\infty) = \mathcal{S}^{n-4}(X^n_\infty). \quad (35)$$

In particular, $\dim_{\mathcal{H}}(\mathcal{S}) \le n-4$*.*

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In particular, $\dim_{\mathcal{H}}(\mathcal{S}) \leq n-4$.

- If a tagent cone $T_pX^n_\infty \equiv \mathbb{R}^{n-3} \times C(Y)$, then $Y \equiv \mathbb{S}^3$ and $T_pX^n_\infty \equiv \mathbb{R}^n$.

Quantitative Symmetry and ϵ-Regularity

Theorem (Cheeger-Naber, 2014)

Given $n \geq 2$, $v > 0$, *there exists* $\epsilon(n, v) > 0$ *such that the following holds. Let* (M^n, g, p) *satisfy* $|\operatorname{Ric}_g| \leq (n-1)\epsilon^2$, $\operatorname{Vol}(B_1(p)) \geq v > 0$ *and* M^n *is* $(n-3, \epsilon, 2)$*-symmetric at* p, *then* $r_h(p) \geq 1$.

Quantitative Symmetry and ϵ-Regularity

Theorem (Cheeger-Naber 2014, Naber-Jiang 2016)

Let (M^n_j, g_j, p_j) *be Einstein manifolds with* $|\operatorname{Ric}| \le n-1$ *and* $\operatorname{Vol}(B_1(p_j)) \ge v$ *such that*

$$(M^n_j, g_j, p_j) \xrightarrow{pGH} (X^n, d_\infty, p_\infty), \tag{36}$$

then the following holds:

- *For every* $q < 2$,

$$\operatorname{Vol}(T_r(\mathcal{B}_r)) \le C(n, v, q) r^{2q}, \tag{37}$$

  where $\mathcal{B}_r \equiv \{x \in M^n | r_{|\operatorname{Rm}|}(x) \le r\}$.
- *(Naber-Jiang)* q *can be improved to* 2.

Quantitative Symmetry and ϵ-Regularity

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- In fact, even in the non-collapsed setting, the cone assumption is unnecessary. Such an improvement is useful in the study of regularity in the collapsed setting. Roughly, in the context of bounded Ricci curvature, non-collapsed limit with $\mathbb{R}^{n-3}$-splitting in effect implies smoothness.

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- In fact, even in the non-collapsed setting, the cone assumption is unnecessary. Such an improvement is useful in the study of regularity in the collapsed setting. Roughly, in the context of bounded Ricci curvature, non-collapsed limit with $\mathbb{R}^{n-3}$-splitting in effect implies smoothness.
- The above improvement mainly follows from a quantitative differentiation argument which is the quantitative version of Cheeger-Colding's metric cone structure theorem.

Quantitative Symmetry and ϵ-Regularity

Theorem (Cheeger-Naber, 2014)

Given $n \geq 2$, $v > 0$, *there exists* $\delta(n, v) > 0$, $r_0(n, v) > 0$ *s.t. if* (M^n, g, p) *satisfies* $|\operatorname{Ric}_g| \leq (n-1)\delta^2$, $\operatorname{Vol}(B_1(p)) \geq v > 0$ *and*

$$d_{GH}(B_2(p), B_2(0^{n-3}, y)) < \delta, \ (0^{n-3}, y) \in \mathbb{R}^{n-3} \times Y, \tag{38}$$

where (Y, y) *is a metric space, then* $r_h(p) \geq r_0 > 0$.

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Some key points in the proof:

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Some key points in the proof:

Definition

Let (X, d, p) be a metric space. For $\alpha \in \mathbb{N}$, let $r_\alpha \equiv 2^{-\alpha} > 0$. Let $\delta > 0$, r_α is called a good scale if X is $(0, \delta, r_\alpha)$-symmetric. Otherwise, r_α is a bad scale.

Quantitative Symmetry and ϵ-Regularity

Theorem (Quantitative Metric Cone Structure Theorem)

Let (M^n, g, p) be a Riemannian manifold with $\mathrm{Ric}_g \geq -(n-1)$ and $\mathrm{Vol}(B_1(p)) \geq v > 0$, then for every $\delta > 0$, there exists $N(\delta, n, v) > 0$ such that every $x \in M^n$ has at most N bad scales.

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- The above theorem immediately implies that for every $\delta > 0$ and $x \in M^n$, there exists $2^{-N-1} < r < 2^{-N}$ such that x is $(0, \delta, r)$-symmetric.

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- The above theorem immediately implies that for every $\delta > 0$ and $x \in M^n$, there exists $2^{-N-1} < r < 2^{-N}$ such that x is $(0, \delta, r)$-symmetric.
- We can choose $\delta > 0$ sufficiently small such that the quantitative $\mathbb{R}^{n-3}$-splitting assumption gives that x is $(n-3, \epsilon_1, r)$-symmetric, where $\epsilon_1 > 0$ is the constant in Cheeger-Naber's ϵ-regularity theorem (symmetric version). Then the proof is complete.

Almost Flat Manifolds

Theorem (Gromov, 1978)

There exists $\epsilon(n) > 0$ and $w(n) < \infty$ such that if (M^n, g) is a closed manifold satisfying

$$\| \sec_g \|_{C^0(M^n)} \cdot \mathrm{diam}_g^2(M^n) < \epsilon, \tag{39}$$

then M^n is finitely covered by a nilmanifold N^n/Γ of order $\leq w(n)$, where N^n is a simply-connected nilpotent Lie group and $\Gamma \leq N^n$.

Almost Flat Manifolds

- Gromov's theorem was improved by E. Ruh.

Theorem (Ruh, 1982)

There exists $\epsilon(n) > 0$ *and* $w(n) < \infty$ *such that if* (M^n, g) *is a closed manifold satisfying*

$$\| \sec_g \|_{C^0(M^n)} \cdot \mathrm{diam}_g^2(M^n) < \epsilon, \tag{40}$$

then M^n *is an infra-nilmanifold. That is, the universal cover* N^n *is a simply-connected nilpotent Lie group and*

$$\Lambda \equiv \pi_1(M^n) \leq N \rtimes \mathrm{Aut}(N). \tag{41}$$

Moreover, $[\Lambda : \Lambda \cap N^n] \leq w(n)$.

Collapse with Bounded Curvature

Theorem (K. Fukaya, Smooth Limit)

Let (M_j^n, g_j) *satisfy* $|\sec_{g_j}| \leq \Lambda_0$ *and* $\mathrm{diam}_{g_j}(M_j^n) \leq D_0$ *such that*

$$(M_j^n, g_j) \xrightarrow{GH} (M_\infty^k, g_\infty), \tag{42}$$

where (M_∞^k, g_∞) *is a smooth Riemannian manifold.*

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where (M_∞^k, g_∞) *is a smooth Riemannian manifold. Then for any sufficiently large* $j \geq J_0(n, \Lambda_0, D_0)$*, there is a fiber bundle map* $f_j : M_j^n \to M_\infty^k$ *with the following properties:*

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1. *For every* $x \in M_\infty^k$, $\mathrm{diam}_{g_j}(f_j^{-1}(x)) \to 0$.

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Let (M_j^n, g_j) *satisfy* $|\sec_{g_j}| \le \Lambda_0$ *and* $\mathrm{diam}_{g_j}(M_j^n) \le D_0$ *such that*

$$(M_j^n, g_j) \xrightarrow{GH} (M_\infty^k, g_\infty), \tag{42}$$

where (M_∞^k, g_∞) *is a smooth Riemannian manifold. Then for any sufficiently large* $j \ge J_0(n, \Lambda_0, D_0)$*, there is a fiber bundle map* $f_j : M_j^n \to M_\infty^k$ *with the following properties:*

1. *For every* $x \in M_\infty^k$, $\mathrm{diam}_{g_j}(f_j^{-1}(x)) \to 0$.
2. *For every* $x \in M_\infty^k$, $\| \mathrm{II}_{f_j^{-1}(x)} \| \le C(n, \Lambda_0, D_0)$.

Collapse with Bounded Curvature

Theorem (K. Fukaya, Smooth Limit)

Let (M_j^n, g_j) satisfy $|\sec_{g_j}| \leq \Lambda_0$ and $\mathrm{diam}_{g_j}(M_j^n) \leq D_0$ such that

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2. *For every $x \in M_\infty^k$, $\| \mathrm{II}_{f_j^{-1}(x)} \| \leq C(n, \Lambda_0, D_0)$.*
3. *(Gromov and Ruh) Each fiber is diffeomorphic to an $(n-k)$-dimensional infra-nilmanifold.*

Collapse with Bounded Curvature

Theorem (K. Fukaya, General Limit)

Let (M_j^n, g_j) be a sequence of closed manifolds with

$$|\sec_{g_j}| \leq \Lambda, \ \operatorname{diam}_{g_j}(M_j^n) \leq D \tag{43}$$

and $(M_j^n, g_j) \xrightarrow{GH} (X_\infty^k, d_\infty)$. Then there is a diagram

$$\begin{array}{ccc} (F(M_j^n), O(n)) & \xrightarrow{eqGH} & (Y_\infty, O(n)) \\ \downarrow \mathrm{pr}_j & & \downarrow \mathrm{pr}_\infty \\ (M_j^n, g_j) & \xrightarrow{GH} & (X_\infty^k, d_\infty) \end{array} \tag{44}$$

such that Y_∞ is a smooth manifold with a $C^{1,\alpha}$-metric.

Collapse with Bounded Curvature

Theorem (Fukaya, General Limit)

Moreover, for each sufficiently large j, there is an $O(n)$-equivariant fiber bundle map

$$\Gamma\backslash N \to F(M_j^n) \xrightarrow{\widehat{F}_j} Y_\infty \tag{45}$$

with nilpotent fibers, which induces a (singlar) infranil fibration

$$N' \to M_j^n \xrightarrow{F_j} X_\infty^k. \tag{46}$$

Collapse with Bounded Curvature

Theorem (J. Cheeger, 1969)

Given $n \geq 2$, $v > 0$ and $D > 0$, there exists $C(n, D, v) > 0$ such that the class of closed manifolds (M^n, g) satisfying

$$|\sec_g| \leq 1, \ \mathrm{diam}_g(M^n) \leq D, \ \mathrm{Vol}_g(M^n) \geq v > 0, \tag{47}$$

contains finite diffeomorphism types of number bounded by $C(n, D, v)$.

Collapse with Bounded Curvature

Theorem (Fukaya)

Let (M^n, g) be a complete Riemannian manifold with $|\sec_g| \leq 1$. There exists $\delta(n) > 0$ such that for every $x \in M^n$, there is some open neighborhood $B_\delta(x) \subset U_x \subset B_{10\delta}(x)$ with the fiber bundle structure

$$\mathbb{D}^k \longrightarrow U_x \longrightarrow N^{n-k}, \tag{48}$$

where N^{n-k} is an infranilmanifold.

Collapse with Bounded Curvature

Definition (Pure Nilpotent Structure)

A pure nilpotent structure is given by the above $O(n)$-invariant fibration structure $\Gamma\backslash N \to F(M^n) \to Y$.

Collapse with Bounded Curvature

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A pure nilpotent structure is given by the above $O(n)$-invariant fibration structure $\Gamma\backslash N \to F(M^n) \to Y$.

Definition (Mixed Nilpotent Structure)

A mixed nilpotent structure $\{(\mathcal{O}_\alpha, \mathcal{N}_\alpha)\}$ is an atlas on M^n such that

- each $(\mathcal{O}_\alpha, \mathcal{N}_\alpha)$ is a pure nilpotent structure
- (compatibility) If $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$, then restricting to $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$, $(\mathcal{O}_\alpha, \mathcal{N}_\alpha)$ is a substructure $(\mathcal{O}_\beta, \mathcal{N}_\beta)$ or vice versa.

Collapse with Bounded Curvature

Theorem (Cheeger-Fukaya-Gromov, 1992)

There exists $v_0(n) > 0$ such that if (M^n, g) is complete with

$$|\sec_g| \leq 1, \; \mathrm{Vol}_g(B_1(x)) < v_0, \; \forall x \in M^n, \tag{49}$$

then there is a mixed $\mathcal{N}$-structure of positive rank on M^n and for every $\epsilon > 0$ there exists an $\mathcal{N}$-invariant g_ϵ nearby g such that

1. $e^{-\epsilon} g < g_\epsilon < e^{\epsilon} g$,
2. $|\nabla^g - \nabla^{g_\epsilon}| < \epsilon$,
3. $|\nabla^k_{g_\epsilon} \mathrm{Rm}_{g_\epsilon}| \leq C_k(n, \epsilon), \forall k \in \mathbb{N}$.

Collapse with Bounded Curvature

Theorem (Q. Cai - X. Rong, 2009)

If (M^n, g) admits an $\mathcal{N}$-structure of positive rank, then there are a family of invariant metrics g_ϵ satisfying

$$|\sec_{g_\epsilon}| \leq 1, \ \mathrm{InjRad}_{g_\epsilon}(x) \leq \epsilon, \ \forall x \in M^n. \tag{50}$$

In particular, $\mathrm{MinVol}(M^n) = 0$ *and all characteristics of* M^n *vanish.*

The Margulis Lemma

Theorem (Margulis)

Let G be a connected Lie group and let G_0 be its identity component, then there is some open neighborhood

$$e \in \mathcal{Z}_e \leq G_0 \tag{51}$$

such that if $\Gamma \leq G$ is discrete, then $\langle \Gamma \cap \mathcal{Z}_e \rangle$ is nilpotent.

- $\mathcal{Z}_e$ is called the Zassenhaus neighborhood.

The Margulis Lemma

Theorem (Margulis)

Let (M^n, g) be a complete manifold with $-1 \leq \sec_g \leq 0$, then there exists $\delta(n) > 0$ and $w(n) > 0$ such that for every $p \in M^n$, the group $\Gamma_\delta(p)$ contains a nilpotent subgroup of index $\leq w(n)$.

The Margulis Lemma

Theorem (Heintze-Margulis)

Let (M^n, g) *be a complete manifold with* $-1 \leq \sec_g < 0$ *and* $\mathrm{InjRad} \to 0$*, then there exists* $\delta(n) > 0$ *and* $p \in M^n$ *such that*

$$\mathrm{InjRad}(p) \geq \delta > 0. \quad (52)$$

The Generalized Margulis Lemma

Theorem (Cao-Cheeger-Rong, 2004)

There exists $\delta(n) > 0$ such that the following holds. Let (M^n, g) be a closed manifold with $\sec_g \leq 0$ and at some point $\mathrm{Ric}_g < 0$. Then for any metric h with $|\sec_h| \leq 1$, there is some $p \in M^n$ such that

$$\mathrm{InjRad}_h(p) \geq \delta > 0. \tag{53}$$

The Generalized Margulis Lemma

Theorem (Cheeger-Tian, 2005)

Let (M^4, g, p) *be a complete Einstein* 4*-manifold with*

$$\mathrm{Ric}_g = \pm 3g \tag{54}$$

and

$$\int_{M^4} |\,\mathrm{Rm}\,|^2 \leq \Lambda_0. \tag{55}$$

Given any $\epsilon > 0$*, there exists* $\delta(\Lambda_0, \epsilon) > 0$ *such that*

$$\frac{\mathrm{Vol}\{x \in M^4 |\, \mathrm{InjRad}(x) \geq \delta\}}{\mathrm{Vol}(M^4)} \geq 1 - \epsilon. \tag{56}$$

Collapse with Locally Bounded Curvature

Definition

We say $U \subset (M^n, g)$ is v_0-collapsed with locally bounded curvature if

$$\mathrm{Vol}(B_{r_{|\mathrm{Rm}|}}(p)) \leq v_0 \cdot (r_{|\mathrm{Rm}|}(p))^n \tag{57}$$

for all $p \in U$.

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Theorem (Cheeger-Tian, 2005)

Let (M^n, g) be Einstein, then there exists $v_0(n) > 0$ such that if $U \subset M^n$ is v_0-collapsed with locally bounded curvature, then U admits an $\mathcal{N}$-structure of positive rank.

Example: Codimension-1 Collapse

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- Let $(\mathbb{R}^2 \times \mathbb{S}^2_\epsilon, g_S)$ be the Schwarzschild space with $\mathrm{Ric}_{g_S} \equiv 0$. Choose $\mathcal{O}_\epsilon \subset \mathbb{R}^2 \times \mathbb{S}^2_\epsilon$ with $\partial \mathcal{O}_\epsilon = \mathbb{S}^1_\epsilon \times \mathbb{S}^2_\epsilon$.

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- Attach $\mathcal{O}_\epsilon$ on N^4, after smoothing, then the resulting manifold (M^4, g_ϵ) is closed with $|\mathrm{Ric}_{g_\epsilon}| \leq 3$ and $|\sec_{g_\epsilon}| \to +\infty$ as $\epsilon \to 0$.

Example: Codimension-1 Collapse of a K3 Surface

L. Foscolo constructed a family of hyperkähler metrics g_ϵ on $\mathrm{K3}$

$$(\mathrm{K3}, g_\epsilon) \xrightarrow{GH} \mathbb{T}^3/\mathbb{Z}_2 \tag{58}$$

with a punctured subset

$$\mathbb{T}^* \equiv \mathbb{T}^3 \setminus \{q_1, \ldots, q_8, p_1, \tau(p_1), \ldots, p_n, \tau(p_n)\} \tag{59}$$

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where n satisfies some "balancing condition" such that:

- There is a non-trivial S^1-fibration over $\mathbb{T}^*$.
- Curvatures blow up around the punctures but uniformly bounded away from the punctures.
- Each bubble is an ALF gravitational instanton.

Example: Codimension-2 Collapse of a K3 Surface

Due to Gross-Wilson, there are a family of hyperkähler metrics g_ϵ on a $\mathrm{K3}$ surface with $(\mathrm{K3}, g_\epsilon) \xrightarrow{GH} (S^2, d_\infty)$:

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- Singular Fibration Structure: There is a holomorphic fibration, $f_\epsilon : K3 \longrightarrow S^2$ with a finite set $\mathcal{S} = \{q_\alpha\}_{\alpha=1}^{24}$ such that

$$\begin{cases} f_\epsilon^{-1}(q) \overset{homeo}{\cong} \mathbb{T}^2, \ q \in \mathbb{S}^2 \setminus \mathcal{S}, \\ f_\epsilon^{-1}(q) \overset{homeo}{\cong} I_1, \ q \in \mathcal{S}. \end{cases} \tag{60}$$

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  - Each bubble is isometric to the Taub-NUT metric.

Example: Codimension-3 Collapse of a $\mathrm{K3}$-Surface

G. Chen-X. Chen constructed a family of hyperkähler metrics g_ϵ on a $\mathrm{K3}$ surface which collapse to a closed interval,

$$(\mathrm{K3}, g_\epsilon) \xrightarrow{GH} ([0,1], dt^2) \tag{61}$$

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- Each bubble X is an ALH gravitational instanton with

$$\int_X |\operatorname{Rm}|^2 = 96\pi^2. \tag{62}$$

Quantitative Splitting Theorem

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Quantitative Splitting Theorem

- Without assuming bounded curvature, a priori there is no fibration structure or $\mathcal{N}$-structure.
- Let $\mathrm{Ric}_g \geq -(n-1)$.
  1. Cheeger and Colding discovered a replacement of the fibration map which controls the collapsing geometry "in the L^2 sense".
  2. In the case of $|\sec_g| \leq 1$, the $\mathcal{N}$-structure contains all the collapsing information. In general, the collapsed information at the level of the fundamental group is controlled by the Generalized Margulis Lemma (Kapovitch-Wilking).

Quantitative Splitting Theorem

Theorem (Cheeger-Colding, 1996)

Let (M_j^n, g_j, p_j) *be a sequence of manifolds with* $\mathrm{Ric}_{g_j} \geq -(n-1)\delta_j^2$ *such that*

$$(M_j^n, g_j, p_j) \xrightarrow{GH} (X_\infty, d_\infty, p_\infty). \tag{63}$$

If X_∞ *admits a line, then* $X_\infty \equiv \mathbb{R}^k \times Y_\infty$ *and* Y_∞ *does not admit any line.*

Quantitative Splitting Theorem

Definition (Cheeger-Colding's ϵ-splitting map)

An ϵ-splitting map $\Phi \equiv (u^{(1)}, \dots, u^{(k)}) : B_r(p) \to \mathbb{R}^k$ is a harmonic map (i.e. $\Delta u^{(\alpha)} = 0$) such that

$$\sum_{\alpha,\beta=1}^{k} ⨍_{B_r(p)} |\langle \nabla u^{(\alpha)}, \nabla u^{(\beta)} \rangle - \delta_{\alpha\beta}| + ⨍_{B_r(p)} |\nabla^2 u^{(\alpha)}|^2 < \epsilon. \quad (64)$$

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- The above gradient and Hessian estimates amount to the “Toponogov Theorem” in the L^2 sense.
- There is some $\Psi(\epsilon|n, r) > 0$ such that

$$\|d_t(p) - \underline{d}_t(p)\|_{L^2} + \|\angle_t - \underline{\angle}_t\|_{L^2} < \Psi. \quad (65)$$

Quantitative Splitting Theorem

Theorem (Cheeger-Colding, 1996)

$\forall\ \epsilon > 0$, $n \geq 2$, $r > 0$, $\exists\ \delta(n, \epsilon, r) > 0$ *such that*

1. *if* $\mathrm{Ric}_g \geq -(n-1)\delta^2$ *and there is an* ϵ*-splitting map* $\Phi \equiv (u^1, \ldots, u^k) : B_{4r}(p) \to \mathbb{R}^k$*, then*

$$d_{GH}(B_r(p), B_r(0^k, x)) < \epsilon r,\ \ B_r(0^k, x) \subset \mathbb{R}^k \times X, \tag{66}$$

   for some complete length space (X, d)*.*

2. *if*

$$\begin{cases} \mathrm{Ric}_g \geq -(n-1)\delta^2 \\ d_{GH}\Big(B_{\delta^{-1}}(p), B_{\delta^{-1}}(0^k, x)\Big) < \delta, \end{cases} \tag{67}$$

   then there is an ϵ*-splitting map* $\Phi : B_{4r}(p) \to \mathbb{R}^k$*.*

Quantitative Splitting Theorem

Theorem (Cheeger-Colding 1996, Cheeger-Colding-Tian 2002)

The ϵ-splitting map $\Phi \equiv (u^{(1)}, \ldots, u^{(k)}) : B_{4R}(p) \to \mathbb{R}^k$ *satisfies:*

1. $\mathrm{Vol}(B_R(0^k) \setminus u(B_R(p))) < \Psi(\epsilon | n, R)$.
2. *Let* $\omega^\ell \equiv du^1 \wedge \ldots \wedge du^\ell$ *for* $1 \le \ell \le k$,

$$\fint_{B_R(p)} |\mathrm{Ric}(\nabla u^{(\alpha)}, \nabla u^{(\alpha)})| + \fint_{B_R(p)} \Big| |\omega^\ell| - 1 \Big| < \Psi(\epsilon | n, R). \tag{68}$$

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1. $\mathrm{Vol}(B_R(0^k) \setminus u(B_R(p))) < \Psi(\epsilon|n, R)$.
2. *Let* $\omega^\ell \equiv du^1 \wedge \ldots \wedge du^\ell$ *for* $1 \leq \ell \leq k$,

$$\fint_{B_R(p)} |\mathrm{Ric}(\nabla u^{(\alpha)}, \nabla u^{(\alpha)})| + \fint_{B_R(p)} \Big||\omega^\ell| - 1\Big| < \Psi(\epsilon|n, R). \tag{68}$$

- Cheeger-Colding-Tian proved the fibers are almost totally geodesic in the L^2-sense.

THANK YOU!