Gravitational collapsing of K3 surfaces I

Jeff Viaclovsky

University of California, Irvine

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Yau's Theorem

Theorem (Yau 1976)

A compact Kähler manifold admits a Ricci-flat Kähler metric $\iff c_1(X) = 0$.

Abstract existence theorem. What do metrics looks like?

Natural families:

- complex structure J
- Kähler class $[\omega]$.

K3 surfaces

$$X = \{f_4(z_0, z_1, z_2, z_3) = 0\} \subset \mathbb{P}^3.$$

Algebraic K3s: 19-dimensional family.

Since K_X is trivial,

$$H^1(X, \Theta) \equiv H^1(X, \Omega^1)$$

so there is actually a $b^{1,1} = 20$-dimensional family of Js.

Each J has a 20-dimensional Kähler cone.

Moduli of Yau's metrics$= 40 + 20 = 60$-dimensional? Overcounted:

Hyperkähler struture

Kähler $\Longrightarrow$ Hol $\subset U(2)$.

K_X trivial $\Longrightarrow \exists \Omega = \omega_J + i\omega_K$ parallel $(2,0)$-form $\Longrightarrow$ Hol $\subset Sp(1) = SU(2)$.

Each of Yau's metrics is Kähler w.r.t,

$$aI + bJ + cK, \ a^2 + b^2 + c^2 = 1,$$

an S^2s worth of complex structures.

Metric moduli $= 58$-dimensional.

General theory

$Ric(g_j) = 0 \Longrightarrow$ Gromov-Hausdorff limit.

- Singularity formation $\Longrightarrow$ curvature blows up.
- Bubbling phenomena: rescaled limits are complete Ricci-flat spaces.
- Volume non-collapsing: $Vol(B_{p_j}(1)) > v_0 > 0 \Longrightarrow$ orbifold limit.
- Volume collapsing $Vol(B_{p_j}(1)) \to 0 \Longrightarrow$ lower-dimensional limit.

Theorem (Cheeger-Tian)

Sequence collapses with uniformly bounded curvature away from finitely many points.

Examples

- Kummer surface: 4-dim limit $= T^4/\mathbb{Z}_2$, with flat metric. At 16 singular points, Eguchi-Hanson metric on $\mathcal{O}_{\mathbb{P}^1}(-2)$ bubbles off. Bubbles are ALE.
- Foscolo: 3-dim limit $= T^3/\mathbb{Z}_2$, with flat metric. At 8 singular points, ALF D_2 metrics bubble off.
- Gross-Wilson: 2-dim limit $= S^2$. Away from 24 singular points, sequence collapses with uniformly bounded curvature, with T^2-fibers being uniformly scaled down. At 24 singular points, Taub-NUT ALF metrics bubble off.

Chen-Chen

Chen-Chen: 1-dim limit $= [0,1]$. Singular points at 0 and 1.

Interior: collapse with unformly bounded curvature, uniform shrinking of flat T^3.

Bubbles are ALH spaces:

$$g = dr^2 + g_{T^3} + O(e^{-\delta r}).$$

as $r \to \infty$, which arise from rational elliptic surfaces:

$$RES = Bl_{p_1,\ldots,p_9}\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^1,$$

and $X = RES \setminus T^2$, where T^2 is a smooth fiber (Tian-Yau).

Chen-Chen produce these examples by gluing together 2 ALH factors with a long cylindrical region in between, using earlier ideas of Kovalev-Singer, Floer.

Tian-Yau metrics

Let DP_b be a degree $1 \leq b \leq 9$ del Pezzo surface. Let $T^2 \subset DP_b$ be a smooth anticanonical divisor.

Theorem (Tian-Yau)

$X_b = DP_b \setminus T^2$ *admits a complete Ricci-flat Kähler metric, which is asymptotic to a Calabi ansatz metric on a punctured disc bundle in* N_{T^2}.

Solution of the form $\omega_g = \frac{i}{2\pi}\Big\{\partial\overline{\partial}(-\log\|S\|^2)^{\frac{3}{2}} + \partial\overline{\partial}\phi\Big\}$.

We would like to "glue" two of these spaces together, but the asymptotic geometry is not cylindrical: need to find appropriate neck region.

Main result

Theorem (Hein-Sun-Viaclovsky-Zhang)

Given any positive integer $1 \leq m \leq 18$, there is a family of hyperkähler metrics g_ϵ on a $K3$ surface which collapse to an interval $[0,1]$,

$$(K3, g_\epsilon) \xrightarrow{GH} ([0,1], dt^2), \ \epsilon \to 0,$$

such that the following topological and regularity properties hold.

- *There exist distinct points $t_i \in (0,1)$, $i = 1 \dots m$, such that at fixed distance away from the t_i, the sequence collapses with uniformly bounded curvature, with regular fibers diffeomorphic to 3-dimensional Heisenberg nilmanifolds or 3-dimensional tori.*

Main result cont'd

Theorem (HSVZ cont'd)

- *There exist points* $x_{\epsilon,i} \to t_i$, *such that* $|Rm_{g_\epsilon}|(x_{\epsilon_i}) \to \infty$ *as* $\epsilon \to 0$, *and rescalings of the metrics near* $x_{\epsilon,i}$ *converge to Taub-NUT metrics.*
- *If* $t = 0$ *or* $t = 1$, *there exist points* $x_{\epsilon,i} \to t$, *such that* $|Rm_{g_\epsilon}|(x_{\epsilon_i}) \to \infty$ *as* $\epsilon \to 0$, *and rescalings of the metrics near* $x_{\epsilon,i}$ *converge to Tian-Yau metrics.*

By varying the choice of neck region, we can arrange that the number of singular points in the interior can be any integer in $[1, b_- + b_+]$. Also, the degrees of the nilmanifolds in the regular collapsing regions can vary from $-b_+$ to b_- and all such degrees can occur.

Heisenberg nilmanifolds

We will assume that the lattice of the torus is $\Lambda = \epsilon\mathbb{Z}\langle 1, \tau\rangle$ in $\mathbb{R}^2_{x,y} = \mathbb{C}$ such that $T^2 = \mathbb{C}/\Lambda$. Let $\tau_1 = Re(\tau)$ and $\tau_2 = Im(\tau)$, and $A = \epsilon^2\tau_2$.

Recall the Heisenberg group $\mathcal{H}^3$ is

$$\begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

for $(x, y, z) \in \mathbb{R}^3$. For $b \in \mathbb{Z}_+$, the Heisenberg nilmanifold $Nil^3_b(\epsilon, \tau)$ is the quotient of $\mathcal{H}^3$ by the action generated by

$$\begin{pmatrix} 1 & \epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \epsilon\tau_1 & 0 \\ 0 & 1 & \epsilon\tau_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \frac{A}{b} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where these elements act on the *left*.

Heisenberg nilmanifolds

Note that these transformations are

$$(x,y,t) \mapsto (x+\epsilon, y, t+\epsilon y)$$
$$(x,y,t) \mapsto (x+\epsilon\tau_1, y+\epsilon\tau_2, t+\epsilon\tau_1 y)$$
$$(x,y,t) \mapsto (x, y, t+\frac{A}{b}).$$

Left-invariant 1-forms:

$$dx, dy, \theta_b \equiv \frac{2\pi b}{A}(dt - x dy)$$

Heisenberg nilmanifolds

Nil^3_b is an S^1-bundle over T^2 of degree b:

$$\begin{tikzcd} S^1 \arrow[r] & Nil^3_b \arrow[d, "\pi"] \\ & T^2. \end{tikzcd}$$

In our main theorem, in the regular collapsing regions, the T^2s and the S^1s shrink at different rates:

$$\begin{aligned} \mathrm{diam}(Nil^3) &\sim \epsilon \\ \mathrm{diam}(S^1) &\sim \epsilon^2 \end{aligned}$$

Heisenberg nilmanifolds of negative degree

For $b \in \mathbb{Z}_+$, we define the Heisenberg nilmanifold Nil^3_{-b} to be the quotient of $\mathcal{H}^3$ by the action generated by

$$
\begin{aligned}
(x, y, t) &\mapsto (x + \epsilon, y, t - \epsilon y) \\
(x, y, t) &\mapsto (x + \epsilon\tau_1, y + \epsilon\tau_2, t - \epsilon\tau_1 y) \\
(x, y, t) &\mapsto (x, y, t - \frac{A}{b}).
\end{aligned}
$$

Note that the generated action is conjugate to the previous action by the mapping $(x, y, t) \mapsto (-x, -y, -t)$.

Left-invariant 1-forms:

$$dx, dy, \theta_{-b} \equiv \frac{2\pi b}{A}(dt + xdy).$$

(Negative degrees are necessary because our gluing procedure needs an orientation-reversing attaching map on one side of the neck.)

The model metric

Gibbons-Hawking ansatz over $U = T^2_{x,y} \times \mathbb{R}_{z>0}$, with

$$V = \frac{2\pi b}{A} z$$

for a positive integer $b > 0$. Total space N has one complete end as $z \to \infty$ and one incomplete end as $z \to 0$.

Choosing the connection form to be $\theta_b = 2\pi(b/A)(dt - xdy)$, we can write

$$g_{model} = \frac{2\pi bz}{A}(dx^2 + dy^2 + dz^2) + \frac{A}{2\pi bz}\theta_b^2,$$

with

- $d\theta = \frac{2\pi b}{A} dvol_{T^2}$,
- The level sets $\{z = constant\}$ are identified with $Nil^3_b(\epsilon, \tau)$, with a left-invariant metric (depending on z).

Hyperkähler triples

The forms

$$\omega_1 = dz \wedge \theta + V dx \wedge dy$$
$$\omega_2 = dx \wedge \theta + V dy \wedge dz$$
$$\omega_3 = dy \wedge \theta + V dz \wedge dx.$$

are a hyperkähler triple,

$$\omega_i \wedge \omega_j = 2\delta_{ij} dvol_g.$$

We will need to construct an approximate hyperkähler triple on the "glued" manifold.

Acharya-Gibbons-Hawking-Hull

ALH_b metrics

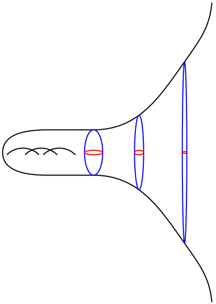
Making the substitution $z = (3/2)s^{2/3}$, and then scaling appropriately, the metric takes the form

$$ds^2 + s^{2/3} g_{T^2} + s^{-2/3}\Big(\frac{A}{3b\pi}\theta_b\Big)^2.$$

- Volume growth is $O(s^{4/3})$.
- $Rm \in L^2$
- $|Rm| = O(s^{-2})$ as $s \to \infty$, but not any better. Thus these asymptotics do not fall under the classification of Chen-Chen.

If $b = 0$ and $V = constant$, this is ALH geometry. For $b \neq 0$, we will therefore refer to this type of geometry as ALH_b geometry.

ALH_b ends



The red circles represent the S^1 fibers, the blue curves represent the T^2s.

Note that, in terms of distance to a basepoint,

$$\mathrm{diam}(Nil_b^3(s)) \sim s^{1/3}$$
$$\mathrm{diam}(S_s^1) \sim s^{-1/3}.$$

Tian-Yau metrics are ALH_b

Theorem (HSVZ)

A Tian-Yau metric on $X_b = DP_b \setminus T^2$ is ALH_b, with

$$g = g_{model,b} + O(e^{-\delta s^{2/3}})$$

as $s \to \infty$, for some $\delta > 0$.

The proof relies on finding good asymptotics for the complex structure, and then using techniques in Hein's thesis and Tian-Yau.

Gauge transformation to make the leading term of the connection our standard choice.

Moreover, there is a hyperkähler triple which is asymptotic to our model hyperkähler triple.

The neck potential

Choose $p_1, \ldots, p_{b_+ + b_-} \in T^2 \times \mathbb{R}$. There exists $V : T^2 \times \mathbb{R} \setminus \mathcal{P} \to \mathbb{R}$ such that

- $\Delta V = 0$
- $V \sim \frac{1}{2r}$ near each monopole point.
- $\frac{1}{2\pi} * dV \in H^2(T^2 \times \mathbb{R} \setminus \mathcal{P}, \mathbb{Z})$.
- $V = O(e^{-\delta|z|}) + \begin{cases} \frac{2\pi}{A} b_- z + c_- & z \ll 0 \\ -\frac{2\pi}{A} b_+ z + c_+ & z \gg 0 \end{cases}$

Proof: in the universal cover, at large distances, V looks like electric potential of a collection of uniformly charged plates. Free to add kz to fix leading terms.

The neck metric

Since $\frac{1}{2\pi} * dV \in H^2(T^2 \times \mathbb{R} \setminus \mathcal{P}, \mathbb{Z})$. there is a corresponding S^1-bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathcal{N} \\ & & \big\downarrow \pi \\ & & T^2 \times \mathbb{R} \setminus \mathcal{P}. \end{array}$$

and a connection form θ so that

$$\Omega = d\theta = *dV.$$

The neck metric:

$$g_{\mathcal{N}} = V(g_{T^2} + dz^2) + V^{-1}\theta^2.$$

Problem

Problem: Gibbons-Hawking requires a *positive* harmonic function, but the above electric potential is negative.

Solution: add a large constant:

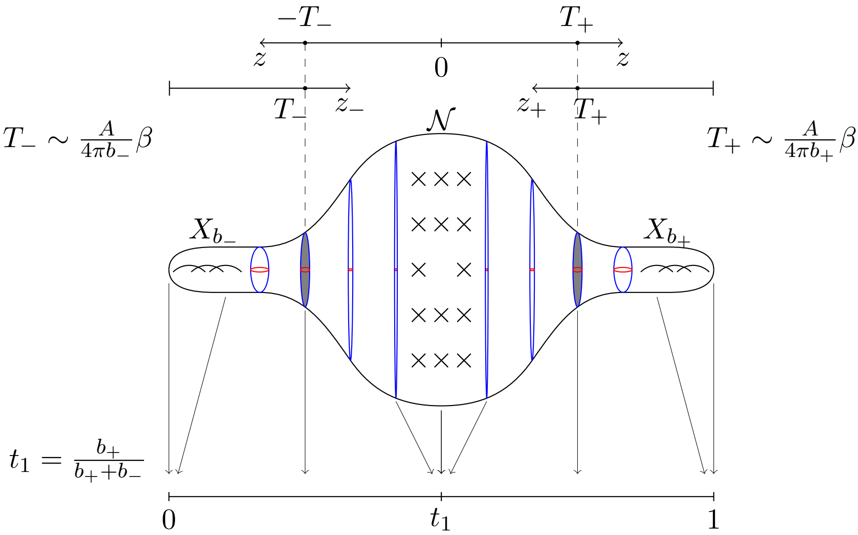
$$V_\beta = V + \beta,$$

where $\beta \gg 0$.

This gives us an *incomplete* metric on the region $\mathcal{N}(T_-, T_+)$, where $-T_- < z < T_+$, where $T_\pm \sim \beta$.

Analogous to Ooguri-Vafa metric, our case is a doubly-periodic analogue of this.

The approximate metric



Curvature of the approximate metric

For $x \in \mathcal{N}(-T_- + 1, T_+ - 1) \subset \mathcal{M}$, there exists constants C, C' so that

$$|Rm|(x) \leq \begin{cases} \beta & r(x) < C'\beta^{-1} \\ \frac{C}{\beta r(x)^2} & r(x) \geq C'\beta^{-1} \end{cases},$$

where $r(x)$ denotes the Euclidean distance to the monopole points.
For $x \in X_{b_\pm}(T_\pm) \subset \mathcal{M}$, there is a constant C so that

$$|Rm|(x) \leq \begin{cases} C & d(x) < \zeta_\pm \\ \frac{C}{d(x)^2} & d(x) \geq \zeta_\pm, \end{cases}$$

where $d(x)$ is metric distance to a base point.
For $x \in DZ_\pm \subset \mathcal{M}$, there is a constant C so that

$$|Rm|(x) \leq C\beta^{-3}.$$

Proof: use the formula $|Rm|^2 = \frac{1}{2}V^{-1}\Delta^2(V^{-1})$.

Models for degenerations of complex structure

- Let $b_+ = b_- = 9$. $X_+ = X_- = \mathbb{P}^2 \setminus \{s_3 = 0\}$
- X = degree 2 K3 surface: $\pi : X \to \mathbb{P}^2$, $2:1$ branched over a sextic s_6.
- Degeneration: $s_6 \to s_3^2$

Parameter counting

- Choice of monpole points: $18 \cdot 3 - 1 = 53$ parameters. We subtract 1 because we can fix the z-coordinate of one of the monopole points to be at 0.
- S^1 rotation when attaching: 1 parameter. (since the neck has a triholomorphic circle action, there is really only 1 rotational parameter.)
- The main gluing parameter β (which determines T_- and T_+): 1 parameter.
- The area of the torus: 1 parameter, which corresponds to an overall scaling of the metric.
- Total of 56 gluing parameters.

Parameter counting

The complex structure on $\mathbb{P}^2 \setminus T^2$ is determined from the choice of cubic, which gives 2 parameters. Note that

$$\chi(X \setminus T^2) = \chi(X) = 3,$$

so $b_2(X \setminus T^2) = 2$. We have $b_2 = b^{2,0} + b^{0,2} = 2$, so $b^{1,1}_{L^2} = 0$, and there are no Kähler deformations of the Tian-Yau metrics (besides scaling, which was already counted above). Adding everything up

$$56 + 2 = 58.$$

Models for degenerations of complex structure

- Let $b_+ = b_- = 8$, with $DP_8 = S^2 \times S^2$.
  $Q_+ = \{q_+ = 0\} \subset \mathbb{P}^3$
  $Q_- = \{q_- = 0\} \subset \mathbb{P}^3$
  $Q_+ \cap Q_- = T^2$.
  $X_+ = Q_+ \setminus T^2, X_- = Q_- \setminus T^2$.
- Degeneration: smooth quartic $q_4 \rightarrow q_2 \cdot q_2'$. Neck is a desingularization of the union of 2 nonsingular quadrics.

Parameter counting

First, there are $3(8+8)-1+1+1+1=50$ gluing parameters. Let X_+ and X_- both arise from quadrics in $\mathbb{P}^3$, which are degree 8. The first quadric, we can assume is the standard diagonal quadric. We can then diagonalize the second quadric. This gives 6 parameters for deformation of complex structure. We have

$$\chi(X \setminus T^2) = \chi(X) = 4,$$

so $b_2(X \setminus T^2) = 3$. Then $b_2 = b^{2,0} + b^{1,1} + b^{0,2} = 2 + b^{1,1}$, so $b^{1,1}_{L^2} = 1$. So each Tian-Yau piece has a 1-dimensional space of Kähler deformations. Adding everything up

$$50 + 6 + 2 = 58.$$

Models for degenerations of complex structure

- Let $b_+ = 3$, $b_- = 9$, with
  $DP_3 = Bl_{p_1,\ldots,p_6}\mathbb{P}^2 = \{q_3 = 0\} \subset \mathbb{P}^3$.
  $DP_9 = \mathbb{P}^2 = \{l_1 = 0\} \subset \mathbb{P}^3$
  $DP_3 \cap DP_9 = T^2$.
  $X_+ = DP_3 \setminus T^2, X_- = \mathbb{P}^2 \setminus T^2$.

- Degeneration: smooth quartic $q_4 \to q_3 \cdot l_1$. Neck is a desingularization of the union of a plane and a nonsingular cubic.

Parameter counting

First, there are $3(9+3)-1+1+1+1=38$ gluing parameters. In this case, X_+ arises from a cubic and X_- arises from a plane in $\mathbb{P}^3$, which are degree 3 and 9, respectively. The cubic is $\mathbb{P}^2$ blown-up at 6 points. We can fix 4 points, so we have 8 dimensions of variation of complex structure. Once this is fixed, the choice of the plane is arbitrary, which gives 6 more parameters. Next,

$$\chi(X_+ \setminus T^2) = \chi(X_+) = 9,$$

so $b_2(X \setminus T^2) = 8$. We have $b_2 = b^{2,0} + b^{1,1} + b^{0,2} = 2 + b^{1,1}$, so $b^{1,1}_{L^2} = 6$. So this has a 6-dimensional space of Kähler deformations. From above, $\mathbb{P}^2 \setminus T^2$ has no Kähler deformations (besides scaling). Adding up,

$$38 + 8 + 6 + 6 = 58.$$

Talk II by Ruobing Zhang

- Analysis of harmonic functions on ALH_b spaces, and Liouville Theorems.
- Analysis of rescaled geometry of approximate metrics.
- Definition of weighted Hölder spaces and weighted Schauder estimate.
- Main blow-up analysis to prove uniform injectivity of linearized operator of hyperkähler triple gluing.

End of Part I

Thank you for your attention.