Gravitational Collapsing of K3 Surfaces II

Ruobing Zhang (Stony Brook University)

Structure of Collapsed Special Holonomy Spaces
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Welcome back to the joint talk!

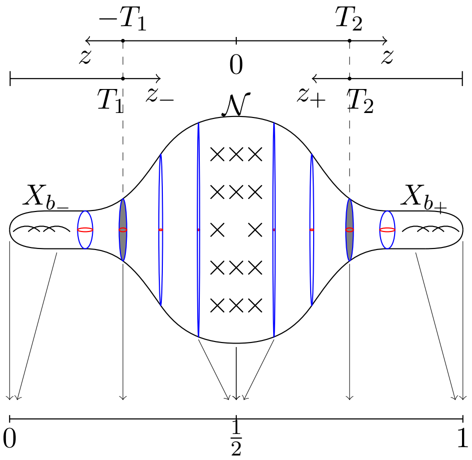

Figure: Red circles: collapsed S^1-fibers, Blue curves: collapsed $\mathbb{T}^2$-fibers, Gray regions: damage zones.

We will focus on the details of the gluing construction and the main technical ingredients in the proof.

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- $\mathrm{K3}$ does not admit collapsing metrics g_ϵ with bounded curvatures.
- Let g_ϵ be a family of Ricci-flat metrics on $\mathrm{K3}$ with

$$(\mathrm{K3}, g_\epsilon) \xrightarrow{GH} (X^k_\infty, d_\infty). \tag{1.1}$$

Main Theorems

Theorem (Fibration Theorem, Hein-Sun-Viaclovsky-Zhang)

There are a family of hyperkähler metrics g_ϵ on $\mathrm{K3}$ which are collapsing to a closed interval, i.e.

$$(\mathrm{K3}, g_\epsilon) \xrightarrow{GH} ([0,1], dt^2), \tag{1.2}$$

such that $\mathrm{K3}$ can be viewed as a singular nilpotent fibration over a closed interval such that for a regular fiber as $\epsilon \to 0$:

- $\mathrm{diam}_{g_\epsilon}(T^2) \sim \epsilon$
- $\mathrm{diam}_{g_\epsilon}(S^1) \sim \epsilon^2$
- $|\mathrm{Rm}_{g_\epsilon}| \sim C_0$ *for some constant C_0.*

Main Theorems

Remark

- Let x be regular, then on the universal cover, $B_{1/10}(\tilde{x}) \approx B_{1/10}(0^4) \subset \mathbb{R}^4$.
- Let x be singular, then $B_{1/10}(\tilde{x})$ is a ball in the Taub-NUT space such that

$$\mathrm{Vol}_g(B_{1/10}(\tilde{x})) \leq C\epsilon^2 \to 0. \quad (1.3)$$

Main Theorems

The singular domain $\mathcal{S}(t_0)$ has 3 connected components: $\mathcal{S}_-(t_0)$, $\mathcal{S}_+(t_0)$ and $\mathcal{S}_{neck}(t_0)$

Theorem (Bubble Classification, HSVZ)

Let g_ϵ be a family of hyperkähler metrics on $\mathrm{K3}$ in the previous theorems, then the bubble limits from $\mathcal{S}(t_0)$ can be classified as follows:

1. *each bubble in $\mathcal{S}_-(t_0)$ and $\mathcal{S}_+(t_0)$ is isometric to a complete hyperkähler Tian-Yau space,*
2. *each bubble in $\mathcal{S}_{neck}(t_0)$ is isometric to the Ricci-flat Taub-NUT space.*

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- *The neck region* $\mathcal{N}^4_{b_- + b_+}$ *satisfies*

$$\begin{cases} \chi(\mathcal{N}^4_{b_- + b_+}) = b_- + b_+, \\ \frac{\mathrm{diam}_g(\mathcal{N}^4_{b_- + b_+})}{\mathrm{diam}_g(\mathcal{M}^4)} \in (\frac{1}{10}, \frac{1}{2}), \end{cases} \tag{1.4}$$

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then there exists a hyperkähler metric on $\mathcal{M}^4$ *and it is diffeomorphic to a* $\mathrm{K3}$ *surface.*

Topological Invariants of the Gluing Space $\mathcal{M}^4$

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	$X^4_{b_-}$	$X^4_{b_+}$	$\mathcal{N}^4_{b_-+b_+}$	$\mathcal{M}^4$
b_0	1	1	1	1
b_1	0	0	2	0
b_2	$11 - b_-$	$11 - b_+$	$b_- + b_+ + 2$	22
b_3	0	0	1	0
b_4	0	0	0	1
χ	$12 - b_-$	$12 - b_+$	$b_- + b_+$	24

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- Equivalently,

$$\mathcal{N}^4_{18} = \Big((\mathbb{R} \setminus \{0\}) \times \mathrm{Nil}^3\Big) \cup \Big(\{0\} \times \mathrm{Sing}^3\Big). \tag{1.5}$$

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- Sing^3 is a singular S^1-bundle over $\mathbb{T}^2$ with 18 singularities.

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0	$\mathbb{C}P^2 \setminus \mathbb{T}^2$	$\mathbb{Z}_3$	0	$\to \infty$
$t \in (0, 1/2)$	Nil^3	$H(1, \mathbb{Z})$	3	$\leq C$
$t = 1/2$	$S^1 \to N^3 \to \mathbb{T}^2$	$\mathbb{Z} \oplus \mathbb{Z}$	2	$\to \infty$
$t \in (1/2, 1)$	Nil^3	$H(1, \mathbb{Z})$	3	$\leq C$
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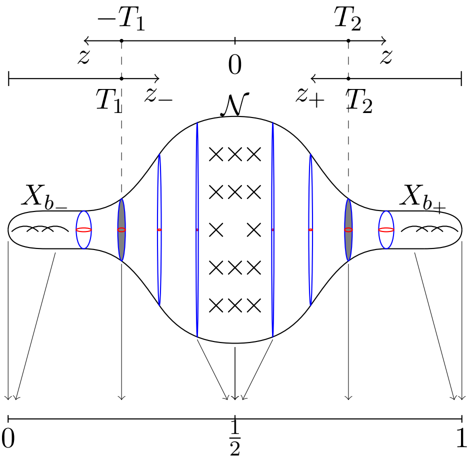
$$H(1, \mathbb{Z}) \equiv \left\{ \begin{bmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} : m, n, p \in \mathbb{Z} \right\}. \qquad (2.1)$$



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General ϵ-Regularity of Collapsed Einstein Manifolds

Theorem (Naber-Zhang, 2016)

Let (M^n, g, p) *be Einstein with* $|\operatorname{Ric}_g| \leq n-1$, *then there exists* $\delta(n) > 0$ *such that if*

$$d_{GH}\big(B_2(p), B_2(0^k)\big) < \delta,\ \ B_2(0^k) \subset \mathbb{R}^k, \tag{2.2}$$

then

$$\operatorname{rank}(\Gamma_\delta(p)) = n-k \iff \sup_{B_1(p)} \|\operatorname{Rm}\| \leq C(n). \tag{2.3}$$

The Model Space

- Given $b \in \mathbb{Z}_+$, our model space $(\mathfrak{M}^4, g_b)$ is a Gibbons-Hawking space over $\mathbb{T}^2 \times \mathbb{R}$ such that

$$g_b = V_b(dx^2 + dy^2 + dz^2) + V_b^{-1}\theta_b^2, \ z > 10^2, \tag{2.4}$$

where

$$\begin{cases} V_b(z) = 2\pi b z \\ \theta_b = b(dt - x dy). \end{cases} \tag{2.5}$$

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- $\mathfrak{M}^4$ is diffeomorphic to $[10^2, +\infty) \times \mathrm{Nil}^3$ and Nil^3 is a $\mathbb{T}^2$-bundle over S^1 with monodromy $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

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$$\Phi : [10^2, +\infty) \times \mathrm{Nil}^3 \xrightarrow{\text{diffeo}} X^4 \setminus K. \tag{2.6}$$

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- The neck region $\mathcal{N}^4_{b_- + b_+}$ is a Gibbons-Hawking space over $\mathbb{T}^2 \times \mathbb{R}$ with $(b_- + b_+)$-monopoles such that

$$\chi(\mathcal{N}^4_{b_- + b_+}) = b_- + b_+, \tag{2.7}$$

and it is also asymptotic to the model space $\mathfrak{M}^4$.

Asymptotics of the Tian-Yau Spaces

Theorem (HSVZ)

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be the diffeomorphism between GH and TY, then

- *(complex structure asymptotics)* $\forall\ k \in \mathbb{N}$ *and* $\epsilon > 0$*,*

$$|\nabla^k_{g_{model}}(\Phi^* J_{TY} - J_{model})|_{g_{model}} = O(e^{(-\frac{1}{2}+\epsilon)z^2}). \tag{2.9}$$

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- *(metric asymptotics)* $\exists\ \delta_0 > 0$ *such that* $\forall\ k \in \mathbb{N}$,

$$|\nabla^k_{g_{model}}(\Phi^* g_{TY} - g_{model})|_{g_{model}} = O(e^{-\delta_0 z}). \tag{2.10}$$

Geometric Structure of $\mathcal{N}^4_{b_-+b_+}$

(1) Given monopoles $\mathcal{P}_k \equiv \{p_1, \ldots, p_k\} \subset \mathbb{T}^2 \times \mathbb{R}$, $\exists$ a global (sign-changing) Green's function $-\Delta_{g_k} V_\infty = 2\pi \sum_{j=1}^{k} \delta_{p_j}$ with

$$V_\infty(x, y, z) \overset{exp}{\approx} \begin{cases} \pi k z, & z \to -\infty, \\ -\pi k z, & z \to +\infty. \end{cases} \quad (2.11)$$

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(2) Let $k \equiv b_- + b_+$, we choose V_β such that $V_\beta = V_\infty + (Lz + \beta)$ and

$$V_\beta(x,y,z) \overset{exp}{\approx} \begin{cases} 2\pi b_- z + \beta, & z \to -\infty, \\ -2\pi b_+ z + \beta, & z \to +\infty. \end{cases} \quad (2.12)$$

The gluing is possible when $\beta \to +\infty$.

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(3) Let $\mathcal{N}^4_k$ be the singular S^1-bundle

$$S^1 \to \mathcal{N}^4_k \xrightarrow{\pi} \mathbb{T}^2 \times \mathbb{R} \tag{2.13}$$

with a connection 1-form θ such that $d\theta = *_{\mathbb{T}^2 \times \mathbb{R}} \circ dV_\beta$.

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(4) Let $\mathcal{N}^4_k$ equip with the Ricci-flat Gibbons-Hawking metric

$$g_k \equiv V_\beta \cdot (dx^2 + dy^2 + dz^2) + (V_\beta)^{-1}\theta^2. \tag{2.14}$$

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(5) To obtain $\mathrm{K3}$, the number of monopoles $k \equiv b_- + b_+$ and hence

$$\begin{cases} \chi(\mathcal{N}^4_{b_-+b_+}) = b_- + b_+ \\ \chi(\mathcal{M}^4) = 24. \end{cases} \quad (2.15)$$

Hyperkähler Triple and Nonlinear PDE

- To prove the existence of a hyperkähler triple, we need to solve the following elliptic system,

$$\begin{cases} d^+\alpha + \zeta = \mathfrak{F}\Big(\operatorname{Id} - Q - (d\alpha)^- \wedge (d\alpha)^- \Big) \\ d^*\alpha = 0, \end{cases} \tag{3.1}$$

where $\alpha \in \Omega^1(\mathcal{M}^4)$ and $\zeta \in \mathcal{H}^+(\mathcal{M}^4)$.

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  where $\alpha \in \Omega^1(\mathcal{M}^4)$ and $\zeta \in \mathcal{H}^+(\mathcal{M}^4)$.
- Denote by

$$\mathscr{D} \equiv d^* + d^+, \tag{3.2}$$

  then the linearized operator is

$$\mathscr{L} \equiv \mathscr{D} \oplus \operatorname{Id} : \Omega_0^1(\mathcal{M}) \oplus \mathcal{H}_+^2(\mathcal{M}) \longrightarrow \Omega_+^2(\mathcal{M}), \tag{3.3}$$

  where $\Omega_0^1(\mathcal{M})$ stands for the space of divergence free 1-forms.

Implicit Function Theorem on Banach Spaces

Lemma (Implicit Function Theorem)

Let $\mathscr{F} : \mathfrak{A} \to \mathfrak{B}$ be a map between two Banach spaces such that $\mathscr{F}(x) - \mathscr{F}(0) = \mathscr{L}(x) + \mathscr{N}(x)$, where $\mathscr{L}$ is the linearized operator.

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- *$\mathscr{L}$ is invertible and $\|\mathscr{L}^{-1}\| \leq C_1$,*
- *$\exists\, r > 0$ s.t. $\forall\, x, y \in B_r(0)$,*
  $\|\mathscr{N}(x) - \mathscr{N}(y)\|_{\mathfrak{B}} \leq C_2 \cdot (\|x\|_{\mathfrak{A}} + \|y\|_{\mathfrak{A}}) \cdot \|x - y\|_{\mathfrak{A}}$,

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  $\|\mathscr{N}(x) - \mathscr{N}(y)\|_{\mathfrak{B}} \leq C_2 \cdot (\|x\|_{\mathfrak{A}} + \|y\|_{\mathfrak{A}}) \cdot \|x - y\|_{\mathfrak{A}}$,
- *$\|\mathscr{F}(0)\|_{\mathfrak{B}} \leq \frac{r}{2C_1}$,*

then there exists a unique solution to $\mathscr{F}(x) = 0$ with

$$\|x\|_{\mathfrak{A}} \leq 2C_1 \cdot \|\mathscr{F}(0)\|_{\mathfrak{B}}. \tag{3.4}$$

Weight Analysis for the Linearized Operator

- The fundamental work is to establish the right weighted space and effective estimate for

$$\mathscr{L} \equiv \mathscr{D} \oplus \mathrm{Id} : \Omega^1_0(\mathcal{M}) \oplus \mathcal{H}^2_+(\mathcal{M}) \longrightarrow \Omega^2_+(\mathcal{M}). \qquad (3.5)$$

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- Goal 1 (uniform elliptic regularity): $\mathscr{D}$ has some effective elliptic regularity.
- Goal 2 (uniform injectivity): there is some uniform constant C (independent of $\beta > 0$),

$$\|\omega\|_{C_*^{1,\alpha}(\mathcal{M})} \leq C\|\mathscr{D}\omega\|_{C_*^{0,\alpha}(\mathcal{M})}, \tag{3.6}$$

for some globally defined weighted norm.

The Natural Rescaled Geometries

- For every $\boldsymbol{x}_j \in \mathcal{M}_j$, we will choose the rescaling factors $\lambda_j > 0$ and the corresponding rescaled metric $\tilde{g}_j = \lambda_j^2 g_j$ we have the convergence

$$(\mathcal{M}_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty). \tag{4.1}$$

The Natural Rescaled Geometries

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$$(\mathcal{M}_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty). \tag{4.1}$$

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- Curvatures are uniformly bounded of almost full volume in the collapsing sequence.
- We rescale the metrics in a “continuous” manner.

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- Let $d_{g_j}(p_m, \boldsymbol{x}_j) \nearrow$, the size of the $\mathbb{T}^2$-fiber will be shrinking such that the rescaled limit will become $\mathbb{T}^2 \times \mathbb{R}$.
- If $\boldsymbol{x}_j \in X^4$, we choose the original scale so that we will obtain a complete Tian-Yau-space.

The Natural Rescaled Geometries

Let $(\mathcal{N}^4_{m_0}, g_\beta)$ be the Gibbon-Hawking space defined by

$$g_\beta \equiv V_\beta g_{\mathbb{T}^2} + V_\beta^{-1}\theta^2, \tag{4.4}$$

such that within the injectivity radius of $\mathbb{T}^2 \times \mathbb{R}$,

$$V_\beta(\boldsymbol{x}) = \frac{1}{2|\boldsymbol{x}|} + \beta + h(\boldsymbol{x}), \tag{4.5}$$

and $h(\boldsymbol{x})$ is bounded harmonic function. Given any positive constant $\sigma > 0$, we define the rescaled metric as follows,

$$\begin{cases} \lambda_{\sigma,\beta} \equiv \sigma \cdot \beta^{\frac{1}{2}} \\ \tilde{g}_{\sigma,\beta} \equiv (\lambda_{\sigma,\beta})^2 g_\beta. \end{cases} \tag{4.6}$$

The Natural Rescaled Geometries

Lemma (Bubble limit of around the monopoles)

Let p_0 be a monopole and let $\sigma > 0$, we have the following C^∞-convergence

$$(\mathcal{N}^4_{m_0}, \tilde{g}_{\sigma,\beta}, p_m) \xrightarrow{C^\infty} (\mathbb{R}^4, \tilde{g}_{\sigma,\infty}, \tilde{p}_{m,\infty}) \text{ as } \beta \to +\infty, \tag{4.7}$$

where $(\mathbb{R}^4, \tilde{g}_{\sigma,\infty}, \tilde{p}_{m,\infty})$ is a Ricci-flat Taub-NUT space with

$$\tilde{g}_{\sigma,\infty} = G_\sigma(dx^2 + dy^2 + dz^2) + (G_\sigma)^{-1}\theta^2 \tag{4.8}$$

and

$$G_\sigma(x,y,z) = \frac{1}{2\sqrt{x^2+y^2+z^2}} + \frac{1}{\sigma^2}. \tag{4.9}$$

The Natural Rescaled Geometries

Denote by $\beta_j \to \infty$ be a sequence of gluing parameters and let $\tilde{g}_j \equiv \lambda_j^2 g_j$ such that

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathcal{X}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty). \tag{4.10}$$

BDD Geometry Regions in $\mathcal{X}_\infty$	Choice of λ_j
Standard Taub-NUT	$\lambda_j \equiv \beta_j^{\frac{1}{2}}$
"collapsing" Taub-NUT	$\lambda_j = (d_{p_j}(\boldsymbol{x}_j))^{-1}$
$\mathbb{R}^3 \setminus \{0^3\}$	$\lambda_j = (d_{p_j}(\boldsymbol{x}_j))^{-1}$
$(\mathbb{T}^2 \times \mathbb{R}) \setminus \{p_1, \ldots, p_{b_-+b_+}\}$	$\lambda_j = (d_{p_j}(\boldsymbol{x}_j))^{-1}$
$(\mathbb{T}^2 \times \mathbb{R}) \setminus \{p_1, \ldots, p_{b_-+b_+}\}$	$\lambda_j \equiv \beta_j^{-\frac{1}{2}}$
$\mathbb{T}^2 \times \mathbb{R}$	$\lambda_j \equiv \frac{1}{(V_\beta(\boldsymbol{x}_j))^{\frac{1}{2}}}$
$X_{b_-}^4, X_{b_+}^4$	$\lambda_j \equiv 1$

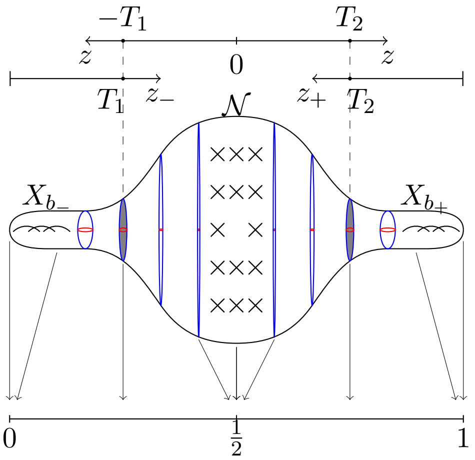

Figure: Red circles: collapsed S^1-fibers, Blue curves: collapsed $\mathbb{T}^2$-fibers, Gray regions: damage zones.

Uniform Elliptic Regularity: the Weighted Schauder Estimate

Proposition (Weighted Schauder Estimate)

Let $(\mathcal{M}, g)$ *be the gluing space with a sufficiently large gluing parameter* $\beta > 0$*. Given* $\alpha \in (0,1)$*, then there exists a uniform constant* $C_0(\alpha) > 0$ *such that for every* $\omega \in \Omega^1(\mathcal{M})$*, it holds that*

$$\|\omega\|_{C^{1,\alpha}_{\delta,\nu,\mu}(\mathcal{M})} \leq C\Big(\|\mathscr{D}_g\omega\|_{C^{0,\alpha}_{\delta,\nu+1,\mu}(\mathcal{M})} + \|\omega\|_{C^{0}_{\delta,\nu,\mu}(\mathcal{M})}\Big). \qquad (4.11)$$

The Uniform Injectivity Estimate for $\mathscr{D}$

Proposition (Uniform Injectivity Estimate)

Let $(\mathcal{M}, g_\beta)$ be the gluing space with a sufficiently large gluing parameter $\beta > 0$. Assume that the parameters δ, μ and ν satisfy

1. $0 < \delta < \delta_0$,
2. $\mu + \nu \in (0, 1)$,

where $\delta_0 \in (0, 10^{-2})$ is independent of $\beta > 0$.
Then for every $\alpha \in (0, 1)$, there exists a uniform constant $C = C(\alpha, \delta, \mu, \nu) > 0$ which is independent of β such that for every $\omega \in \Omega^1(\mathcal{M})$ it holds that

$$\|\omega\|_{C^{1,\alpha}_{\delta,\nu,\mu}(\mathcal{M})} \leq C \cdot \|\mathscr{D}\omega\|_{C^{0,\alpha}_{\delta,\nu+1,\mu}(\mathcal{M})}. \tag{4.12}$$

The Rescaling Argument

- By rescaling, we obtain the scenario that there is a rescaled "limiting" 1-form $\tilde{\omega}_\infty$ in some weighted space which satisfies (after rescaling),

$$\begin{cases} \widetilde{\mathscr{D}}_\infty \tilde{\omega}_\infty \equiv 0 \\ \left| \rho^{(0)}_{\delta,\nu,\mu}(x_0) \cdot \tilde{\omega}_\infty(x_0) \right| = 1. \end{cases} \tag{4.13}$$

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- Need to prove a Liouville type theorem:

$$\begin{cases} \widetilde{\mathscr{D}}_\infty \tilde{\omega}_\infty \equiv 0 \\ |\tilde{\omega}_\infty(x)| \leq (\tilde{\rho}_\infty)^{-1}(x) \end{cases} \tag{4.14}$$

implies $\tilde{\omega}_\infty \equiv 0$. So the contradiction arises.

The Weight Functions

- Consistent with the different collapsing behaviors, there are 9 different pieces in the definition of weight function.

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- Consistent with the different collapsing behaviors, there are 9 different pieces in the definition of weight function.
- For simplicity, we only list the rescaled limits and the rescaled weight function.

Bdd Geo. Region in $(\mathcal{M}_\infty, \tilde{g}_\infty, x_\infty)$	Rescaled Weight
Taub-NUT	$\tilde{\rho} = (d_{p_j})^\mu$, $\mu > 0$
"collapsing" Taub-NUT	$\tilde{\rho} = (d_{p_j})^\mu$, $\mu > 0$
$\mathbb{R}^3 \setminus \{0^3\}$	$\tilde{\rho} = (d_{p_j})^\mu$, $\mu > 0$
$(\mathbb{T}^2 \times \mathbb{R}) \setminus \{p_1, \ldots, p_{b_- + b_+}\}$	$\tilde{\rho} = \exp(\delta z)$, $\delta > 0$
$\mathbb{T}^2 \times \mathbb{R}$	$\tilde{\rho} = \exp(\delta z)$, $\delta > 0$
$X^4_{b_-}$, $X^4_{b_+}$	$\tilde{\rho} = \exp(\delta z)$, $\delta > 0$

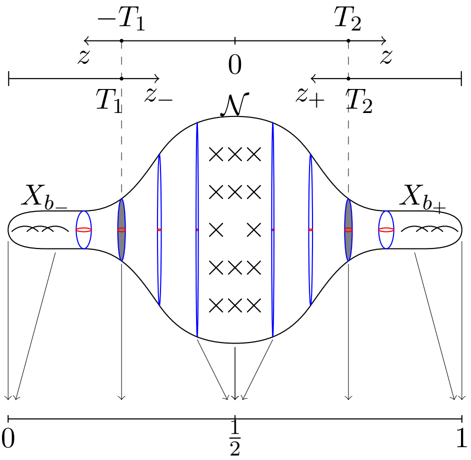

Figure: Red circles: collapsed S^1-fibers, Blue curves: collapsed $\mathbb{T}^2$-fibers, Gray regions: damage zones.

The Weight Functions

For complexity, we list the weight functions at the original scale.

$$\rho_{\delta,\nu,\mu}^{(k+\alpha)}(\boldsymbol{x}) \equiv \begin{cases} e^{\delta\cdot(2T_-)}\cdot(\beta^{-1})^{\frac{2\mu+\nu+k+\alpha}{2}}, & \boldsymbol{x}\in \mathrm{I} \\ e^{\delta\cdot(2T_-)}\cdot(\beta^{-\frac{\mu}{2}})\cdot d_m^{\mu+\nu+k+\alpha}(\boldsymbol{x}), & \boldsymbol{x}\in \mathrm{II} \\ e^{\delta\cdot(2T_-)}\cdot(\beta)^{\frac{\nu+k+\alpha}{2}}, & \boldsymbol{x}\in \mathrm{III} \\ e^{\delta\cdot(z(\boldsymbol{x})+2T_-)}\cdot(V_\beta(\boldsymbol{x}))^{\frac{\nu+k+\alpha}{2}} & \boldsymbol{x}\in \mathrm{IV}_- \\ e^{\delta\cdot(z(\boldsymbol{x})+2T_-)}\cdot(V_\beta(\boldsymbol{x}))^{\frac{\nu+k+\alpha}{2}} & \boldsymbol{x}\in \mathrm{IV}_+ \\ e^{\delta\cdot(z_-(\boldsymbol{x}))}\cdot(V_-(\boldsymbol{x}))^{\frac{\nu+k+\alpha}{2}}, & \boldsymbol{x}\in \mathrm{V}_- \\ e^{\delta\cdot(-z_+(\boldsymbol{x})+2T_-+2T_+)}\cdot(V_+(\boldsymbol{x}))^{\frac{\nu+k+\alpha}{2}}, & \boldsymbol{x}\in \mathrm{V}_+ \\ e^{\delta\cdot\zeta_0^-}\cdot\Big(V_-(\zeta_0^-)\Big)^{\frac{\nu+k+\alpha}{2}}, & \boldsymbol{x}\in \mathrm{VI}_- \\ e^{\delta(-\zeta_0^++2T_-+2T_+)}\cdot\Big(V_+(\zeta_0^+)\Big)^{\frac{\nu+k+\alpha}{2}}, & \boldsymbol{x}\in \mathrm{VI}_+\,. \end{cases}$$

The Weighted Space

Definition (Weighted Norm)

Let $\omega \in \Omega^l(\mathcal{M})$,

$$\|\omega\|_{C^k_{\delta,\nu,\mu}(\mathcal{M})} \equiv \sum_{j=0}^{k} \left\| \rho^{(j)}_{\delta,\nu,\mu} \cdot \nabla^j \omega \right\|_{C^0(\mathcal{M})}. \tag{4.15}$$

The Liouville Theorem for Differential 1-Forms

Theorem (HSVZ)

Let (X^4, g) *be a complete Tian-Yau space, then there exists* $\delta_1 \in (0,1)$ *such that if* $\omega \in \Omega^1(X)$ *satisfies*

$$\begin{cases} \mathscr{D}\omega = 0, \\ |\omega| = O(e^{\delta_1 z}), \end{cases} \tag{4.16}$$

then

$$\omega \equiv 0. \tag{4.17}$$

Functional Liouville Theorem

Theorem (HSVZ)

Let (X^4, g) be a complete Tian-Yau space, then there is some constant $\ell_0 \in (0, 1)$ such that the following holds: Let u satisfy

$$\begin{cases} \Delta_g u = 0 \\ u = O(e^{\ell_0 z}), \end{cases} \tag{4.18}$$

then

$$u \equiv const. \tag{4.19}$$

Proof of the Functional Liouville Theorem

- The essence of the proof is to accurately understand the asymptotic behavior of a harmonic function u in X^4.

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Proof of the Functional Liouville Theorem

- The essence of the proof is to accurately understand the asymptotic behavior of a harmonic function u in X^4.
- Apply the asymptotic geometry of the Tian-Yau space to show that u is almost "harmonic" with respect to the model metric.
- Let $\delta \in (0, \delta_0/2)$ such that u satisfies

$$\begin{cases} \Delta_{TY} u = 0 \\ u = O(e^{\delta z}), \end{cases} \tag{4.20}$$

then for every fixed $k \in \mathbb{Z}_+$, we have

$$\|\nabla^k \Delta_{model} u(\boldsymbol{x})\| \leq C(k, g) \cdot e^{-\frac{\delta_0 z(\boldsymbol{x})}{2}}. \tag{4.21}$$

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- (Linear Analysis 2) If $\ell_0 > 0$ is sufficiently small, then
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- The harmonic 1-form du decays at infinity $\Longrightarrow |du| \equiv 0$.

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- $\Delta_{model} u = \xi \Longrightarrow$

$$\frac{d^2 u_k(z)}{dz^2} - (j_k^2 z^2 + \frac{\lambda_k}{2}) u_k(z) = \frac{\xi_k(z) \cdot z}{2}. \tag{4.26}$$

Proof of the Functional Liouville Theorem: Linear Analysis

- In terms of the growth rate, solutions to

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3. slower than $e^z \implies u_k = az + b + O(e^{-\delta z})$

- Given a function $\zeta = O(e^{-\eta_0 z})$, there is a solution to the Possion equation $\Delta_{model} v = u$ such that

$$v = O(e^{-\eta' z}),\ 0 < \eta' < \eta. \tag{4.28}$$

The Proof of the Liouville Theorem for Differential 1-Forms

Theorem (HSVZ)

Let (X^4, g) be a complete Tian-Yau space, then there exists $\delta_1 \in (0,1)$ such that if $\omega \in \Omega^1(X)$ satisfies

$$\begin{cases} \mathscr{D}\gamma = 0, \\ |\gamma| = O(e^{\delta_1 z}), \end{cases} \tag{4.29}$$

then

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The Proof of the Liouville Theorem for Differential 1-Forms

- We write $\gamma = \gamma^{1,0} + \gamma^{0,1}$.
- By Kähler identities,

$$d^+\gamma = 0 \iff \begin{cases} \bar\partial\gamma^{0,1} = 0 \\ i(\bar\partial^*\gamma^{0,1} - \partial^*\gamma^{1,0}) = 0 \end{cases} \tag{4.31}$$

$$d^*\gamma = 0 \iff \bar\partial^*\gamma^{0,1} + \partial^*\gamma^{1,0} = 0. \tag{4.32}$$

- So $d^+\gamma = d^*\gamma = 0$ is equivalent to

$$\bar\partial\gamma = \bar\partial^*\gamma = 0. \tag{4.33}$$

The Proof of the Liouville Theorem for Differential 1-Forms

Main Strategy:

- Find a function $f = u + iv$ such that $\bar{\partial} f = \gamma$ and

$$\Delta_{\omega_{TY}} u = \Delta_{\omega_{TY}} v = 0. \tag{4.34}$$

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- A priori the potential function f might have unsatisfying growth rate.
- The key point is to improve the growth rate to

$$u = O(e^{\delta z}),\ v = O(e^{\delta z}) \tag{4.35}$$

for some sufficiently small $\delta > 0$.

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- Let $f = u + iv$, then

$$\Delta_{\omega_{TY}} u = \Delta_{\omega_{TY}} v = 0. \tag{4.36}$$

Step 2: Improvement of the Growth Rate of f

- (The advantage of the complex structure)

$$\bar{\partial} f = \gamma \Longrightarrow Re(\gamma) = du + J_{TY} dv. \tag{4.37}$$

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- Changing to the complex structure of model metric,

$$\begin{cases} |J_{TY} - J_{model}| = O(e^{(-\frac{1}{2}+\epsilon)z^2}) \\ |\nabla^k dJ_{model} du| = O(e^{\epsilon z}) \\ |\nabla^k dJ_{model} dv| = O(e^{\epsilon z}). \end{cases} \tag{4.38}$$

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- Taking trace, then we obtain

$$\begin{cases} |\nabla^k \Delta_{model} u| = O(e^{\epsilon z}) \\ |\nabla^k \Delta_{model} v| = O(e^{\epsilon z}). \end{cases} \tag{4.39}$$

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- We write

$$\begin{cases} u \equiv u_p + u_h \\ v \equiv v_p + v_h \end{cases} \tag{4.40}$$

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- $u_h = O(e^{\epsilon z^2})$ and $v_h = O(e^{\epsilon z^2})$.
- Further improvement: $u_h = O(e^{\epsilon z})$ and $v_h = O(e^{\epsilon z})$.

Step 2: Improvement of the Growth Rate of f

- We write

$$\begin{cases} u \equiv u_p + u_h \\ v \equiv v_p + v_h \end{cases} \tag{4.40}$$

such that

$$\begin{cases} \Delta_{model} u_p = \Delta_{model} u \\ u_p = O(e^{\epsilon' z}), \ \epsilon' \in (\epsilon, \frac{3\epsilon}{2}). \end{cases} \tag{4.41}$$

- $u_h = O(e^{\epsilon z^2})$ and $v_h = O(e^{\epsilon z^2})$.
- Further improvement: $u_h = O(e^{\epsilon z})$ and $v_h = O(e^{\epsilon z})$.
- u and v are constants

Thank you!