Quantitative Nilpotent Structure and Regularity Theorems of Collapsed Einstein Manifolds

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Structure of Collapsed Special Holonomy Spaces
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Ricci Limit Spaces

Definition (Ricci-Limit Space)

We call a metric measure space (X, d_X, ν_X, x) a Ricci-limit space if there exists a sequence of manifolds (M_j^n, g_j, ν_j, p_j) with $\mathrm{Ric}_{g_j} \geq -(n-1)\lambda$ and

$$\nu_j = \frac{\mathrm{dvol}_{g_j}}{\mathrm{Vol}(B_1(p_j))} \tag{1}$$

such that

$$\left(M_j^n, g_j, \nu_j, p_j\right) \xrightarrow{GH} \left(X, d_X, \nu_X, x\right), \tag{2}$$

where $\nu_X \equiv \lim_j \nu_j$ is a Radon measure and called the limiting renormalized measure.

Ricci Limit Spaces

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and $\mathcal{R} \equiv \bigcup_k \mathcal{R}_k$.
- $\mathcal{S} \equiv X \setminus \mathcal{R}$.

Theorem (Cheeger-Colding, 1997)

Let (X, d_X, ν_X, p) be a Ricci-limit space, then $\nu_X(\mathcal{S}) = 0$.

Ricci Limit Spaces

Theorem (Colding-Naber, 2011)

Let (X, d, ν_X, x) be a Ricci-limit space. Then there is a unique integer $k \geq 0$ such that $\nu_X(X \setminus \mathcal{R}_k) = 0$. The above unique integer k is called the limiting dimension of X.

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- In fact, if $\dim(X) = k$, then

$$\mathcal{R} = \bigcup_{j \leq k} \mathcal{R}_j. \tag{4}$$

Ricci Limit Spaces

Theorem (Cheeger-Colding, 2000)

Let (X, d_X, ν_X, p) be a Ricci-limit space for some non-collapsing sequence, then $\mathrm{Isom}(X)$ is a Lie group.

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Theorem (Colding-Naber, 2011)

Let (X, d_X, ν_X, p) be any Ricci-limit space, then $\mathrm{Isom}(X)$ is a Lie group.

Nilpotent Groups

Definition (Nilpotent Group and Nilpotent Rank)

A group Γ is called nilpotent if there exists a finite descending lower central series

$$\Gamma \equiv \Gamma_0 \rhd \Gamma_1 \rhd \ldots \rhd \Gamma_k \equiv \{e\}, \tag{5}$$

where $\Gamma_j \equiv [\Gamma, \Gamma_{j-1}]$ for all $1 \leq j \leq k$ and k is called the step of Γ.

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- For each $1 \leq j \leq k$, $A_j \equiv \Gamma_{j-1}/\Gamma_j$ is abelian.
- If Γ is finitely generated, then A_j is finitely generated for each $1 \leq j \leq k$ and

$$\mathrm{rank}(\Gamma) \equiv \sum_{j=1}^{k} \mathrm{rank}(A_j). \tag{6}$$

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Definition

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Definition (Rank of Almost Nilpotent Group)

Let $N_0 \leq \Gamma$ and $N_1 \leq \Gamma$ be nilpotent such that

$$[\Gamma : N_0] < \infty \text{ and } [\Gamma : N_1] < \infty, \tag{8}$$

then $\operatorname{rank}(N_0) = \operatorname{rank}(N_1)$.
The common number is a defined as the nilpotent rank of Γ.

The Margulis Lemma

Definition (Fibered Fundamental Group)

Let (M^n, g, p) be a Riemannian manifold, then we define

$$\Gamma_\delta(p) \equiv \mathrm{Image}[\pi_1(B_\delta(p)) \to \pi_1(B_2(p))]. \tag{9}$$

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- $\Gamma_\delta(p)$ can be viewed as the subgroup of $\pi_1(B_2(p))$ which is generated by the small loops of length $\leq 2\delta$.
- If we identify $\pi_1(B_2(p))$ with the deck transformation group of $\widetilde{B_2(p)}$, then $\Gamma_\delta(p)$ is generated by the elements with short dispalcement.

The Generalized Margulis Lemma

Theorem (Fukaya-Yamaguchi, 1992)

There exists $\delta(n) > 0$ *and* $w(n) > 0$ *such that if* (M^n, g, p) *satisfy* $\sec_g \geq -1$, *then*

- $\Gamma_\delta(p)$ *has a solvable subgroup* Λ *with* $[\Gamma_\delta(p) : \Lambda] \leq w$.
- $\Gamma_\delta(p)$ *has a nilpotent subgroup* N *with* $[\Gamma_\delta(p) : N] < \infty$.

Generalized Margulis Lemma

Definition ((C, m)-Nilpotency)

A finitely generated group Γ is called (C, m)-nilpotent if there is a nilpotent subgroup $N \leq \Gamma$ with

$$\operatorname{rank}(N) \leq m, \; [\Gamma : N] \leq C. \tag{10}$$

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Theorem (Kapovitch-Petrunin-Tuschmann, 2010)

There are $\epsilon(n) > 0$, $w(n) < \infty$ such that if (M^n, g, p) is complete with $\sec_g \geq -1$, then for any $p \in M^n$, the group

$$\Gamma_\epsilon(p) \equiv \operatorname{Image}[\pi_1(B_\epsilon(p)) \to \pi_1(B_1(p))] \tag{11}$$

is $(w(n), n)$-nilpotent.

The Generalized Margulis Lemma

Theorem (Kapovitch-Wilking, 2011)

There are $\epsilon(n) > 0$, $w(n) < \infty$ *such that if* (M^n, g, p) *is complete with* $\mathrm{Ric} \geq -(n-1)$, *then for any* $p \in M^n$, *the group*

$$\Gamma_\epsilon(p) \equiv \mathrm{Image}[\pi_1(B_\epsilon(p)) \to \pi_1(B_1(p))] \tag{12}$$

is $(w(n), n)$*-nilpotent.*

The Generalized Margulis Lemma

Theorem (Naber-Zhang, 2015)

Let (Z^k, z^k) *be a Ricci-limit with* $\dim Z^k = k$*, then there exists* $\epsilon_0 > 0$*,* $w_0 > 0$*,* $K_0 > 0$ *such that for each* $0 < \delta \leq \epsilon_0$*, if* (M^n, g) *has* $\mathrm{Ric} \geq -(n-1)$ *and*

$$d_{GH}(B_2(p), B_2(z^k)) < \delta, \tag{13}$$

then $\Gamma_\delta(p)$ *has a nilpotent subgroup* $\mathcal{N}$ *such that:*

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then $\Gamma_\delta(p)$ has a nilpotent subgroup $\mathcal{N}$ such that:

1. $[\Gamma_\delta(p) : \mathcal{N}] \le w_0,\ \mathrm{rank}(\mathcal{N}) \le n - k =$ *collapsed dimension.*

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then $\Gamma_\delta(p)$ *has a nilpotent subgroup* $\mathcal{N}$ *such that:*

1. $[\Gamma_\delta(p) : \mathcal{N}] \leq w_0, \ \mathrm{rank}(\mathcal{N}) \leq n - k =$ *collapsed dimension.*
2. $\mathcal{N}$ *has an effective polycyclic refinement,*

$$\mathcal{N} \equiv \mathcal{N}_m \overset{\langle\sigma_m\rangle}{\rhd} \mathcal{N}_{m-1} \overset{\langle\sigma_{m-1}\rangle}{\rhd} \ldots \overset{\langle\sigma_1\rangle}{\rhd} \mathcal{N}_0 = \{e\} \tag{14}$$

such that $d(\sigma_\alpha \cdot \hat{p}, \hat{p}) < K_0 \cdot \delta$ *for each* $1 \leq \alpha \leq m$.

Statements of the Main Theorems

Theorem (Naber-Zhang, 2015)

For $n \geq 4$ *and each Ricci-limit space* (Z^ℓ, z^ℓ) *with* $\ell \leq 3$*, there are constants* $\delta > 0$, $C_0 > 0$ *such that if* (M^n, g, p) *is Einstein with* $|\,\mathrm{Ric}_g\,| \leq n-1$ *and*

$$d_{GH}\big(B_2(p), B_2(0^{k-\ell}, z^\ell)\big) < \delta,\; B_2(0^{k-\ell}, z^\ell) \subset \mathbb{R}^{k-\ell} \times Z^\ell, \quad (15)$$

then $\Gamma_\delta(p)$ *is almost nilpotent with* $\mathrm{rank}(\Gamma_\delta(p)) \leq n-k$*. Furthermore,*

$$\mathrm{rank}(\Gamma_\delta(p)) = n-k \implies \sup_{B_1(p)} |\,\mathrm{Rm}\,| \leq C_0. \quad (16)$$

Statements of the Main Theorems

Theorem (Naber-Zhang, 2015)

In particular, if $\ell = 0$*, i.e.* $\mathbb{R}^{k-\ell} \times Z^{\ell} \equiv \mathbb{R}^{k}$*,*

$$\mathrm{rank}(\Gamma_\delta(p)) = n - k \iff \sup_{B_1(p)} \|\,\mathrm{Rm}\,\| \leq C_0. \tag{17}$$

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In particular, if $\ell = 0$, *i.e.* $\mathbb{R}^{k-\ell} \times Z^\ell \equiv \mathbb{R}^k$,

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- If we drop the Einstein condition, the curvature estimate can be replaced with bounded $C^{1,\alpha}$-covering geometry for any $0 < \alpha < 1$.

Statements of the Main Theorems

Example

Let (M^4, g_{EH}) be the Ricci-flat Eguchi-Hanson manifold. Take $\lambda_j \to 0$ such that

$$\Big(\mathbb{T}^{n-4} \times M^4, \lambda_j^2 \cdot (g_0 \oplus g_{EH}), p\Big) \xrightarrow{GH} \Big(\mathbb{R}^4/\mathbb{Z}^2, d_0, 0^*\Big). \quad (18)$$

Then $\mathrm{Ric}_{g_j} \equiv 0$ and $\mathrm{rank}(\Gamma_\delta(p)) = n - 4$ which is maximal, but curvatures blow up around the cone tip.

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- This example tells us that the assumption "$\ell \leq 3$" in the theorem is necessary.

Statements of the Main Theorems

Example

Let $M^3 \equiv \mathbb{R}^2 \times S^1$. Let $\mathbb{Z}_k$ be a finite cyclic group of rotations which acts isometrically and freely on M^3.

- As $k \to \infty$,

$$(M^3/\mathbb{Z}_k, g_k, p_k) \xrightarrow{GH} (\mathbb{R}^1_+, dt^2, 0^1). \tag{19}$$

- $\sec_{g_k} \equiv 0$.
- $\mathrm{rank}(\Gamma_\delta(p_k)) = 1$.

Statements of the Main Theorems

Theorem (Naber-Zhang, 2015)

Let $(M_j^n, g_j, p_j) \xrightarrow{GH} (X_\infty^k, d_\infty, x_\infty)$ *satisfy* $|\operatorname{Ric}_{g_j}| \leq n-1$ *and for some* $\delta > 0$, $\operatorname{rank}(\Gamma_\delta(p_j)) = n-k$, *then the following holds:*

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Theorem (Naber-Zhang, 2015)

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1. *There is some open subset* $\mathcal{O} \subset B_1(x_\infty)$ *with* $\nu(B_1(x_\infty) \setminus \mathcal{O}) = 0$ *which is a* $C^{1,\alpha}$*-Riemannian manifold for any* $0 < \alpha < 1$.

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- If (M_j^n, g_j) are Einstein, then $C^{1,\alpha}$ can be improved to C^∞.

Statements of the Main Theorems

Theorem (Naber-Zhang, 2015)

Given $\epsilon > 0$, $n \geq 4$ *and a Ricci-limit space* (X^k, x^k), *there exists* $\delta > 0$, $C > 0$ *such that if* (M^n, g, p) *is a Riemannian manifold with* $|\,\mathrm{Ric}_g\,| \leq n-1$ *satisfying*

$$\begin{cases} d_{GH}(B_2(p), B_2(x^k)) < \delta, \\ \Gamma_\delta(p) = n-k, \end{cases} \tag{20}$$

then $\exists\, \mathcal{O} \subset B_1(p)$ *such that* $\frac{\mathrm{Vol}(\mathcal{O})}{\mathrm{Vol}(B_1(p))} > 1-\epsilon$ *and*

$$\sup_{\mathcal{O}} |\,\mathrm{Rm}\,| \leq C. \tag{21}$$

A Toy Model: Codimension-1 Collapse

- Let (M_j^n, g_j, p_j) be a sequence of Einstein manifolds satisfying $\mathrm{Ric}_{g_j} \geq -(n-1)\delta_j^2$ and $\delta_j \to 0$.

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- Assume that

$$(M_j^n, g_j, p_j) \xrightarrow{GH} \mathbb{R}^{n-1} \tag{22}$$

and

$$\Gamma_{\delta_j}(p_j) \equiv \mathbb{Z}. \tag{23}$$

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- We will prove

$$\sup_{B_R(p_j)} |\,\mathrm{Rm}\,| \leq C_0(n, R), \tag{24}$$

where $C_0 > 0$ does not depend on j.

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  1. The universal covers $(\widetilde{M}^n_j, \tilde{g}_j, \tilde{p}_j)$ are non-collapsing.
  2. There is an extra $\mathbb{R}$-splitting in the unwrapped limit:

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- Once we manage to prove (25), the uniform curvature estimate follows from the ϵ-regularity for non-collapsing manifolds (which is due to Cheeger-Colding).

Quantitative Splitting Theorem

Theorem (Cheeger-Colding, 1996)

Let (M_j^n, g_j, p_j) *be a sequence of manifolds with* $\mathrm{Ric}_{g_j} \geq -(n-1)\delta_j^2$ *such that*

$$(M_j^n, g_j, p_j) \xrightarrow{GH} (X_\infty, d_\infty, p_\infty). \tag{26}$$

If X_∞ *admits a line, then* $X_\infty \equiv \mathbb{R}^k \times Y_\infty$ *and* Y_∞ *does not admit any line.*

Quantitative Splitting Theorem

Definition (Cheeger-Colding's ϵ-splitting map)

An ϵ-splitting map $\Phi \equiv (u^{(1)}, \ldots, u^{(k)}) : B_r(p) \to \mathbb{R}^k$ is a harmonic map (i.e. $\Delta u^{(\alpha)} = 0$) such that

$$\sum_{\alpha,\beta=1}^{k} ⨍_{B_r(p)} |\langle \nabla u^{(\alpha)}, \nabla u^{(\beta)} \rangle - \delta_{\alpha\beta}| + ⨍_{B_r(p)} |\nabla^2 u^{(\alpha)}|^2 < \epsilon. \quad (27)$$

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- The above gradient and Hessian estimates amount to the "Toponogov Theorem" in the L^2 sense.
- There is some $\Psi(\epsilon|n, r) > 0$ such that

$$\|d_t(p) - \underline{d}_t(p)\|_{L^2} + \|\angle_t - \underline{\angle}_t\|_{L^2} < \Psi. \quad (28)$$

Quantitative Splitting Theorem

Theorem (Cheeger-Colding, 1996)

$\forall\ \epsilon > 0$, $n \geq 2$, $r > 0$, $\exists\ \delta(n, \epsilon, r) > 0$ *such that*

1. *if* $\mathrm{Ric}_g \geq -(n-1)\delta^2$ *and there is an* ϵ*-splitting map* $\Phi \equiv (u^1, \ldots, u^k) : B_{4r}(p) \to \mathbb{R}^k$, *then*

$$d_{GH}(B_r(p), B_r(0^k, x)) < \epsilon r,\ \ B_r(0^k, x) \subset \mathbb{R}^k \times X, \tag{29}$$

   for some complete length space (X, d).

2. *if*

$$\begin{cases} \mathrm{Ric}_g \geq -(n-1)\delta^2 \\ d_{GH}\Big(B_{\delta^{-1}}(p), B_{\delta^{-1}}(0^k, x)\Big) < \delta, \end{cases} \tag{30}$$

   then there is an ϵ*-splitting map* $\Phi : B_{4r}(p) \to \mathbb{R}^k$.

A Toy Model: Codimension-1 Collapse

- By Cheeger-Colding's Quantitative Splitting Theorem, there are harmonic maps

$$\Phi_j = (u_j^{(1)}, \ldots, u_j^{(n-1)}) : B_{10R}(p_j) \to \mathbb{R}^{n-1} \tag{31}$$

such that

$$\sum_{\alpha,\beta=1}^{n-1} ⨍_{B_{5R}(p_j)} |\langle \nabla u_j^{(\alpha)}, \nabla u_j^{(\beta)} \rangle - \delta_{\alpha\beta}| < \Psi(\delta_j | n, R) \tag{32}$$

and

$$\sum_{\alpha=1}^{n-1} ⨍_{B_{5R}(p_j)} |\nabla^2 u_j^{(\alpha)}|^2 < \Psi(\delta_j | n, R). \tag{33}$$

A Toy Model: Codimension-1 Collapse

- Applying the volume comparison theorem,

$$\sum_{\alpha,\beta=1}^{n-1} \fint_{B_{5R}(\tilde{p}_j)} |\langle \nabla \tilde{u}_j^{(\alpha)}, \nabla \tilde{u}_j^{(\beta)} \rangle - \delta_{\alpha\beta}| < \Psi(\delta_j | n, R) \qquad (34)$$

  and

$$\sum_{\alpha=1}^{n-1} \fint_{B_{5R}(\tilde{p}_j)} |\nabla^2 \tilde{u}_j^{\alpha}|^2 < \Psi(\delta_j | n, R). \qquad (35)$$

  Therefore,

$$(\widetilde{M_j^n}, \tilde{g}_j, \tilde{p}_j) \xrightarrow{GH} \mathbb{R}^{n-1} \times Y \qquad (36)$$

  and $Y = \lim_{j\to\infty} \widetilde{\Phi}_j^{-1}(0^{n-1})$.

A Toy Model: Codimension-1 Collapse

- Let $\Gamma_j \equiv \pi_1(M^n_j)$, then

$$\begin{array}{ccc} \left(\widetilde{M^n_j}, \Gamma_j, \tilde{p}_j\right) & \xrightarrow{\ eqGH\ } & \left(\mathbb{R}^{n-1} \times Y, \Gamma_\infty, \tilde{p}_\infty\right) \\ \Big\downarrow{\scriptstyle \mathrm{pr}_i} & & \Big\downarrow{\scriptstyle \mathrm{pr}_\infty} \\ \left(M^n_j, g_j, p_j\right) & \xrightarrow{\ GH\ } & \mathbb{R}^{n-1} \times \{0\}. \end{array} \tag{37}$$

A Toy Model: Codimension-1 Collapse

- Let $\Gamma_j \equiv \pi_1(M^n_j)$, then

$$\begin{array}{ccc} \left(\widetilde{M^n_j}, \Gamma_j, \tilde{p}_j\right) & \xrightarrow{\quad eqGH \quad} & \left(\mathbb{R}^{n-1} \times Y, \Gamma_\infty, \tilde{p}_\infty\right) \\ \mathrm{pr}_i \Big\downarrow & & \Big\downarrow \mathrm{pr}_\infty \\ \left(M^n_j, g_j, p_j\right) & \xrightarrow{\quad GH \quad} & \mathbb{R}^{n-1} \times \{0\}. \end{array} \tag{37}$$

- $\Gamma_\delta(p_j) \equiv \mathbb{Z} \Longrightarrow Y = \lim_{j\to\infty} \widetilde{\Phi}_j^{-1}(0^{n-1})$ is non-compact.

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- $\Gamma_\infty \leq \mathrm{Isom}(Y)$ acts homogeneously on Y.
- Y is a line $\Longrightarrow Y \equiv \mathbb{R}$.

The Main Technical Theorem

Theorem (Naber-Zhang, 2015)

Given a Ricci-limit space Z^ℓ*, there exists* $v_0(n, Z^\ell)) > 0$*, and for every* $\epsilon > 0$ *there exists* $\delta_0(\epsilon, n, Z^\ell) > 0$ *such that if* (M^n, g, p) *is a manifold with* $\mathrm{Ric}_g \geq -(n-1)$ *satisfying*

- $d_{GH}(B_2(p), B_2(0^{k-\ell}, z^\ell)) < \delta_0,\ B_2(0^{k-\ell}, z^\ell) \subset \mathbb{R}^{k-\ell} \times Z^\ell$,
- $\mathrm{rank}(\Gamma_{\delta_0}(p)) = n - k$,

then on the universal cover $\widetilde{B_2(p)}$ *we have*
(1) $\mathrm{Vol}(B_1(\tilde{p})) \geq v_0(n, B_1(z^\ell)) > 0$,
(2) $\exists\, r \in (\delta_0, 1)$ *such that*

$$d_{GH}(B_r(\tilde{x}), B_r(0^{n-\ell}, \hat{z})) < r\epsilon,\ B_r(0^{n-\ell}, \hat{z}) \subset \mathbb{R}^{n-\ell} \times C(\widehat{Z}), \quad (38)$$

where $\hat{z}$ *is the cone tip of* $C(\widehat{Z})$.

The Proof of the Main Technical Theorem

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2. The universal cover of $B_2(p)$ is non-collapsing.
3. The quantitative splitting holds in general (the limit is $\mathbb{R}^{k-\ell} \times Z^\ell$).

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- The crucial point is to show $Y_\infty \equiv \mathbb{R}^{n-k}$.

Step 1: The Collapsed Limit Space is $\mathbb{R}^k$

Lemma

Let (X, d) be complete non-compact. If X is C_0-homogeneous, i.e., $\exists\, C_0 > 0$ s.t. $\forall\, x, y \in X$, there is an isometry $f \in \mathrm{Isom}(X)$ with

$$d_X(y, f(x)) \leq C_0, \tag{41}$$

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- X/G is compact and $G \le \mathrm{Isom}(X) \Longrightarrow X$ admits a line.
- The above lemma and Cheeger-Colding's Splitting Theorem (in the limit) tell us that

$$Y_\infty \equiv \mathbb{R}^d \times K_\infty \tag{42}$$

for some $d \ge 0$ and some compact space K_∞.

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Subtlety: $\mathbb{R}^d/\mathcal{N}_\infty$ might be non-compact.

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Lemma (Nonlocalness Lemma)

Given $n \geq 2$, $\epsilon > 0$, $R \geq 1$, *there exists*

$$\Psi_1 = \Psi_1(\epsilon, R, n) > 0,\ N_1 = N_1(\epsilon, R, n) < \infty \tag{44}$$

such that the following properties hold: Let (M^n, g, p) *satisfy* $\mathrm{Ric} \geq -(n-1)$. *If* $\gamma \in \mathrm{Isom}(M^n)$ *satisfies*

$$d(\gamma \cdot p, p) < \Psi_1(\epsilon, R, n), \tag{45}$$

then there is some positive integer $1 \leq d \leq N_1$ *such that for all* $x \in B_R(\hat{p})$

$$d(\gamma^d \cdot x, x) < \epsilon. \tag{46}$$

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Idea: $\mathcal{N}_\infty$ is a nilpotent Lie group and $\dim_{\mathcal{H}}(\mathcal{N}_\infty \cdot 0^d) = n - k$
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- $\sigma_\infty \neq \{e\}$ and $\sigma_\infty \in (\mathcal{N}_\infty)_0$ (the identity component of $\mathcal{N}_\infty$)

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- The general case follows from inductive arguments.

Step 2: The Universal Cover is Non-Collapsing

Goal: Let $\mathrm{pr}_j : \widetilde{B_2(p_j)} \to B_2(p_j)$ be the universal cover,

$$\begin{cases} d_{GH}(B_2(p_j), B_2(z^k)) < \delta, \quad B_2(z^k) \subset Z^k, \\ \mathrm{rank}(\Gamma_\delta(p_j)) = n - k \end{cases} \tag{48}$$

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- $\dim(Z^k) = k \overset{\text{Colding-Naber}}{\Longrightarrow}$ Let $q_\infty \in B_1(z^k)$ be a regular point such that $\forall\ \eta > 0$, $\exists\ s(\delta, n, B_1(z^k)) > 0$ such that

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- Let $q_j \in B_1(p_j)$ and $q_j \to q_\infty$, then

$$d_{GH}(B_s(q_j), B_s(0^k)) < 2s \cdot \eta. \tag{50}$$

Step 2: The Universal Cover is Non-Collapsing

- Now it is reduced to Step 1, $\mathrm{pr}_j^{-1}(B_s(q_j))$ contains a smaller ball $B_{s_0}(\tilde{q}_j)$ of definite scale radius which is very close to $B_{s_0}(0^n) \subset \mathbb{R}^n$ (in the Gromov-Hausdorff sense).

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- Applying Colding's Volume Convergence Theorem,

$$\mathrm{Vol}(B_{s_0}(\tilde{q}_j)) \geq (1 - 10^{-3})\,\mathrm{Vol}(B_{s_0}(0^n)) \geq \frac{1}{2}\omega_n s_0^n > 0. \quad (51)$$

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- There exist a sequence of collapsing manifolds with positive Ricci curvature and with maximal nilpotent rank, but the limit space of their universal covers does not split off any line.
- This collapsing sequence is based on an example of positively Ricci curved manifold constructed by Guofang Wei (1989).

An Example

Wei constructed an example of a complete non-compact manifold (M^n, g) which satisfies the following property:

- with $\mathrm{Ric}_g > 0$
- a simply-connected nilpotent Lie group $\mathcal{N}^k$ acts freely and isometrically on M^n.
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We choose a sequence of co-compact lattices $\Gamma_j \in \mathcal{N}^k$ such that $\Gamma_j \to \mathcal{N}^k$, then

- $\mathrm{rank}(\Gamma_j) = \dim \mathcal{N}^k = k$,
- $$\begin{array}{ccc} M^n & = & M^n \\ \downarrow{\scriptstyle \mathrm{pr}_i} & & \downarrow{\scriptstyle \mathrm{pr}_\infty} \\ M^n/\Gamma_j & \longrightarrow & \mathbb{R}^{n-k} \equiv M^n/\mathcal{N}^k. \end{array} \tag{52}$$

Step 3: Line Splitting v.s. Cone Splitting

Theorem (Cheeger-Colding's Line Splitting, 1996)

Let $(M_i^n, g_i, p_i) \xrightarrow{GH} (X, d, p)$ *with* $\mathrm{Ric} \geq -\delta_i \to 0$. *Assume that* (X, d, p) *admits a line, then* $X \cong \mathbb{R}^k \times Y$ *such that* Y *does not admit any line.*

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Theorem (Cone Splitting Principle)

Let $(C(Y), y^*)$ *be a metric cone with vertex* y^**. Assume that there is another cone tip* $\bar{y}^* \in C(Y)$*, then there exists a line connecting* y^* *and* $\bar{y}^*$*. Moreover,*

$$C(Y) \equiv \mathbb{R}^1 \times C(W). \tag{53}$$

General Collapsed Limit $\mathbb{R}^{k-\ell} \times Z^{\ell}$

- Step 2 gives a sequence of non-collapsing universal covers:

$$(\widetilde{M}_j^n, \tilde{p}_j) \xrightarrow{GH} (\mathbb{R}^{k-\ell} \times \widetilde{Z}^{n-k+\ell}, (0^{k-\ell}, \tilde{z}^{n-k+\ell})). \qquad (54)$$

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- There is a blowing-up sequence $\lambda_j \to \infty$ such that

$$(\lambda_j \cdot \widetilde{M}^n_j, \tilde{p}_j, \mathcal{N}_j) \xrightarrow{eqGH} (\mathbb{R}^{k-\ell} \times C(\widetilde{Y}), (0^{k-\ell}, y^*), \mathcal{N}_\infty), \quad (55)$$

where $C(\widetilde{Y}) \equiv \mathbb{R}^d \times C(\widehat{W})$ and the integer $d \geq 0$ is chosen such that $(C(\widehat{W}), w^*)$ does not split off any Euclidean factor.

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- The key is to show that $d = n - k$.

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- Cone Splitting Principle $\Longrightarrow \mathcal{N}_\infty \cdot \hat{z} \subset \mathbb{R}^d$, and thus

$$d \geq n - k. \tag{57}$$

THANK YOU!