Collapsing geometry of hyperkähler manifolds in dimension four

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Background

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  - volume collapsing if $\mathrm{Vol}_{g_j}(B_1(x_j)) \to 0$, equivalently, $\dim_{\mathcal{H}}(X_\infty) < n$

Motivation: compactification of the K3 moduli

- Let $\mathcal{K}$ be the K3 manifold and we denote

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- Consider the Gromov-Hausdorff compactification $\overline{\mathfrak{M}}^{GH}$. $\overline{\mathfrak{M}}^{GH} \setminus \mathfrak{M}^*$ consists of volume collapsed limits of hyperkähler metrics whose periods diverge to infinity in $\mathcal{D}$.

Summary of the main results

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Theorem (Sun-Z, 2021)

Any element $(X_\infty, d_\infty) \in \overline{\mathfrak{M}}^{GH} \setminus \mathfrak{M}^*$ *has an integer Hausdorff dimension, and* (X_∞, d_∞) *is isometric to one of the following:*

- *(dim 3) a flat orbifold* $\mathbb{T}^3/\mathbb{Z}_2$*,*
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Theorem (Sun-Z, 2021)

Let (X^4, g) *be a complete non-compact hyperkähler manifold satisfying* $\int_{X^4} |\mathrm{Rm}|^2 < \infty$. *Then* (X^4, g) *has a unique asymptotic cone. Moreover,* X^4 *has an asymptotic model end.*

Dimension of a Ricci limit space

Let $(X^n_j, g_j, p_j) \xrightarrow{GH} (X_\infty, d_\infty, p_\infty)$ satisfy $\mathrm{Ric}_{g_j} \geq \Lambda$ and $\nu_j \equiv (\mathrm{Vol}_{g_j}(B_1(p_j)))^{-1} \, \mathrm{dvol}_{g_j}$.

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We define $\dim_{\mathrm{ess}}(X_\infty) \equiv k$ the essential dimension of X_∞. In general, $\dim_{\mathrm{ess}}(X_\infty) \leq \dim_{\mathcal{H}}(X_\infty)$. There are examples of Ricci-limit spaces with $1 < \dim_{\mathcal{H}}(X_\infty) < 2$ (Pan-Wei, 2021).

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- In general, very little is known in the collapsing case

Collapsing v.s. non-collapsing

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Local structure	metric cone structure	?
Regularity theory	✓	?
Regular set	smooth Einstein	?
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- Hope to obtain "almost symmetries" over "regular regions" in collapsing Einstein manifolds.

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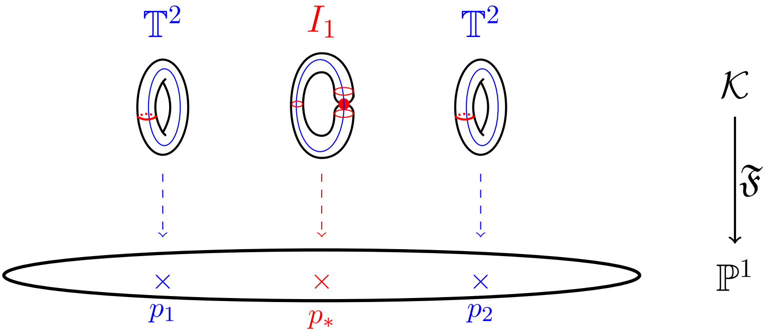
Furthermore, $\operatorname{rank}(N) = 4 - k \Longrightarrow \sup_{B_1(p_j)} |\operatorname{Rm}| \leq C_0$.

Motivating picture: elliptic fibration of K3



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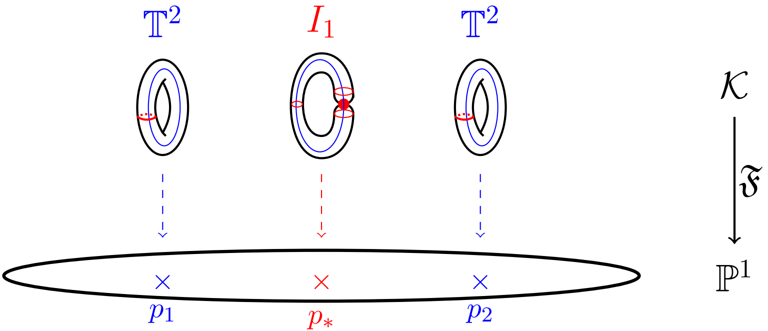

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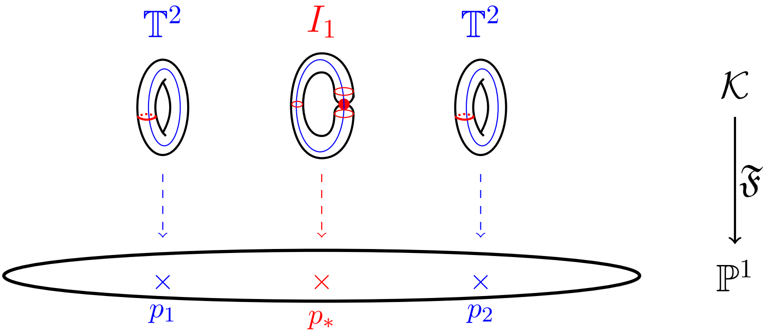

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- Refined geometry of the singular region other than π_1?

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- Obtained an N_∞-invariant hyperkähler metric on $\widehat{\mathcal{U}}$

The Gibbons-Hawking ansatz

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Then we have a principal S^1-bundle $\mathcal{M}$ with a connection $-\sqrt{-1}\Theta$ such that $d\Theta = *_{\mathbb{R}^3} \circ dV$,

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- Key idea: extra order $r^{-\delta}$ gives control on holonomy at infinity and excludes inhomogeneous collapsing

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Let g_j be hyperkähler metrics on the K3 manifold $\mathcal{K}$ such that

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Remark: Measure-topology correspondence was discovered in Sun-Zhang 2019 and Honda-Sun-Zhang 2019

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- $d = 2$
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If (X^4, g) has a unique asymptotic cone Y_∞ s.t. $r^{-1}X^4 \xrightarrow{GH} Y_\infty$ as $r \to \infty$, then we define $\dim_\infty(X^4) \equiv \dim_{\text{ess}}(Y_\infty)$.

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Moreover, in the ALH / ALH* *case, the asymptotics has an exponentially decaying order.*

Asymptotic models

- Let $(X^4, r^{-2} \cdot g, p) \xrightarrow{GH} (Y^d_\infty, \hat{d}_\infty, \hat{p}_\infty)$, $\quad d < 4, \quad r \to \infty$.

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- AL$\mathfrak{X}$ model metrics are $\mathcal{N}$-invariant for $\mathfrak{X} \in \{F, G, G^*, H, H^*\}$
- Old friends: ALF / G / H, new friends: ALG^* and ALH^*

Asymptotic models of type ALF/G/H

Model geometry on (X^4, g): $\exists$ diffeo. $\Phi : \mathscr{M}^4 \to X^4 \setminus K$ s.t.

1. $\mathcal{N}^{4-d} \to \mathscr{M}^4 \xrightarrow{F} A_{R,\infty}(\mathbf{0}) \subset C(\Sigma^{d-1})$ flat cone, $1 \leq d \leq 3$,
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- (ALF model)

ALF-A$_k$: $C(\Sigma^2) \equiv \mathbb{R}^3$, g_{model} given by applying the Gibbons-Hawking ansatz to $V = 1 + \frac{k+1}{2r}$.

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- (ALH model) $\mathcal{N}^3 = \mathbb{T}^3$, quotient $= [R, \infty)$, $g_{model} = dz^2 + g_{\mathbb{T}^3}$.

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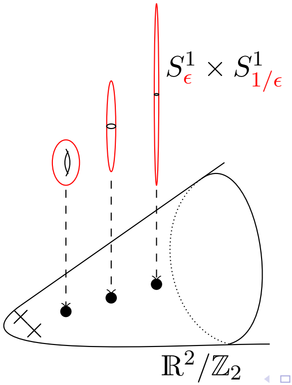

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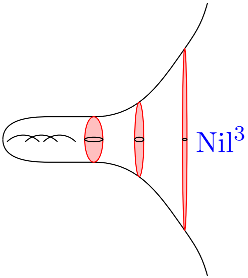

Gravitational instantons appear as bubble limits

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A motivating example

- Taub-NUT space (singular S^1-bundle) $S^1 \to \mathbb{C}^2 \xrightarrow{F} \mathbb{R}^3$:
  $g_{TN} = V(dx^2 + dy^2 + dz^2) + V^{-1}\Theta^2, \quad V = 1 + (2r)^{-1}$,
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- Classical theory only provides the collapsing S^1-bundle (0th order geometry), and we need the 1st order geometry to retrieve the degree of F.

Thank you for your attention!